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INDEX.

PAGE

Agnew, Ralph Palmer. On summability of multiple sequences, 62

Albert, A. Adrian. Normal division algebras of degree 4 over F of characteristic 2, 75

Alden, H. H. Solution of $f(x, y)\frac{\partial z}{\partial x} + g(x, y)\frac{\partial z}{\partial y} = 0$ in a neighborhood of a singular point, 593

Barber, Sherburne F. Planar Cremona transformations, 109

Bell, Clifford. Alternant surfaces, 42

Blumenthal, Leonard M. A chain of determinant theorems arising from the characterization of pseudo r-spheric (S_n) sets, 225

Brahana, H. R. On isomorphisms of the abelian group of type $1, 1, \cdots$, 53

Brahana, H. R. On the metabelian groups which contain a given group H as a maximal invariant abelian subgroup, 490

Carroll-Rusk, Evelyn. Cremona involutions defined by a pencil of cubic surfaces, 96

Chanler, Josephine H. Poristic double binary forms, 529

Coble, Arthur B. Cremona's diophantine equations, 459

Cope, Frances Thorndike. Formal solutions of irregular linear differential equations. Part I, 411

Dickson, L. E. Two-fold generalizations of Cauchy's lemma, 513

Godeaux, Lucien. Sur quelques transformations birationnelles involutives associées à une cubique gauche, 214

Goeritz, Lebrecht. Die Betti'schen Zahlen der zyklischen Überlagerungsräume der Knotenaussenräume, 194

Gupta, Hansraj. A problem in diophantine analysis, 269

Harry, C. H. Concerning the geometry of acyclic sets, 233

Haviland, E. K. and Wintner, Aurel. On the Fourier-Stieltjes transform, 1

Haviland, E. K. and Wintner, Aurel. A note on the Kronecker-Weyl theorem, 17

Haviland, E. K. On the theory of absolutely additive distribution functions, 625

James, R. D. The representation of integers as sums of values of cubic polynomials, 303

Johnson, Roberta F. Involutions of order two associated with the surfaces of genera $p_a = p_g = 0$, $P_2 = 1$, $P_3 = 0$, 199

Kimball, B. F. Application of Bernoulli polynomials of negative order to differencing (Second Paper), 147

PAGE

Latimer, Claiborne G. On the units in a cyclic field, ... 69

Levy, Harry. Linearly connected spaces and ennuples of curves, . 381

Lewis, Daniel C., Jr. On certain periodic motions of dynamical systems with more than two degrees of freedom, ... 25

Lewis, D. C., Jr. and Wintner, Aurel. On the distribution of successive images in the Poincaré transformation problem of a circle into itself, ... 407

Lewis, Daniel C., Jr. On the distribution of conjugate points on a closed geodesic, ... 613

Lowan, Arnold N. Note on the cooling of a radioactive sphere, . . 254

Lowan, Arnold N. Heat conduction in a semi-infinite radioactive solid, 396

Lunn, A. C. and Senior, J. K. A method of determining all the solvable groups of given order and its application to the orders $16p$ and $32p$, ... 319

Lunn, A. C. See Senior and Lunn, page 328.

Lunn, A. C. and Senior, J. K. Note on a unique representation for every solvable group, ... 511

Mayor, John R. A generalization of the Veronese and Steiner surfaces, 372

Miller, G. A. Groups involving four operators which are squares, . . 47

Miller, William I. Fundamental regions for the simple group of order 168 in S_4, ... 316

Montgomery, Deane. Properties of plane sets and functions of two variables, ... 569

Morrey, Charles B., Jr. A class of representations of manifolds. Part II, ... 275

Murray, F. H. Optical paths in the ionosphere, ... 259

Rutt, N. E. Some theorems on triodic continua, ... 122

Senior, J. K. See Lunn and Senior, page 319.

Senior, J. K. and Lunn, A. C. Determination of the groups of orders 101-161, omitting order 128, ... 328

Senior, J. K. See Lunn and Senior, page 511.

Spencer, H. Earl. On convergence and oscillation of transforms of sequences of vectors, ... 445

Steenrod, Norman E. Characterizations of certain finite curve-sums, 558

Thielman, Henry P. On the invariance of a generalized Gramian in a Riemannian function space, ... 438

Trott, G. Richard. On the canonical form of a non-singular pencil of Hermitian matrices, ... 359

Vanderslice, John L. Non-holonomic geometries, ... 153

Wade, Thomas L., Jr. Syzygies for Weitzenböck's irreducible complete system of euclidean concomitants for the conic, with an algebraically complete system of such concomitants, ... 349

PAGE

WARD, MORGAN. The representation of Stirling's numbers and Stirling's polynomials as sums of factorials, ... 87

WHYBURN, G. T. Cyclic elements of higher orders, ... 133

WHYBURN, G. T. Non-alternating transformations, ... 294

WHYBURN, WILLIAM M. Matrix differential equations, ... 587

WILDER, R. L. Concerning irreducibly connected sets and irreducible regular connexes, ... 547

WILLIAMSON, J. The covariants of two quadratic forms in n variables, 339

WINTNER, AUREL. See Haviland and Wintner, page 1.

WINTNER, AUREL. On the addition of independent distributions, . . 8

WINTNER, AUREL. See Haviland and Wintner, page 17.

WINTNER, AUREL. On the asymptotic differential distribution of almost-periodic and related functions, ... 401

WINTNER, AUREL. See Lewis and Wintner, page 407.

WINTNER, AUREL. On analytic convolutions of Bernoulli distributions, 659

WONG, B. C. On a certain rational V_n^{2n+1} in S_{2n+1}, ... 219

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ON THE FOURIER-STIELTJES TRANSFORM.

By E. K. HAVILAND AND AUREL WINTNER.

Let $\rho(\xi)$, $-\infty < \xi < +\infty$, be continuous and of bounded variation in the infinite range $[-\infty, +\infty]$ and let

$$(1) \qquad L(t;\rho) = \int_{-\infty}^{+\infty} \exp it\xi \, d\rho(\xi), \qquad -\infty < t < +\infty$$

denote the Fourier-Stieltjes transform of ρ. If ρ is absolutely continuous, then by the Riemann-Lebesgue lemma

$$\int_{-a}^{a} \exp it\xi \, d\rho(\xi) \to 0, \qquad t \to \pm\infty$$

where $a > 0$ is arbitrarily large, so that since $|\exp| = 1$ and $\int_{-\infty}^{+\infty} |d\rho(\xi)| < +\infty$

$$(2) \qquad L(t;\rho) \to 0, \qquad t \to \pm\infty$$

whereas (2) need not hold if the continuous function ρ is not absolutely continuous.[†] On the other hand, the curve $z = L(t;\rho)$, $-\infty < t < +\infty$ shows an asymptotic tendency toward the origin of the z-plane even if (2) does not hold. For on placing

$$(3) \qquad \mathfrak{M}(f(\cdot)) = \lim_{T=+\infty} (2T)^{-1} \int_{-T}^{T} f(t)\,dt$$

one has [‡]

$$(4) \qquad \mathfrak{M}(|L(\cdot;\rho)|^2) = 0$$

for any continuous ρ. In the present note, this asymptotic tendency of $z(t) = L(t;\rho)$ will be formulated in a form which is sharper than (4). Also it will be shown that $L(t;\rho)$ affects the statistical distribution of the z-values by introducing a corresponding [§] sharp diffusion when added to an almost-periodic Fourier-Stieltjes transform, which of itself never has a diffused distribution.[¶] The refinements in question are made possible by the theory of distribution functions.[‖]

[†] Cf. T. Carleman, *loc. cit.* The references are collected at the end of the paper.

[‡] Cf. I. Schoenberg, *loc. cit.*, where reference is made to a paper of N. Wiener. Since then, (4) has often been rediscovered in connection with the unitary dynamics of Carleman and Koopman.

[§] The results as to the diffusing effect of $L(t;\rho)$ proven in the researches mentioned in the previous footnote are not sharper than (4) itself.

[¶] By a diffused distribution is meant a distribution which is not a *proper* distribution; cf. p. 6 of the present note.

[‖] A. Wintner, I and II, and E. K. Haviland, I.

Use will be made of the following results:[†] Let $z(t) = x(t) + iy(t)$, $-\infty < t < +\infty$, be a bounded continuous function for which all time averages $\mathfrak{M}([x(t)]^n [y(t)]^m)$, $n, m = 0, 1, 2, \cdots$, exist. Let R denote a rectangle parallel to the coördinate axes of the z-plane and such that none of the four lines of R lies on an at most denumerable sequence of lines $x = a_j$ or $y = b_k$ which, if they exist at all, depend only on $z(t)$. An R having none of its sides on one of the lines $x = a_j$, $y = b_k$ is called a non-singular rectangle. Let $\{R; T\}$ denote the set of those values t in the interval $-T \leqq t \leqq T$ for which the point $z(t)$ is a point of R, so that

$$(2T)^{-1}\operatorname{meas}\{R; T\}$$

represents the "probability of R between the dates $t = -T$ and $t = T$," viz., the relative amount of time spent by the curve $z = z(t)$ in the rectangle R during the time between $t = -T$ and $t = T$. Now there exists a monotone absolutely additive set function[‡] $\phi = \phi(E)$ represented for $E = R$ by

$$\phi(R) = \lim_{T=+\infty} (2T)^{-1}\operatorname{meas}\{R; T\}. \tag{5}$$

This set function, called the distribution function of $z(t) = x(t) + iy(t)$, satisfies the momentum conditions

$$\int\!\!\int_{-\infty}^{+\infty} x^n y^m d_{xy}\phi(E) = \mathfrak{M}([x(\cdot)]^n [y(\cdot)]^m), \quad (n, m = 0, 1, 2, \cdots) \tag{6}$$

and every monotone solution ϕ of the momentum problem (6) is, up to an additive constant, the distribution function (5) of $z(t) = x(t) + iy(t)$. The additive constant is determined by the condition that $\phi(E) = 0$ if E lies without a sufficiently large circle $|z| = r$. Two set functions ϕ which are identical for all non-singular rectangles are not considered as distinct.

There exist for every function $\sigma(\xi)$, $-\infty < \xi < +\infty$, satisfying the conditions[§]

[†] *Ibid.*

[‡] For the definition of a monotone absolutely additive set function, cf. J. Radon, *loc. cit.* The Stieltjes integral (6) is meant in the sense of Radon and is, in reality, not an improper integral, $\phi(E)$ being clearly independent of E if E lies without a sufficiently large circle.

[§] The second of the conditions (7) is automatically satisfied up to an at most denumerable set of ξ-values.

$$(7)\qquad \int_{-\infty}^{+\infty} |\, d\sigma(\xi)\, | < +\infty, \qquad \sigma(\xi) = \tfrac{1}{2}[\sigma(\xi+0) + \sigma(\xi-0)]$$

precisely one pair of functions $\sigma' = \sigma'(\xi)$, $\sigma'' = \sigma''(\xi)$ such that (7) is satisfied both by $\sigma = \sigma'$ and by $\sigma = \sigma''$, and

$$(8)\qquad \sigma = \sigma' + \sigma''$$

where σ' is everywhere continuous and σ'' is a step-function such that $\sigma''(-\infty) = 0$. From (1) and (8) one has

$$(9)\qquad L(t;\sigma) = L(t;\sigma') + L(t;\sigma'')$$

where $L(t;\sigma'')$ is, in virtue of (7), an almost-periodic function in the sense of Bohr † inasmuch as $\sigma''(\xi) \equiv 0$ or is a true step-function according as $\sigma(\xi)$ is or is not everywhere continuous. On placing

$$(10)\qquad \begin{aligned} \sigma_I(\xi) &= \tfrac{1}{2}[\sigma(\xi) - \sigma(-\xi)] \\ \sigma_{II}(\xi) &= \tfrac{1}{2}[\sigma(\xi) + \sigma(-\xi)] \end{aligned}$$

so that (7) is satisfied by both $\sigma = \sigma_I$ and $\sigma = \sigma_{II}$, the formula

$$(11)\qquad L(t;\sigma) = L(t;\sigma_I) + iL(t;\sigma_{II})$$

gives the decomposition of $L(t;\sigma)$ into real and imaginary parts, inasmuch as the conjugated complex value of $L(t;\sigma)$ is ‡

$$\int_{-\infty}^{+\infty} e^{-it\xi}\, d\sigma(\xi) = \int_{-\infty}^{+\infty} e^{it\xi}\, d[-\sigma(-\xi)].$$

Also,

$$(12)\qquad \begin{aligned} L(t;\sigma_I) &= L(t;\sigma'_I) + L(t;\sigma''_I) \\ L(t;\sigma_{II}) &= L(t;\sigma'_{II}) + L(t;\sigma''_{II}) \end{aligned}$$

where σ'_I, σ'_{II} and σ''_I, σ''_{II} belong to σ' and σ'' in the same way that σ_I, σ_{II} belong to σ.

Besides the above notations, we shall use the following facts: § If (7) is satisfied both by $\sigma = \alpha = \alpha(\xi)$ and by $\sigma = \beta = \beta(\xi)$, then there exists exactly one function of ξ satisfying (7) and represented at all its continuity points ξ by the formula

---

† H. Bohr, *op. cit.*, p. 33.

‡ For convenience it is supposed that $\sigma(\xi)$ is real-valued.

§ Cf. A. Wintner, III and E. K. Haviland, II. The proofs given there suppose that the functions are not only of bounded variation but monotone as well. The latter restriction may, however, be removed in the results to be used by decomposing the function of bounded variation into its monotone components.

$$\int_{-\infty}^{+\infty} \alpha(\xi-\eta)\,d\beta(\eta).$$

This function of ξ, called the Faltung of α and β, is usually denoted by $\alpha * \beta$ and satisfies the relation

$$L(t;\alpha*\beta)=L(t;\alpha)L(t;\beta). \tag{13}$$

Furthermore, (13) is a *characteristic* property of the Faltung $\alpha*\beta$, inasmuch as a function satisfying (7) is, up to an additive constant, uniquely determined † by its Fourier-Stieltjes transform. For the same reason, it is clear from (13) that

$$\alpha*\beta=\beta*\alpha \text{ and } (\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$$

if γ is a further function satisfying (7). The function satisfying (7) and the relation

$$L(t;[\alpha*]^h[\beta*]^j)=[L(t;\alpha)]^h[L(t;\beta)]^j$$

will be denoted by

$$[\alpha*]^h[\beta*]^j \qquad (h,j=0,1,2,\cdots). \tag{14}$$

If (7) is satisfied by $\sigma=\rho$ and if ρ is everywhere continuous, then

$$\mathfrak{M}(L(\cdot\,;\rho))=0. \tag{15}$$

This is clear from (4) by Schwarz's inequality.‡

---

† Cf., e. g., N. Wiener, *loc. cit.*

‡ We shall not use (4) but only the relation (15), the proof of which proceeds as follows (Cf. I. Schoenberg, *loc. cit.*): Since the contribution of the integration domains $[-\infty,-a]$, $[a,+\infty]$ to (1) has for all values of t a modulus $<\epsilon$ if $a>0$ is sufficiently large, it is sufficient to prove that for any fixed value of a and for sufficiently large values of T, the modulus of

$$J=(2T)^{-1}\int_{-T}^{T}\{\int_{-a}^{a}\exp it\xi\,d\rho(\xi)\}dt$$

is $<\epsilon$. The order of the integrations in J is interchangeable (Cf. L. Lichtenstein, *loc. cit.*). Hence

$$J=\int_{-a}^{a}=\left[\int_{-a}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{a}\right]s(T\xi)\,d\rho(\xi)=\sum_{k=1}^{3}J_k(T,\delta)$$

where $s(\eta)=\eta^{-1}\sin\eta$ and $0<\delta<a$. Now from $|s(\eta)|<|\eta|^{-1}$

$$|J_1(T,\delta)+J_3(T,\delta)|\leqq T^{-1}\int_{-\infty}^{+\infty}|d\rho(\xi)|/\delta\to 0, \qquad (T\to+\infty)$$

and from $|s(\eta)|\leqq 1$

$$|J_2(T,\delta)|\leqq\int_{-\delta}^{\delta}|d\rho(\xi)|\to 0, \qquad (\delta\to 0)$$

ρ being continuous at $\xi=0$.

Since σ'_I and σ'_{II} are everywhere continuous in virtue of the continuity of σ' and since a Faltung of functions $\alpha, \beta, \cdots$ at least one of which is everywhere continuous is everywhere continuous,[†] (15) is satisfied by[‡]

$$\rho = [\sigma'_I *]^k [\sigma''_I *]^{n-k} [\sigma'_{II} *]^l [\sigma''_{II} *]^{m-l},$$
$$0 \leqq k \leqq n \geqq 0, \qquad 0 \leqq l \leqq m \geqq 0$$

if at least one of the numbers k, l is $\neq 0$, i. e., if $k + l > 0$. Hence from (13)

$$(16) \qquad \mathfrak{M}([L(\cdot\,;\sigma'_I)]^k [L(\cdot\,;\sigma''_I)]^{n-k} [L(\cdot\,;\sigma'_{II})]^l [L(\cdot\,;\sigma''_{II})]^{m-l}) = 0$$

if $k + l > 0$. The expression (16) need not vanish but certainly exists if $k = l = 0$, inasmuch as each of the functions σ''_I, σ''_{II} is either a step function or $\equiv 0$, so that $L(t;\sigma''_I)^n L(t;\sigma''_{II})^m$ is, as mentioned above, a product of $n + m$ almost-periodic functions and has therefore an $\mathfrak{M}$-value.[§] Since (16) exists for $k = l = 0$ also, it follows from (12) that

$$\mathfrak{M}([L(\cdot\,;\sigma_I)]^n [L(\cdot\,;\sigma_{II})]^m) \qquad (n, m = 0, 1, 2, \cdots), \text{ exists.}$$

This assures[¶] the existence of a distribution function $\phi(E)$ for the function (11), which is a continuous and bounded function of t in virtue of (7) and (1). Furthermore,

$$(17) \qquad \mathfrak{M}([L(\cdot\,;\sigma_I)]^n [L(\cdot\,;\sigma_{II})]^m) = \mathfrak{M}([L(\cdot\,;\sigma''_I)]^n [L(\cdot\,;\sigma''_{II}]^m),$$
$$(n, m = 0, 1, 2, \cdots).$$

For on introducing (12) into the expression on the left of (17) and applying the binomial formula twice, $(n + 1)(m + 1) - 1$ of the resulting $(n + 1)(m + 1)$ terms vanish in virtue of (16). On comparing (17) with the test (6) we see that $z(t) = L(t;\sigma)$ and $z(t) = L(t;\sigma'')$ have one and the same distribution function ϕ.

The Fourier-Stieltjes transform $z(t) = L(t;\sigma)$ *of a function of bounded variation always possess a distribution function* $\phi(E)$. *This function* $\phi(E)$ *is identical with the distribution function of the almost-periodic component* $L(t;\sigma'')$ *of* $L(t;\sigma) = L(t;\sigma') + L(t;\sigma'')$ *so that the complementary component* $L(t;\sigma')$ *does not affect the asymptotic repartition of the points passed by the curve* $z = L(t;\sigma)$, $-\infty < t < +\infty$, *in the z-plane although this curve is, if* $\sigma' \not\equiv 0$, *not identical with the curve* $z = L(t;\sigma'')$. *In other words,*

[†] Cf. A. Wintner, III.

[‡] We are using the notation (14).

[§] H. Bohr, *loc. cit.*, pp. 33 and 34.

[¶] Cf. p. 2 above.

$L(t;\sigma')$ and $L(t;\sigma'')$ have statistically independent distributions,[†] *the distribution function $\psi(R)$ of $L(t;\sigma')$ being $=1$ or $=0$ according as the origin $z=0$ is or is not in the rectangle R.*

The latter statement, which is a precision of (4), may be proven as follows. If $\sigma=\rho$ then $\sigma''=0$, so that

$$L(t;\sigma''_I)\equiv 0, \qquad L(t;\sigma''_{II})\equiv 0,$$

hence

$$\mathfrak{M}([L(\,\cdot\,;\rho_I)]^n[L(\,\cdot\,;\rho_{II})]^m)=0, \qquad (n+m>0)$$

in virtue of (17), where $\sigma=\rho$. On the other hand,

$$\int\!\!\int_{-\infty}^{+\infty} x^n y^m d_{xy}\psi(E)=0, \qquad (n+m>0)$$

if $\psi(R)=0$ for any rectangle R not containing the origin $x=y=0$. Hence (6) is satisfied by placing

$$x(t)+iy(t)=L(t;\rho_I)+iL(t;\rho_{II})=L(t;\rho) \text{ and } \phi=\psi$$

so that ψ is the distribution function of $L(t;\rho)$.

Let $\{z(t)\}$ denote the set of those points in the z-plane which are cluster points of the curve $z=z(t)$, so that $\{z(t)\}$ is a perfect set if the continuous function $z(t)$ is not independent of t. If $z(t)$ possesses a distribution function $\phi(E)$, let $\{z(t)\}_0$ denote the set of those points p for which $\phi(R)\neq 0$ whenever p is within the rectangle R. It follows from our theorem that

$$\{L(t;\sigma')+L(t;\sigma'')\}_0=\{L(t;\sigma'')\}_0$$

although

$$\{L(t;\sigma')+L(t;\sigma'')\}=\{L(t;\sigma'')\}$$

need not hold, of course. It is clear from (5) and from the definition of $\{z(t)\}_0$ that $\{z(t)\}_0$ is a closed subset of $\{z(t)\}$. We may say that $z=z(t)$, $-\infty<t<+\infty$, is *properly* distributed[‡] if $\{z(t)\}_0=\{z(t)\}$. It follows from the definition of the set function $\psi(E)$ that if the continuous function $\sigma'(\xi)$ is not independent of ξ, then $L(t;\sigma')$ is not properly distributed. On the other hand, it may be proven that every almost-periodic function, hence also $L(t;\sigma'')$, is properly distributed.

[†] Cf. A. Wintner, IV and E. K. Haviland, II.

[‡] Cf. A. Wintner, II.

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ON THE ADDITION OF INDEPENDENT DISTRIBUTIONS.

By Aurel Wintner.

The object of the present note is the proof and the precision of some simple theorems which are often tacitly supposed in Calculus of Probability. Although the validity of these theorems is suggested by statistical interpretation (addition of probabilities), satisfactory proofs cannot be obtained in this way. In fact, the statements are purely analytical theorems on a certain Stieltjes integral, so that the question regarding their validity is in reality equivalent to the one whether the usual statistical interpretation of those Stieltjes expressions is always justified or not. Hence a detailed use of the Stieltjes integration theory, for instance the use of a general uniqueness theorem (furnished by the Lemma on p. 12) cannot be avoided.

A function $\rho(\xi)$ is termed a distribution function if it is defined for every real ξ, is monotone in the whole range $-\infty < \xi < +\infty$, and satisfies the boundary conditions $\rho(-\infty)=0$, $\rho(+\infty)=1$. If for two distribution functions ρ_1, ρ_2

$$\text{(1a)} \qquad \rho_1(\xi)=\rho_2(\xi) \quad \text{at} \quad \xi=X_i \qquad (i=1,2,\cdots)$$

where $\{X_i\}$ is an everywhere dense sequence on the real axis then

$$\text{(1b)} \qquad \rho_1(\xi+0)=\rho_2(\xi+0),\ \rho_1(\xi-0)=\rho_2(\xi-0);\ -\infty<\xi<+\infty$$

inasmuch as the limits $\rho_k(\xi \pm 0)$ may be obtained by using one of those monotone subsequences which have the limit $\xi \pm 0$ where ξ is any real number. It is clear from (1b) and from the Stieltjes integral definition that if

$$K_i = \int_{-\infty}^{+\infty} f(\xi)\, d\rho_i(\xi)$$

exists for $i=1$ then it exists for $i=2$ also and $K_1=K_2$, where $f(\xi)$ is any monotone function. On placing

$$\text{(2)} \qquad \Delta\rho(\xi)=\rho(\xi+0)-\rho(\xi-0) \geqq 0$$

so that $\Delta\rho(\xi)=0$ holds if and only if ξ be a continuity point of ρ, we see from (1b) that $\Delta\rho_1(\xi)=\Delta\rho_2(\xi)$ at every ξ, i. e. that ρ_1 and ρ_2 have the same continuity points and the same discontinuity points, and at the latter ones the same jumps $\Delta\rho$. Furthermore, from (1b), $\rho_1(\xi) = \rho_2(\xi)$ holds at every continuity point and therefore up to an at most denumerable set. For these reasons, two distribution functions will be considered as identical if they satisfy the condition (1a), i. e., we shall write $\rho_1 = \rho_2$ if (1b) holds for every ξ. This agreement will be important in what follows.

By the vectorial sum $R_1 + R_2$ of two sets R_i of numbers is understood † the set of those numbers x which may be represented in at least one way in the form $x = x_1 + x_2$ where x_i is an element of R_i. In connection with (5) we shall need the symbol $R_1 + R_2$ also in the case where R_1 or R_2 or both do not contain any element. In this case, viz. if at least one of the two sets R_i be empty, we shall understand by $R_1 + R_2$ the empty set. According to this convention, which is essential for our purposes, the set $R_1 + R_2$ may be empty although R_1 is not empty. We shall often use the evident fact that the vectorial sum of two closed sets is a closed set.

In accordance with the Hilbert theory of bounded Hermitian matrices ‡ we shall designate as spectrum $S = S(\rho)$ of the distribution function $\rho(\xi)$ the set of those numbers x for which there does not exist a $\delta = \delta_x > 0$ such that $\rho(\xi)$ is constant in the interval $x - \delta < \xi < x + \delta$. Correspondingly,§ the set $P = P(\rho)$ of the discontinuity points of ρ may be termed the point-spectrum of the distribution function. $P(\rho)$ is a subset of $S(\rho)$ and $P(\rho)$ is at most denumerable and may be the empty set whereas $S(\rho)$ contains at least one point inasmuch as the total variation of ρ is distinct from zero. $S(\rho)$ is always, $P(\rho)$ not necessarily a closed set.

We shall prove the following theorem: *For every pair of distribution functions $\sigma_1(x)$, $\sigma_2(x)$ there exists exactly one* ¶ *distribution function $\sigma(x)$ such that the integral ("Faltung")* ‖

$$(3) \qquad \int_{-\infty}^{+\infty} \sigma_1(x - \xi)\, d\sigma_2(\xi)$$

exists and $= \sigma(x)$ at all points x not contained in $P(\sigma_1) + P(\sigma_2)$. The integral representation (3) *of this uniquely determined distribution function $\sigma(x)$ breaks down at all points of $P(\sigma_1) + P(\sigma_2)$; i. e. $\sigma_1(x - \xi)$ is not*

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† Cf. H. Bohr and B. Jessen, "Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver," *D. Kgl. Danske Vidensk. Selsk. Skrifter*, ser. 8, no. 3, vol. 12 (1929), pp. 331-332.

‡ Regarding the theorem in question, cf. p. 178 of the author's book, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929.

§ Cf. *ibid.*, p. 194, etc.

¶ Cf. the agreement (p. 9) regarding $\rho_1 = \rho_2$.

‖ Cf. e. g. R. Deltheil, *Erreurs et moindres carrés*, Paris, 1930.

integrable with respect to $\sigma_2(\xi)$ *in these points* x. *The spectrum and the point spectrum of* σ *may a priori be calculated from those of* σ_1 *and* σ_2 *by means of the formulae*

$$S(\sigma) = S(\sigma_1) + S(\sigma_2) \tag{4}$$

and

$$P(\sigma) = P(\sigma_1) + P(\sigma_2). \tag{5}$$

Finally, the function σ *remains unaltered*[†] *if in its definition one interchanges the two given functions* σ_1, σ_2.[‡] *The integral* $\int_a^b$ *occurring in the definition*

$$\int_a^b \to \int_{-\infty}^{+\infty} \qquad (a \to -\infty,\ b \to +\infty) \tag{6}$$

of (3) *is understood in the so-called Riemann-Stieltjes sense.*[§]

A necessary and sufficient condition for the existence of the Riemann-Stieltjes integral

$$\int_a^b \beta_1(\xi)\, d\beta_2(\xi) \qquad (-\infty < a < b < +\infty) \tag{7}$$

is well known.[¶] In the special case where both functions β_i are of bounded variation[‖] the condition referred to is easily seen to be satisfied if and only if at every point ξ at least one of the two functions β_i is continuous. Now ξ is a common discontinuity point of the two functions

$$\beta_1(\xi) = \sigma_1(x - \xi), \qquad \beta_2(\xi) = \sigma_2(\xi) \tag{7a}$$

if and only if x may be represented in the form $x = \xi_1 + \xi_2$ where ξ_i denotes

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[†] Cf. the agreement (p. 9) regarding $\rho_1 = \rho_2$.

[‡] This follows immediately from the partial integration formula on p. 15 inasmuch as $\sigma_i(-\infty) = 0$, $\sigma_i(+\infty) = 1$; $i = 1, 2$, hence $c = 0$.

[§] Cf. E. W. Hobson, *Theory of Functions of a Real Variable*, 2nd Edition, vol. 1, pp. 507-508. There is, however, an error at this point. In the definitions of $\overline{S}$ and $\underline{S}$ in § 377, it is necessary to replace $\sum_{r=1}^{m} f(x_r)\{\phi(x_r + 0) - \phi(x_r - 0)\}$ by

$$\sum_{r=0}^{m} f(x_r)\{\phi(x_r + 0) - \phi(x_r - 0)\},$$

with the provision that $\phi(x_0 - 0) = \phi(a)$. Otherwise the definitions favor one end-point above the other and are not consistent with the definition of p. 507. I am obliged for this remark to Dr. Haviland.

[¶] *Ibid.*, p. 509.

[‖] Regarding this special case cf. also R. Schmidt, "Ueber divergente Folgen und lineare Mittelbildungen," *Mathematische Zeitschrift*, vol. 22 (1925), pp. 123-125.

a discontinuity point of $\sigma_i(\xi)$; $i=1,2$. Consequently, the integral (7) occurring in the definition (6) of (3) exists for sufficiently large values of $-a$ and b if and only if x is not a point of $P(\sigma_1)+P(\sigma_2)$. Hence (6) and therefore (3) is meaningless if x is a point of $P(\sigma_1)+P(\sigma_2)$. On the other hand, if x is not a point of $P(\sigma_1)+P(\sigma_2)$, then not only does $\int_a^b$ exist for all values of a and b but so also does the limit (6) inasmuch as the σ_i are distribution functions so that $0 \leqq \sigma_1 \leqq 1$ and

$$(8) \qquad 0 \leqq \pm \int_{\pm t}^{\pm\infty} |\, d\sigma_2(\xi)\,| \leqq \epsilon \quad \text{when} \quad t \geqq T_\epsilon > 0.$$

Let $\sigma_0(x)$ denote the value of the integral (3) provided that the latter exists, and let Q be the set of those points which are not contained in $P(\sigma_1)+P(\sigma_2)$. The function $\sigma_0(x)$ is accordingly not defined for every x but only on the set Q. Since the complement $P(\sigma_1)+P(\sigma_2)$ of Q is at most denumerable, Q is everywhere dense so that the process $x \to \lambda$ where x is a point of Q is meaningful for every real point λ and also for $\lambda = \pm\infty$. With this meaning of the symbol $\to$ we have

$$(9) \qquad \sigma_0(x) \to 0,\ x \to -\infty; \quad \sigma_0(x) \to 1,\ x \to +\infty.$$

In fact, since σ_1 is a distribution function and therefore $\sigma_1(-\infty)=0$, we may choose for every $\epsilon > 0$ the number T_ϵ so large that not only (8) is satisfied but also

$$(10) \qquad \sigma_1(x-\xi) \leqq \epsilon \quad \text{whenever} \quad |\,\xi\,| \leqq T_\epsilon \quad \text{and} \quad x \leqq -2T_\epsilon.$$

Since σ_1 and σ_2 are distribution functions and therefore $\geqq 0$, $\leqq 1$ and monotone non-decreasing, it is clear from the definition (3) of σ_0 and from (8) and (10) that

$$0 \leqq \sigma_0(x) \leqq \int_{-\infty}^{-T_\epsilon} d\sigma_2(\xi) + \int_{-T_\epsilon}^{T_\epsilon} \epsilon\, d\sigma_2(\xi) + \int_{T_\epsilon}^{+\infty} d\sigma_2(\xi) \leqq 3\epsilon$$

for every point $x \leqq -2T_\epsilon$ of Q. The second of the relations (9) is similarly proven from $\sigma_1(+\infty)=1$. Furthermore, if x_1 and $x_2 > x_1$ denote two points of Q then $\sigma_0(x_2)-\sigma_0(x_1)$ is, according to (3),

$$= \int_{-\infty}^{+\infty} [\sigma_1(x_2-\xi) - \sigma_1(x_1-\xi)]\, d\sigma_2(\xi)$$

and therefore $\geqq 0$ inasmuch as $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are non-decreasing functions. Hence $\sigma_0(x)$ is monotone on the everywhere dense set Q on which it is defined. Consequently we may define for all values of x a function $\sigma(x)$ by placing

$\sigma(x) = \sigma_0(x)$ on Q and $\sigma(x) = \sigma_0(x+0)$ on the complement $P(\sigma_1) + P(\sigma_2)$ of Q. This monotone function $\sigma(x)$ is, by virtue of (9), a distribution function. Finally, since Q is everywhere dense, there cannot exist another distribution function which is equal to $\sigma_0(x)$ at all points of Q (cf. p. 9).

In order to prove (4) we need the following

LEMMA. Let $\beta_1(\xi)$ and $\beta_2(\xi)$ be two functions of bounded variation without a common discontinuity point in the finite range $a \leqq \xi \leqq b$ so that the integral

$$I = \int_a^b \beta_1(\xi)\, d\beta_2(\xi)$$

exists. Suppose that β_2 is monotone and that β_1 is either everywhere $\geqq 0$ or everywhere $\leqq 0$, finally that $I = 0$. Then at least one of the numbers[†] $\beta_1(x)$, $\beta_1(x-0)$, $\beta_1(x+0)$ is zero if, for arbitrarily small values of ϵ, the value of $\beta_2(\xi)$ is not independent of ξ in the ϵ-vicinity[‡] of x. The same holds in the case $a = -\infty$, $b = +\infty$, provided that the integral I defined by (6) exists.

Since $I = 0$, we may confine the proof to the case where β_1 is non-negative and β_2 non-decreasing. On denoting by $V_2(\epsilon; x) \geqq 0$ the total variation of $\beta_2(\xi)$ in the ϵ-vicinity of the arbitrary point x, and by $\lambda_1(\epsilon; x)$ the greatest lower bound of $\beta_1(\xi)$ in the same vicinity, we have

$$0 = I \geqq \lambda_1(\epsilon; x) V_2(\epsilon; x) \geqq 0,$$

i. e. either $V_2(\epsilon; x) = 0$ or else $\lambda_1(\epsilon; x) = 0$. Suppose that there does not exist an ϵ-vicinity of x such that $\beta_2(\xi)$ is independent of ξ in this vicinity. For these values of x and for every $\epsilon > 0$ we have $V_2(\epsilon; x) > 0$. Hence $\lambda_1(\epsilon; x) = 0$ so that either $\beta_1(x) = 0$ or else the zeros of β_1 are clustering at x. Since in the latter case either $\beta_1(x+0) = 0$ or else $\beta_1(x-0) = 0$, the Lemma is proven.[§]

The Lemma enables us to prove the less easy half of the theorem (4), viz. the fact that the vectorial sum $S(\sigma_1) + S(\sigma_2)$ of the component spectra is a subset of the spectrum $S(\sigma)$ of the resulting distribution function σ. Let y be a point not contained in $S(\sigma)$. We have to prove that y cannot be a point of $S(\sigma_1) + S(\sigma_2)$. Since the points not contained in the at most denumerable set $P(\sigma_1) + P(\sigma_2)$ lie everywhere dense, and since $S(\sigma)$ is a closed set, we may suppose that y is not a point of $P(\sigma_1)+P(\sigma_2)$, i. e. that the integral representation (3) of $\sigma(x)$ exists at $x=y$. Since y is not a point of $S(\sigma)$, there exists an $\epsilon>0$ such that $\sigma(y)-\sigma(\eta)=0$ when $|y-\eta|<\epsilon$. We consider at present only those values η in this ϵ-vicinity of y which are not contained in the at most denumerable set $P(\sigma_1)+P(\sigma_2)$. Then (3) exists at $x=\eta$ and we have

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[†] One of these three numbers is undefined, and should be omitted in the above statement, if $x = a$ or $x = b$.

[‡] By this we understand the one-sided ϵ-vicinity of x if and only if $x = a$ or $x = b$.

[§] The lemma may be interpreted as a refinement of the Hilfssatz 2 of O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig, 1913, p. 368.

$$0=\sigma(y)-\sigma(\eta)=\int_{-\infty}^{+\infty}[\sigma_1(y-\xi)-\sigma_1(\eta-\xi)]d\sigma_2(\xi).$$

Consequently, on placing with fixed values of y and η

$$\beta_1(\xi)=\sigma_1(y-\xi)-\sigma_1(\eta-\xi), \qquad \beta_2(\xi)=\sigma_2(\xi),$$

the premises of the Lemma are satisfied inasmuch as $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are monotone. Hence if x be a point of $S(\sigma_2)$ then at least one of the three numbers $\beta_1(x)$, $\beta_1(x+0)$, $\beta_1(x-0)$ is zero, i. e. at least one of the three equations

$$\begin{aligned}\sigma_1(\eta-x)&=\sigma_1(y-x),\\ \sigma_1(\eta-x-0)&=\sigma_1(y-x-0),\\ \sigma_1(\eta-x+0)&=\sigma_1(y-x+0)\end{aligned}$$

is true. Since this holds for all those values of η in the ϵ-vicinity of the fixed point y which are not contained in the at most denumerable set $P(\sigma_1)+P(\sigma_2)$ and since $\sigma_1(\xi)$ is a monotone function, it is clear that the value of $\sigma_1(\eta-x)$ for the fixed value of x is independent of η when $|y-\eta|<\epsilon$, even if η be a point of $P(\sigma_1)+P(\sigma_2)$. Hence the value of $\sigma_1(\xi)$ is independent of ξ if $|y-x-\xi|<\epsilon$, i. e. $y-x$ cannot be a point of $S_1(\sigma_1)$. Now x was any fixed point of $S(\sigma_2)$. Consequently y cannot be a point of $S(\sigma_1)+S(\sigma_2)$, q. e. d.

We now prove the converse, viz. that $S(\sigma)$ is a subset of $S(\sigma_1)+S(\sigma_2)$. Let y be any point not contained in $S(\sigma_1)+S(\sigma_2)$. We have to prove that y is not a point of $S(\sigma)$. Let $C=C_\gamma$ denote the closed interval whose end-points are $y-\gamma$ and $y+\gamma$ where $\gamma>0$. Since y is not a point of the closed set $S(\sigma_1)+S(\sigma_2)$ we may choose γ so small that $S(\sigma_1)+S(\sigma_2)$ and C have no point in common. There exists then a constant $\alpha>0$ such that

$$|x_1+x_2-\zeta|\geqq\alpha \tag{11}$$

where x_1, x_2, ζ denote arbitrary points of $S(\sigma_1)$, $S(\sigma_2)$, C respectively. Since y and ζ are not contained in $S(\sigma_1)+S(\sigma_2)$ and since $P(\sigma_i)$ is a subset of $S(\sigma_i)$ where $i=1,2$, the points y and ζ are not contained in $P(\sigma_1)+P(\sigma_2)$. Hence the integral representation (3) of $\sigma(x)$ exists at $x=y$ and at $x=\zeta$ so that

$$(12) \qquad \sigma(\zeta)-\sigma(y)=\int_{-\infty}^{+\infty}[\sigma_1(\zeta-\xi)-\sigma_1(y-\xi)]d\sigma_2(\xi).$$

Since $S(\sigma_2)$ is a closed set, the contribution of the immediate vicinity of ξ to the integral (12) is nothing if ξ be not a point of $S(\sigma_2)$. Hence if we prove that for sufficiently small values of $|y-\zeta|$ the contribution of the immediate vicinity of ξ to the integral (12) is nothing even if ξ be a point of $S(\sigma_2)$, it will be proven that $\sigma(\zeta)-\sigma(y)=0$ for sufficiently small values of $|y-\zeta|$, i. e. that y cannot be a point of $S(\sigma)$. Let now ξ be a fixed point of $S(\sigma_2)$ and let η denote any point in the ϵ-vicinity of ξ. We choose $\epsilon<\alpha$ where α denotes the same constant as in (11). Suppose that there exists in the ϵ-vicinity $|\xi-\eta|\leqq\epsilon$ of ξ a point η such that $x_1=\zeta-\eta$ is a point of $S(\sigma_1)$ where ζ denotes a suitably chosen point in the interval C. Since $x_2=\xi$ is a point of $S(\sigma_2)$ we have then

$$|x_1+x_2-\zeta|=|(\zeta-\eta)+\xi-\eta|=|\xi-\eta|\leqq\epsilon<\alpha,$$

in contradiction to (11). Our supposition, viz. that $\zeta-\eta$ represents for suitably chosen values of ζ and η a point of $S(\sigma_1)$, is therefore wrong, i. e., if $|y-\zeta|\leqq\gamma$ and $|\xi-\eta|\leqq\epsilon$ then $\zeta-\eta$ cannot be a point of $S(\sigma_1)$. Since these inequalities are satisfied by $\zeta=y$, $\eta=\xi$ respectively, the point $y-\xi$ is not a point of $S(\sigma_1)$, i. e. the function σ_1 is constant in the vicinity of the point $y-\xi$. Hence there exists a $\delta>0$ such that $\sigma_1(\zeta-x)-\sigma_1(y-x)=0$ when $|\xi-x|\leqq\delta$ and $|\zeta-y|\leqq\delta$. The contribution of the δ-vicinity of the point ξ of $S(\sigma_1)$ to the integral (12) is therefore

$$=\int_{\xi-\delta}^{\xi+\delta}[\sigma_1(\zeta-x)-\sigma_1(y-x)]d\sigma_2(x)=\int_{\xi-\delta}^{\xi+\delta}0\cdot d\sigma_2(x)=0$$

for sufficiently small (viz. $\leqq\delta$) values of $|\zeta-y|$, q. e. d. This completes the proof of (4).

In order to prove (5), suppose first that σ_1 is everywhere continuous. Then $\sigma_1(\xi)$ is uniformly continuous in the infinite range $-\infty<\xi<+\infty$ inasmuch as $\sigma_1(-\infty)=0$, $\sigma_1(+\infty)=1$. Since

$$|x'-x''|=|(x'-\xi)-(x''-\xi)|,$$

there exists therefore for every $\epsilon>0$ a $\delta=\delta_\epsilon>0$ such that

$$|\sigma_1(x'-\xi)-\sigma_1(x''-\xi)|\leqq\epsilon, \qquad -\infty<\xi<+\infty,$$

whenever $|x'-x''|\leqq\delta$. Moreover, σ_1 being everywhere continuous, σ_1 and σ_2 do not possess a common discontinuity point so that the integral representation (3) of $\sigma(x)$ holds for every x. Accordingly

$$|\sigma(x')-\sigma(x'')|=\left|\int_{-\infty}^{+\infty}[\sigma_1(x'-\xi)-\sigma_1(x''-\xi)]d\sigma_2(\xi)\right|\leqq\int_{-\infty}^{+\infty}\epsilon d\sigma_2(\xi)=\epsilon$$

whenever $|x'-x''|\leqq\delta$. Hence σ is everywhere continuous if σ_1 be everywhere continuous, although σ_2 may have discontinuity points. Furthermore, σ is everywhere continuous if not σ_1 but σ_2 be everywhere continuous inasmuch as

$$\sigma(x)=\int_{-\infty}^{+\infty}\sigma_1(x-\xi)\,d\sigma_2(\xi)=c+\int_{-\infty}^{+\infty}\sigma_2(x-\xi)\,d\sigma_1(\xi),$$
$$c=\sigma_1(-\infty)\sigma_2(+\infty)-\sigma_1(+\infty)\sigma_2(-\infty)$$

by virtue of the Stieltjes formula for partial integration.[†] The proofs obviously hold also in the case where σ_1 and σ_2 are bounded and monotone without being distribution functions.

Due to the definition of the vectorial addition of empty sets (p. 9), the relation (5) is thus proven for the case where at least one of the functions σ_1, σ_2 is everywhere continuous. Suppose therefore that both functions have discontinuity points, and let $\{\xi_{ik}\}$ denote the finite or infinite sequence of discontinuity points of σ_i where $i=1,2$ so that according to (2)

$$(13)\qquad \Delta\sigma_1(\xi_{1n})>0,\qquad \Delta\sigma_2(\xi_{2m})>0.$$

On placing

$$(14)\qquad \sigma_i^*(\xi)=\sum_{\xi}\Delta\sigma_i(\xi_{ik})\quad\text{and}\quad\sigma_i^{**}(\xi)=\sigma_i(\xi)-\sigma_i^*(\xi)$$

where the summation runs through all those values of k for which $\xi_{ik}<\xi$, all four functions σ_i^*, σ_i^{**} are monotone and bounded, and both functions σ_i^{**} are everywhere continuous[‡] ($i=1,2$). In order to prove that $P(\sigma)$ is a subset of $P(\sigma_1)+P(\sigma_2)$ let y denote a point not contained in $P(\sigma_1)+P(\sigma_2)$. We have to prove that y is not a point of $P(\sigma)$. Since y is not a point of $P(\sigma_1)+P(\sigma_2)$, the integral representation (3) of $\sigma(x)$ exists at $x=y$ and may be written, by virtue of (14), as the sum of four analogous integrals. Three of them are everywhere continuous by virtue of the last remark in the previous paragraph inasmuch as σ_1^{**} and σ_2^{**} do not possess discontinuity points. The fourth integral is, according to (14) and (2),

$$\int_{-\infty}^{+\infty}\sigma_1^*(x-\xi)\,d\sigma_2^*(\xi)=\sum_k\sigma_1^*(x-\xi_{2k})\Delta\sigma_2(\xi_{2k})$$

[†] Cf. e. g. E. W. Hobson, *op. cit.*, p. 507.

[‡] That is to mean, the distribution function $\sigma_i(x)$ may be modified at its discontinuity points $x=\xi_{ik}$ in such a way that the function $\sigma_i^{**}(x)$ belonging to the modified distribution function $\sigma_i(x)$ be everywhere continuous. The modified function is identical with the original function by virtue of the agreement regarding $\rho_1=\rho_2$ (p. 9).

where the summation runs through all values of k. Let $\phi(x)$ denote this series. $\phi(x)$ is uniformly convergent in the infinite range $-\infty < x < +\infty$ inasmuch as $\sigma_1{}^*$ is a bounded function and the sum of all jumps $\Delta\sigma_2(\xi_{2k}) > 0$ of the monotone bounded function σ_2 is finite. Hence at a discontinuity point of $\phi(x)$ at least one of the functions $\phi_k(x) = \sigma_1{}^*(x - \xi_{2k})$ is discontinuous. Now y is by supposition not a point of $P(\sigma_1) + P(\sigma_2)$, i. e. y cannot be represented in the form $y = \xi_{1l} + \xi_{2k}$. Hence $x = y - \xi_{2k}$ is not a discontinuity point of $\sigma_1(x)$ and therefore not a discontinuity point of

$$\sigma_1{}^*(x) = \sigma_1(x) - \sigma_1{}^{**}(x)$$

inasmuch as $\sigma_1{}^{**}$ is everywhere continuous. Consequently every $\phi_k(x)$ and therefore $\phi(x)$ itself is continuous at $x = y$. Since $\sigma(x) - \phi(x)$ is the sum of the three everywhere continuous integrals, $\sigma(x)$ is continuous at $x = y$, i. e. y is not a point of $P(\sigma)$, q. e. d.

In order to prove the converse, viz. that $P(\sigma_1) + P(\sigma_2)$ is a subset of $P(\sigma)$, we have to prove that if $y = \xi_{1n} + \xi_{2m}$ where n and m are fixed, then σ is discontinuous at y, i. e. that

$$\Delta\sigma(y) \neq 0; \qquad y = \xi_{1n} + \xi_{2m}. \tag{15}$$

Since the set $P(\sigma_1) + P(\sigma_2)$ is at most denumerable and has therefore an everywhere dense complement, we may choose a sequence $\{\epsilon_j\}$ such that, on the one hand,

$$0 < \epsilon_j \to 0 \quad \text{as} \quad j \to +\infty, \tag{16}$$

and, on the other hand, neither $y + \epsilon_j$ nor $y - \epsilon_j$ is contained in $P(\sigma_1) + P(\sigma_2)$ where $j = 1, 2, \cdots$. The integral representation (3) of $\sigma(x)$ exists then at every point $x = y \pm \epsilon_j$ and yields

$$\begin{aligned} \sigma(y + \epsilon_j) - \sigma(y - \epsilon_j) &= \int_{-\infty}^{+\infty} [\sigma_1(y + \epsilon_j - \xi) - \sigma_1(y - \epsilon_j - \xi)]\, d\sigma_2(\xi) \\ &\geqq [\sigma_1(y + \epsilon_j - \xi_{2m}) - \sigma_1(y - \epsilon_j - \xi_{2m})]\, [\sigma_2(\xi_{2m} + 0) - \sigma_2(\xi_{2m} - 0)] \\ &= [\sigma_1(\xi_{1n} + \epsilon_j) - \sigma_1(\xi_{1n} - \epsilon_j)]\Delta\sigma_2(\xi_{2m}) \end{aligned}$$

inasmuch as $y = \xi_{1n} + \xi_{2m}$. Accordingly from (2) and (16)

$$\Delta\sigma(y) \geqq \Delta\sigma_1(\xi_{1m})\Delta\sigma_2(\xi_{2m}).$$

Hence (15) follows by virtue of (13).

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A NOTE ON THE KRONECKER-WEYL THEOREM.

By E. K. Haviland and Aurel Wintner.

Let $x_k(t)$, $k = 1, 2, \cdots n$, be n real functions defined for $-\infty < t < +\infty$. For simplicity, we assume them continuous. Let $\mathfrak{T}$ denote the n-dimensional torus $0 \leqq x_k < 1$ resulting from the n-dimensional Euclidean space by reduction mod 1 and let $\mathfrak{C}$ denote the curve $x_k = x_k(t)$ on this torus. It is supposed also that all time-averages

$$\mathfrak{M}[\exp\{2\pi i \sum_{k=1}^{n} m_k x_k(t)\}], \qquad (m_1, \cdots m_n = 0, \pm 1, \pm 2, \cdots)$$

where

$$\mathfrak{M}(\cdots) = \lim_{T=+\infty} 1/2T \int_{-T}^{T} \cdots dt$$

exist.

In the present paper it will be shown that to every region $\mathfrak{R}$: $(a_k \leqq x_k \leqq b_k)$ on $\mathfrak{T}$ there belongs an asymptotic probability $\phi(\mathfrak{R})$ representing the average time spent by $\mathfrak{C}$ in $\mathfrak{R}$, provided that $\mathfrak{R}$ is non-singular in the sense that the boundary numbers a_k and b_k of $\mathfrak{R}$ do not belong to an at most denumerable set.* This monotone set function ϕ satisfies the momentum conditions

$$\mathfrak{M}[\exp\{2\pi i \sum_{k=1}^{n} m_k x_k(t)\}] = \int_{\mathfrak{T}} \exp\{2\pi i \sum_{k=1}^{n} m_k x_k\} d\phi(E)$$

and is in the main uniquely determined by these conditions. The assumption that all time-momenta $\mathfrak{M}$ exist seems to be a rather restrictive hypothesis. In reality, it turns out to be not only sufficient but also necessary for the existence of a distribution function ϕ. In the particular case $x_k(t) = \lambda_k t + \alpha_k$, where the λ_k and α_k are real numbers and the λ_k linearly independent, our theorem yields the Kronecker-Weyl theorem according to which in this linear case $\mathfrak{C}$ is asymptotically "gleichverteilt" on $\mathfrak{T}$.† In fact, the time-momenta $\mathfrak{M}$ all exist and are zero unless all m_k are zero. Hence $\phi(\mathfrak{R}) =$ meas $\mathfrak{R}$ satisfies the momentum conditions and is therefore the distribution function of the asymptotic probability. The existence of the distribution function ϕ is, however, assured by our theorem also in the case $x_k(t) = \lambda_k t + \psi_k(t)$, where $\psi_k(t)$ is any real almost periodic function, although there need not then be an asymptotic equipartition of $\mathfrak{C}$ on $\mathfrak{T}$. Also it is not necessary that $\exp[ix_k(t)]$ be almost periodic. In the particular case where the monotone set function $\phi(\mathfrak{R})$ is absolutely continuous and may therefore be represented as a Lebesgue integral[*]

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* That these exceptional lines may actually occur is shown by an example of H. Bohr, "Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. II," *Det. Kgl. Danske Videnskabernes Selskab. Meddelelser*, vol. 10 (1930), no. 10.

† Cf. H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins," *Mathematische Annalen*, vol. 77 (1916), pp. 319-320.

$$\phi(\mathfrak{R}) = \int_{\mathfrak{R}} f(x_1, \cdots x_n)\, dx_1 \cdots dx_n, \qquad (f \geqq 0),$$

f is a function of the position on the torus and represents the density of probability ($f \equiv 1$ in the case of equipartition). ϕ is, however, not always absolutely continuous. If ϕ is absolutely continuous, every $\mathfrak{R}$ is non-singular in the sense mentioned above, although the converse is not necessarily true.

The condition that the $x_k(t)$ be continuous may be replaced by the more general one that $x_k(t)$ be measurable in every finite t-interval. Hence it is allowed that $x_k(t)$ be a step-function constant in every interval $m \leqq t < m+1$, so that all considerations hold also if the curve $x_1 = x_1(t), \cdots x_n = x_n(t)$ is replaced by a sequence $x_1 = x_1(m), \cdots x_n = x_n(m)$, $m = 0, \pm 1, \pm 2, \cdots$ as considered, in the particular case of an equipartition, by Weyl.[†]

The proof of our theorem closely parallels that given for almost-periodic complex functions of a real variable[‡] and is an extension of the theorem proved in the case of one dimension.[§] We restrict our considerations for convenience to the case $n = 2$ and first prove the

UNIQUENESS THEOREM (I). *Let there be given an absolutely additive set function $\phi(E)$ such that $\phi(E) = 0$ if E lies outside the fixed rectangle J: $0 \leqq x < 1$; $0 \leqq y < 1$* and

$$(1) \quad \int\!\!\int_{-\infty}^{+\infty} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 2\pi nx \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 2\pi my\, d_{xy}\phi(E) = \int\!\!\int_{-\infty}^{+\infty} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 2\pi nx \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\, 2\pi my\, d_{xy}\phi(E) = 0$$

for all non-negative integral values of n, m. Then

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[*] Cf., e. g., E. W. Hobson, *Theory of Functions of a Real Variable*, 2nd Edition, vol. 1 (1920), p. 545.

[†] H. Weyl, *loc. cit.*, pp. 318-319.

[‡] E. K. Haviland, "On statistical methods in the theory of almost-periodic functions," *Proceedings of the National Academy of Sciences*, vol. 19 (1933), pp. 549-555. See also C. A. Fischer, "Linear functionals of N-spreads," *Annals of Mathematics*, ser. 2, vol. 19 (1917-1918), pp. 37-43, or S. Bochner, "Monotone Funktionen," *Mathematische Annalen*, vol. 108 (1933), pp. 378-410.

[§] A. Wintner, "On the distribution function of almost-periodic angular variables," *American Journal of Mathematics*, vol. 55 (1933), pp. 606-610.

$$\int\int_R d_{xy}\phi(E) = 0$$

for every non-singular[§] *rectangle R.*

Proof. If $e: (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$ be a rectangle whose side $y = y_1$ is a non-singular line of ϕ, then it is known[*] that

$$\lim_{y_2 = y_1} \int\int_e | d_{xy}\phi(E) | = 0$$

and a similar result holds if the rôles of x and y be interchanged. From (1) it follows by the Weierstrass Approximation Theorem[†] that

$$\int\int_{-\infty}^{+\infty} f(x, y)\, d_{xy}\phi(E) = \int\int_J f(x, y)\, d_{xy}\phi(E) = 0$$

for all periodic functions continuous in x, y together. In particular, we consider the continuous function $f_n(x, y)$ defined as follows: Let $R: (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$ be the non-singular rectangle referred to in the statement of the theorem. Let S_n be the rectangle[‡]

$$(x_1 - n^{-1} \leqq x < x_2 + n^{-1};\ y_1 - n^{-1} \leqq y < y_2 + n^{-1})$$

where n is a positive integer. In R and on its boundary $f_n(x, y) \equiv 1$. Without S_n, $f_n(x, y) \equiv 0$. In $S_n - R$, $f_n(x, y)$ shall be represented by that point of the truncated pyramid having S_n as base and R as top whose projection is (x, y). Then

$$\int\int_{-\infty}^{+\infty} f_n(x, y)\, d_{xy}\phi(E) = \int\int_R d_{xy}\phi(E) + \sum_{k=1}^{4} \int\int_{e_{nk}} f_n(x, y)\, d_{xy}\phi(E)$$

where the e_{nk}, $k = 1, \cdots 4$, are four non-overlapping rectangles which compose $S_n - R$.[§] The integral on the left is zero by hypothesis. Moreover,

$$| \int\int_{e_{nk}} f_n(x, y)\, d_{xy}\phi(E) \quad < \int\int_{e_{nk}} | d_{xy}\phi(E) | < \tfrac{1}{4}\epsilon, \quad (n > N,\ k = 1, \cdots 4).$$

Hence $| \int\int_R d_{xy}\phi(E) | < \epsilon$ for $n > N$, so $\int\int_R d_{xy}\phi(E) = 0$, q. e. d.

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[*] E. K. Haviland, *loc. cit.*, p. 550.

[†] Cf. L. Tonelli, *Serie Trigonometriche*, Bologna, 1928, p. 494.

[‡] It is supposed that the images of S_n, mod 1, do not overlap. This involves no restriction on R.

[§] E. K. Haviland, *loc. cit.*, p. 551.

With the help of the foregoing theorem, we prove the

MOMENTUM THEOREM (II). *Let there be given a continuum of monotone absolutely additive set functions* ϕ_T *of uniformly bounded variation and such that* $\phi_T(E) = 0$ *for all* T *and for all* E *outside the fixed rectangle* $J\colon 0 \leqq x < 1;\ 0 \leqq y < 1$. *Then if as* T *becomes infinite*

$$\lim \int\!\!\int_{-\infty}^{+\infty} \cos 2\pi nx \cos 2\pi my \, d_{xy}\phi_T(E) = \mu_{nm}^{(1)},$$

$$\lim \int\!\!\int_{-\infty}^{+\infty} \cos 2\pi nx \sin 2\pi my \, d_{xy}\phi_T(E) = \mu_{nm}^{(2)},$$

$$\lim \int\!\!\int_{-\infty}^{+\infty} \sin 2\pi nx \cos 2\pi my \, d_{xy}\phi_T(E) = \mu_{nm}^{(3)},$$

and

$$\lim \int\!\!\int_{-\infty}^{+\infty} \sin 2\pi nx \sin 2\pi my \, d_{xy}\phi_T(E) = \mu_{nm}^{(4)},$$

for all non-negative integers n, m, *then* $\lim \phi_T(R) = \phi(R)$, *where* $\phi(E)$ *has the same properties as* $\phi_T(E)$ *and* R *is any rectangle non-singular with respect to* $\phi(E)$.

Proof. It follows from the Compactness Theorem for monotone absolutely additive set functions* that there exists a sequence of functions $\{\phi_r{}^{\mathrm{I}}(R)\}$ such that $\lim \phi_r{}^{\mathrm{I}}(R) = \phi^{\mathrm{I}}(R)$, where $\phi^{\mathrm{I}}(R)$ possesses the same properties as ϕ_T. If it were possible to select a second sequence $\{\phi_r{}^{\mathrm{II}}(R)\}$ converging to $\phi^{\mathrm{II}}(R)$, we could apply the Term by Term Integration Theorem for Radon integrals to

$$\lim_{r=\infty} \int\!\!\int_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi my \, d_{xy}\phi_r{}^{\mathrm{I}}(E) = \lim_{r=\infty} \int\!\!\int_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi my \, d_{xy}\phi_r{}^{\mathrm{II}}(E)$$

obtaining

$$\int\!\!\int_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi my \, d_{xy}\phi^{\mathrm{I}}(E) = \int\!\!\int_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi my \, d_{xy}\phi^{\mathrm{II}}(E)$$

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* For proof, see J. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften*, Wien, vol. 122 (1913), pp. 1337-1342. Revised, *ibid.*, vol. 128 (1919), pp. 1093-1094. A statement of this theorem and of the Term by Term Integration Theorem more convenient for our purposes is to be found in E. K. Haviland, *loc. cit.*, pp. 551-552.

or
$$\iint\limits_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi my\; d_{xy}\psi(E) = 0$$

where $\psi = \phi^{\mathrm{I}} - \phi^{\mathrm{II}}$ is an absolutely additive set function. Similarly

$$\iint\limits_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi nx \begin{matrix}\sin\\ \cos\end{matrix}\, 2\pi my\; d_{xy}\psi(E) = 0.$$

It follows from our Uniqueness Theorem that $\phi^{\mathrm{I}}(R) = \phi^{\mathrm{II}}(R)$ on all rectangles R non-singular with respect to ψ, ϕ^{I} and ϕ^{II}, and hence on every rectangle R non-singular with respect to $\phi^{\mathrm{I}} = \phi^{\mathrm{II}}$.* On placing $\phi^{\mathrm{I}} = \phi$, we obtain $\lim \phi_T(R) = \phi(R)$ on all non-singular rectangles of ϕ, q. e. d.

We next consider

THEOREM III. *Let $x(t)$, $y(t)$ be two single-valued real functions of the real variable t defined in $-\infty < t < +\infty$ and measurable in every finite interval. Let E be any set in the x, y-plane such that if T is a positive number, (E, T), the set of those points t for which $-T \leqq t \leqq T$ and the curve $x = x(t)$, $y = y(t)$ reduced mod 1 belongs to E, is measurable. Then the monotone set function $\phi_T(E) = 1/2T$ meas (E, T), which is easily seen to be absolutely additive on the sets E above defined,† is such that*

$$(2)\quad 1/2T \int_{-T}^{T} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi[nx(t) + my(t)]\,dt = \iint\limits_{-\infty}^{+\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi(nx + my)\, d_{xy}\phi_T(E)$$

where n, m are real integers or zero.

Proof. Since $\cos 2\pi[nx(t) + my(t)]$ is a bounded measurable function of t,‡ the Lebesgue integral on the left of (2) exists and the limit of its Lebesgue approximating sums is

$$(3)\qquad \lim \sum_{\nu=1}^{p} \cos 2\pi(nx_\nu + my_\nu) \text{ meas } \{\eta_{\nu-1} \leqq f(t) < \eta_\nu\}$$

where $A = \eta_0 < \eta_1 \cdots < \eta_p = B$ is a subdivision of $[A, B]$, where $A < f(t) < B$ and $\cos 2\pi(nx_\nu + my_\nu)$ is a value of $f(t) = \cos 2\pi[nx(t) + my(t)]$ in $[\eta_{\nu-1}, \eta_\nu]$ if such exists; otherwise meas $\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} = 0$ so the definition is not affected. Again

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* E. K. Haviland, *loc. cit.*, p. 550.

† Cf. J. Radon, *loc. cit.*, p. 1299.

‡ Cf., e. g., E. W. Hobson, *op. cit.*, vol. 1, p. 518.

$$\operatorname{meas}\{\eta_{\nu-1} \leqq f(t) < \eta_\nu\} = \sum_{i=1}^{r}\sum_{j=1}^{s} \operatorname{meas}\left\{\begin{array}{l} \eta_{\nu-1} \leqq f(t) < \eta_\nu \\ x_{i-1} \leqq (x) < x_i \\ y_{j-1} \leqq (y) < y_j \end{array}\right\} = \sum_{i=1}^{r}\sum_{j=1}^{s} \operatorname{meas}\{R_{\nu ij}\}$$

since the sets on the right are disjoint, the $[x_{i-1}, x_i)$ representing a division of $[0, 1)$ into r equal parts and the $[y_{j-1}, y_j)$ a similar division of $[0, 1)$ and (x) denotes as usual the fractional part $x - [x]$ of x and likewise $(y) = y - [y]$.

Then (3) may be written

$$\lim \sum_{\nu=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{s} \cos 2\pi(nx_{\nu ij} + my_{\nu ij}) \operatorname{meas}\{R_{\nu ij}\} \tag{4}$$

where $(x_{\nu ij}, y_{\nu ij})$ is a point of $(x_{i-1} \leqq x < x_i;\ y_{j-1} \leqq y < y_j)$ for which $\eta_{\nu-1} \leqq f(t) < \eta_\nu$, by virtue of the periodicity of the cosine. Furthermore, (4) may be rewritten as

$$2T \lim \sum_{\nu=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{s} \cos 2\pi(nx_{\nu ij} + my_{\nu ij})\phi_T(R_{\nu ij}), \tag{5}$$

where $R_{\nu ij}$ denotes the set of those points (x, y) for which the three conditions

$$x_{i-1} \leqq x < x_i,\ y_{j-1} \leqq y < y_j \quad \text{and} \quad \eta_{\nu-1} \leqq \cos 2\pi(nx + my) < \eta_\nu$$

are fulfilled, and $2T\phi_T(R_{\nu ij})$ is the measure of the set of those points t in $[-T, T]$ such that $x = x(t)$, $y = y(t)$, reduced mod 1, satisfy these three conditions. That the set of points t corresponding to $R_{\nu ij}$ is actually measurable (so that $R_{\nu ij}$ belongs to the domain of definition of ϕ_T) follows from the fact that it is the intersection of an at most denumerable number of measurable sets, viz. those points t in $[-T, T]$ for which $\eta_{\nu-1} \leqq f(t) < \eta_\nu$, those for which $M + x_{i-1} \leqq x(t) < M + x_i$ and those for which $N + y_{j-1} \leqq y(t) < N + y_j$; $M, N = 0, \pm 1, \pm 2, \cdots$.* Since

$$\sum_{\nu=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{s} R_{\nu ij} = J\colon\ (0 \leqq x < 1;\ 0 \leqq y < 1) \quad \text{and} \quad \lim_{p,r,s=\infty} \delta(R_{\nu ij}) = 0$$

and since $\cos 2\pi(nx + my)$ is uniformly continuous in $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$ and hence a fortiori in J, the Radon integral on the right of (2) exists, $\phi_T(E)$ being zero for sets E lying outside J. As the integral is independent of the mode of subdivision of J, provided only the latter be sufficiently fine, it follows that this integral is given by (5) divided by $2T$, thus proving (2). A similar result holds if cosine is replaced by sine.

We now immediately obtain

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* Cf., e. g., E. Kamke, *Das Lebesguesche Integral* (1925), p. 61, Theorem 9.

THEOREM IV. *If $x(t)$, $y(t)$, as defined in the previous theorem, are such that*

$$\lim_{T=\infty} 1/2T \int_{-T}^{T} \exp\,[2\pi i\{nx(t) + my(t)\}]dt = \mathfrak{M}\{\exp\,[2\pi i(nx + my)]\}$$

exists for all integral values of n and m, and if $\phi_T(E)$ is the set function there defined, there exists a monotone absolutely additive set function $\phi(E)$ such that as T becomes infinite, lim $\phi_T(E) = \phi(E)$ for all non-singular rectangles of ϕ and $\phi(E) = 0$ for all sets E lying outside J. Furthermore,

$$\mathrm{M}_{nm}(\phi) = \mathfrak{M}\{\exp\,[2\pi i(nx + my)]\}$$

where

$$\mathrm{M}_{nm}(\phi) = \int\!\!\int_{-\infty}^{+\infty} \exp\,[2\pi i(nx + my)]\; d_{xy}\phi(E)$$

and $\phi(E)$ is uniquely determined by its trigonometrical momenta save on its singular lines.

Proof. From the preceding theorem it follows that

$$(6)\qquad 1/2T \int_{-T}^{T} \exp[2\pi i\{nx(t) + my(t)\}]dt = \int\!\!\int_{-\infty}^{+\infty} \exp[2\pi i(nx + my)]d_{xy}\phi_T(E).$$

Hence as T becomes infinite,

$$\lim \int\!\!\int_{-\infty}^{+\infty} \exp[2\pi i(nx + my)]d_{xy}\phi_T(E)$$

exists. Consequently

$$\lim \int\!\!\int_{-\infty}^{+\infty} \cos 2\pi(nx + my)\, d_{xy}\phi_T(E) \quad\text{and}\quad \lim \int\!\!\int_{-\infty}^{+\infty} \cos 2\pi(nx - my)\, d_{xy}\phi_T(E)$$

exist and hence

$$\lim \int\!\!\int_{-\infty}^{+\infty} \cos 2\pi nx \cos 2\pi my\; d_{xy}\phi_T(E) = \mu_{nm}^{(1)}$$

exists and the existence of $\mu_{nm}^{(i)}$, $i = 2, 3, 4$ follows similarly. The Momentum Theorem then shows that lim $\phi_T(R) = \phi(R)$ for all rectangles R non-singular with respect to ϕ. Applying the Term by Term Integration Theorem to the right-hand side of (6), we obtain

$$\lim_{T=\infty} 1/2T \int_{-T}^{T} \exp[2\pi i\{nx(t)+my(t)\}]dt = \int_{-\infty}^{+\infty}\int \exp[2\pi i(nx+my)]d_{xy}\phi(E)$$

i. e.,

$$\mathfrak{M}\{\exp[2\pi i(nx+my)]\} = \mathrm{M}_{nm}(\phi)$$

where it follows from the Uniqueness Theorem that ϕ is completely determined up to its singular lines save for an additive constant.

Conversely, we may state

THEOREM V. *If the curve $x = x(t)$, $y = y(t)$, reduced mod 1, possesses a distribution function $\phi(E)$, then*

$$\lim_{T=\infty} 1/2T \int_{-T}^{T} \exp[2\pi i\{nx(t) + my(t)\}]dt$$

exists for all integral values of n, m.

Proof. By hypothesis, the functions $\phi_T(E)$ exist and $\lim_{T=\infty} \phi_T(E)$ exists, $= \phi(E)$, where E is any set of points in the x, y-plane such that the points t in $[-T, T]$, T arbitrarily large, for which $(x(t), y(t))$, reduced mod 1, lies in E are measurable. Under these conditions, Theorem III holds and applying the Term by Term Integration Theorem to the right-hand side of (2) (the integration being effectively over $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$), we obtain the result that

$$\lim_{T=\infty} 1/2T \int_{-T}^{T} {\cos \atop \sin} 2\pi[nx(t) + my(t)]dt$$

exists, which implies the existence of

$$\lim_{T=\infty} 1/2T \int_{-T}^{T} \exp[2\pi i\{nx(t) + my(t)\}]dt \qquad \text{q. e. d.}$$

THE JOHNS HOPKINS UNIVERSITY.

ON CERTAIN PERIODIC MOTIONS OF DYNAMICAL SYSTEMS WITH MORE THAN TWO DEGREES OF FREEDOM.

By Daniel C. Lewis, Jr.[*]

1. *Introduction.* It is well known that the motions near a given periodic motion of a dynamical system with $n+1$ degrees of freedom can be studied with the help of a Hamiltonian system of order $2n$:

$$(1.1)\qquad dx_j/dt=\partial H/\partial y_j,\quad dy_j/dt=-\partial H/\partial x_j\qquad (j=1,2,\cdots n).$$

This reduction from the original $2n+2$-th order system is carried out with the help of the energy integral, a certain change of variables, and the elimination of the time. The t which occurs in (1.1) is not the time but an angular coördinate. H is an analytic function of $x_1, y_1, x_2, y_2, \cdots x_n, y_n$, and t; and admits the period 2π in t. The given periodic motion appears as a generalized equlibrium point, $x_j=y_j=0$ $(j=1,2,\cdots n)$, and any further periodic solutions of (1.1), near this equilibrium point and having a period which is an integral multiple of 2π, correspond to periodic motions in the original $2n+2$-th order system.

In case the given periodic motion is of general stable type, the existence of an infinite number of further periodic motions in the neighborhood of the given motion was proved in a joint paper by Birkhoff and myself.[†] In the non-integrable case of two degrees of freedom it is known that infinitely many of these periodic motions are of stable type and infinitely many are of unstable type.[‡] It is purposed here to obtain an analogous result for the present case of $n+1$ degrees of freedom.

It is found that, if a certain symmetric matrix of invariants of the differential equations is the matrix of a definite quadratic form, there will "in general" exist infinitely many periodic motions of stable type and infinitely many of each unstable type for which the characteristic exponents are all real or pure imaginary. In case the invariant matrix does not yield a definite form, it appears possible that none of the periodic motions may be of stable type, while some (though never all) will appear as periodic motions with complex characteristic exponents, $a + bi$ $(a \neq 0, b \neq 0)$.

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[†] "On the periodic motions near a given periodic motion of a dynamical system," *Annali di Matematica*, serie 4, tomo 12, pp. 117-133. This paper will hereafter be referred to as BL.

[‡] G. D. Birkhoff, "Dynamical systems," *American Mathematical Society Colloquium Publications*, vol. 9, chapter VIII, 2.

The weak part of the paper lies in the hypothesis that the "general case" as defined in § 5 may actually occur. It is freely admitted that it would be highly impracticable to test an example for the fulfillment of the conditions of the definition. The hypothesis, the fulfillment of which appears to the writer to be entirely plausible, is submitted without further apology.

We assume a preliminary well known normalization of the function H.* Namely, we write H in the form,

$$H = \sum_{j=1}^{n} \lambda_j x_j y_j + P_\mu(x_1 y_1, \cdots x_n y_n) + K_\mu, \tag{1.2}$$

where the λ_j are constants (the characteristic exponents or multipliers). P_μ is a polynomial in the n products $x_1y_1, \cdots x_ny_n$, of degree $\mu + 1$, at most, and without linear terms. The coefficients of P_μ are constants. K_μ is a convergent power series in $x_1, y_1, \cdots x_n, y_n$, beginning with terms of degree not lower than $2\mu + 3$, the coefficients being analytic periodic functions of t. This normal form can always be attained for an arbitrarily chosen $\mu \geqq 1$, provided that there are no homogeneous linear relations with integral coefficients, not all zero, connecting the λ_j and $i (= \sqrt{-1})$.

The given periodic motion is assumed to be of general stable type: That is, the λ_j are pure imaginary and admit no such commensurability relations, and the determinant of the coefficients of the quadratic terms in P_μ is not zero. It may be observed in this connection that the matrix of these coefficients is independent of the choice of μ; and, in fact, if $\mu' > \mu$, $P'_{\mu'}$ and P_μ do not differ in terms up to the $\mu + 1$-th degree inclusive.

It is also convenient to note here the well-known fact that in this case of pure imaginary characteristic exponents, the Hamiltonian conjugate variables (x_j, y_j) occurring in (1.2) are conjugate imaginary, while the coefficients of P_μ are pure imaginary.

Using the normal form for H, the solution of (1.1) which takes on the initial values x_{j0}, y_{j0}, for $t = 0$, may be written in the form,

$$\begin{aligned} x_j &= x_{j0} \exp\left[+ itM_j(x_0y_0)/(2\pi)\right] + X_j(t, x_0, y_0) \\ y_j &= y_{j0} \exp\left[- itM_j(x_0y_0)/(2\pi)\right] + Y_j(t, x_0, y_0) \\ & \qquad (j = 1, 2, \cdots n), \end{aligned} \tag{1.3}$$

where

$$M_j(x_0y_0) = (2\pi/i)\left[\lambda_j + \partial P_\mu/\partial(x_{j0}y_{j0})\right], \tag{1.4}$$

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* G. D. Birkhoff, "Dynamical systems," *American Mathematical Society Colloquium Publications*, vol. 9, chapter III, 7, 8, 9.

and $X_j\ (t, x_0, y_0)$ and $Y_j\ (t, x_0, y_0)$ are power series in $x_{10}, y_{10}, \cdots x_{n0}, y_{n0}$, beginning with terms of degree not lower than $2\mu + 1$, the coefficients being analytic functions of t. These series converge absolutely and uniformly for any fixed range of values for t, if the $x_{10}, y_{10}, \cdots x_{n0}, y_{n0}$ are taken sufficiently small.

2. *The transformation T and some of its properties.* We now set $t = 2\pi$ in equations (1. 3), and replace the x_j and y_j by x_{j1} and y_{j1}, respectively:

$$x_{j1} = x_{j0} \exp\left[+ iM_j(x_0 y_0)\right] + X_j(2\pi, x_0, y_0)$$
$$y_{j1} = y_{j0} \exp\left[- iM_j(x_0 y_0)\right] + Y_j(2\pi, x_0, y_0).$$

These equations define a transformation T of the neighborhood of the origin into itself; and evidently, since H is assumed to admit the period 2π in t, there is a one-to-one correspondence between the periodic solutions of (1. 1) of period $2m\pi$ and the points that are invariant under T^m, the m-th iterate of T. The existence of such invariant points was proved in BL.

It is more convenient to write T in terms of the modified polar coördinates $(u_1, \theta_1, u_2, \theta_2, \cdots u_n, \theta_n)$, where

$$u_j = x_j y_j\,;\ \ \theta_j = \tan^{-1}[i(y_j - x_j)/(x_j + y_j)].$$

Since x_j and y_j were conjugate imaginary, it is easily seen that these new variables are real. The transformation T now appears in the form:

$$\begin{aligned} u_{j1} &= u_{j0} + U_j(u_0, \theta_0) \\ \theta_{j1} &= \theta_{j0} + M_j(u_0) + \Theta_j(u_0, \theta_0), \end{aligned} \tag{2. 1}$$

where $U_j(u, \theta)$ may be represented as a convergent power series in $u_1^{1/2}, \cdots u_n^{1/2}$, beginning with terms of degree not lower than $2\mu + 2$, the coefficients being analytic periodic functions of $\theta_1, \theta_2, \cdots \theta_n$. $\Theta_j(u, \theta)$ is an abbreviation for an expression of the form,

$$f_j(u, \theta)/[u_j^{1/2} - g_j(u, \theta)],$$

where f_j and g_j are convergent power series in the $u_j^{1/2}$ beginning with terms of degree not lower than $2\mu + 1$, the coefficients being analytic periodic functions of the θ's.

Let $u_{1m}, \theta_{1m}, \cdots u_{nm}, \theta_{nm}$, represent the point into which the point $u_{10}, \theta_{10}, \cdots u_{n0}, \theta_{n0}$ is carried by T^m; $m = 1, 2, \cdots$.

It is easily shown, on account of the Hamiltonian form of the differential equations in x_j, y_j, that not only is

$$\sum_{j=1}^{n} (u_{jm} d\theta_{jm} - u_{j0} d\theta_{j0})$$

an exact differential but its indefinite integral (thought of as a function of the u_{j0} and θ_{j0}) is periodic in the θ_{j0} with period 2π. Incidentally this is in part a mere reflection of the well known fact that the Hamiltonian form of the differential equations is preserved by the change from the variables (x_j, y_j) to the new variables (u_j, θ_j).

The greater part of the analysis in BL was directed toward the proof of the following

FUNDAMENTAL THEOREM: *There exists an infinite number of manifolds in the space of the $2n$ variables, $u_{10}, \theta_{10}, \cdots u_{n0}, \theta_{n0}$, each manifold defined by equations of the type,*

$$u_{j0} = B_j(\theta_{10}, \theta_{20}, \cdots \theta_{n0}), \qquad (j = 1, 2, \cdots n), \tag{2.2}$$

along which, for a suitable choice of the integer m, the θ_{jm} differ from the θ_{j0} by integral multiples of 2π. The B_j are analytic single valued non-vanishing periodic functions of period 2π in the θ_0's.

Furthermore a manifold of this type may be taken in such a way that, given a positive number $\alpha < \frac{1}{2}$, the u_{j0} along the manifold satisfy the following relations:

1. $u_{j0}/u_{k0} \geqq 2\alpha$ *for all pairs of indices,* $j, k = 1, 2, \cdots n$.
2. $\sum_{j=1}^{n} u_{j0}^2 \; [\equiv \zeta_0^2]$ *is arbitrarily small.*
3. $0 < m \leqq K\zeta_0^{1-\mu/4}$, *where K is independent of the u_{j0}.*

These results were obtained under the assumption that the transformation T (or the original differential equations) had been normalized to the extent that $\mu \geqq 8n + 4$.

We shall call a manifold of the above type a manifold B.

We now consider the indefinite integral, $\int \sum_{j=1}^{n} (u_{jm} d\theta_{jm} - u_{j0} d\theta_{j0})$, extended over a manifold B. Since a manifold B is essentially characterized by the existence of a set of integers $k_1, k_2, \cdots k_n$, such that $\theta_{jm} = \theta_{j0} + 2k_j\pi$ along B, we have $d\theta_{jm} = d\theta_{j0}$; and we write

$$J(\theta_{10}, \theta_{20}, \cdots \theta_{n0}) = \int \sum_{j=1}^{n} (u_{jm} - u_{j0}) d\theta_{j0}, \tag{2.3}$$

where J, uniquely defined save for an additive constant, is periodic in the θ_0's.

The existence of invariant points under T^m now follows immediately from the remark due to Birkhoff[*] that any point on B corresponding to a critical value of J is necessarily invariant under T^m. This is obvious, since $\delta J/\delta\theta_{i0} = u_{im} - u_{i0} = 0$[†] at a critical point, while, along B, $\theta_{im} \equiv \theta_{i0}$ (mod 2π). In a fundamental domain, J must have at least 2^n critical points—provided that multiple critical points be counted the proper number of times.[‡]

It is one of our main purposes to study the relationship existing between the character of these critical points and the type of stability or instability possessed by the corresponding periodic motion.

In the case of $n = 1$, J is a function of a single variable and can have only two kinds of critical points, maxima and minima; in as much as a horizontal point of inflection counts as a multiple critical point. If J be not a constant, it must possess at least one maximum and one minimum. Since non-integrability means essentially that J can not be a constant, Birkhoff's result, already referred to in § 1, can be interpreted to mean that a maximum of J always corresponds to a periodic motion of unstable type and a minimum to a motion of stable type; or vice versa, according to the sign of a certain invariant of the differential equations.

For $n > 1$, the simple geometrical methods of Birkhoff do not appear to be applicable. This will necessitate a direct appraisal of the characteristic exponents in question. In order to obtain concrete results we eventually confine attention to the "general case" as defined in § 5. This is to some extent analogous to Birkhoff's restriction to the non-integrable case though of a much more complicated character. Furthermore instead of the sign of a single invariant which figures in the above interpretation of Birkhoff's result, we shall be concerned with a whole square matrix, C, of invariants. The matrix in question is $4\pi/\sqrt{-1}$ times the matrix of coefficients of the quadratic terms in the polynomial P_μ which appears in the normal form of the differential equations. From this definition of C, it is important for later purposes to note that C is obviously symmetric. If we let $c_{jk}(u) =$

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[*] G. D. Birkhoff, "Une généralisation à n dimensions du dernier théorème de géométrie de Poincaré," *Comptes Rendus*, vol. 192 (1931), pp. 196-198.

[†] Here, as elsewhere, the notation $\delta/\delta\theta$ is used to denote partial differentiation along a manifold B.

[‡] M. Morse, "Relations between the critical points of a real function of n independent variables," *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 345-396; and A. B. Brown, "Relations between the critical points of a real analytic function of N independent variables," *American Journal of Mathematics*, vol. 52 (1930), pp. 251-270. Here the connectivity numbers are the binomial coefficients $\binom{n}{k}$ $k = 1, 2, \cdots, n$.

$\partial M_j/\partial u_k$, it appears from (1.4) that the elements of C are precisely the numbers $c_{jk}(0)$. We also observe that the determinant $| c_{jk}(u) |$ is not zero near the origin since we start with a periodic motion of general stable type.

Before proceeding further with these ideas we must devote the next few pages to some fundamental inequalities.

3. *Some fundamental inequalities.* We shall use the following notation:

$$\zeta = (\sum_{k=1}^{n} u_k^2)^{1/2}, \qquad \zeta_m = (\sum_{k=1}^{n} u_{km}^2)^{1/2}$$

$\alpha =$ an arbitrary positive number $< \frac{1}{2}$.

The capital letter A, followed sometimes by a subscript, is used throughout the paper to denote a suitably chosen positive quantity, dependent perhaps upon α, the equations (1.1), and μ (i. e. the extent to which the equations have been normalized), but independent of $u_1, \theta_1, u_2, \theta_2, \cdots u_n, \theta_n$.

In BL it was shown that the following inequalities hold as long as ζ is sufficiently small and $u_k/u_j \geqq \alpha$ for all pairs of indices $k, j = 1, 2, \cdots n$:

$$(3.1) \qquad \begin{array}{lll} | U_j(u, \theta) | < A\zeta^{\mu+1} & | \partial U_j/\partial u_k | < A\zeta^{\mu} & | \partial U_j/\partial \theta_k | < A\zeta^{\mu+1} \\ | \Theta_j(u, \theta) | < A\zeta^{\mu} & | \partial \Theta_j/\partial u_k | < A\zeta^{\mu-1} & | \partial \Theta_j/\partial \theta_k | < A\zeta^{\mu}. \end{array}$$

Let $R(\eta, \beta)$ denote the collection of points (u, θ) for which $\zeta \leqq \eta$ and $u_j/u_k \geqq \beta$ for all ordered pairs of indices j and k. Then the following important result was obtained in BL:

THEOREM I. *If (u_0, θ_0) is a point of $R(\eta, 2\alpha)$, then the image point (u_m, θ_m) under T^m must lie within $R(n\eta, \alpha)$ as long as $m \leqq A_1\zeta_0^{-\mu}$, provided that ζ_0 is sufficiently small, at least for the case that $n \geqq 2$.*

We write the m-th iterate of T in the form,

$$(3.2) \qquad \begin{array}{ll} u_{jm} = u_{j0} + U_{jm}(u_0, \theta_0) & \\ \theta_{jm} = \theta_{j0} + mM_j(u_0) + \Theta_{jm}(u_0, \theta_0) & \qquad j = 1, 2, \cdots n \end{array}$$

suggested by (2.1), and we proceed to prove the following theorem.

THEOREM II. *If (u_0, θ_0) is a point of $R(\eta, 2\alpha)$, with η sufficiently small, then the following inequalities hold as long as $m \leqq K\zeta_0^{1-\mu/4}$:*

$$|\partial U_{jm}/\partial u_{k0} | \leqq A\zeta_0^{1+\mu/2}, \ | \partial U_{jm}/\partial \theta_{k0} | \leqq A\zeta_0^{3\mu/4}, \ | \partial \Theta_{jm}/\partial u_{k0} | \leqq A\zeta_0^{2+\mu/4},$$
$$| \partial \Theta_{jm}/\partial \theta_{k0} | \leqq A\zeta_0^{1+\mu/2}.$$

Here K is an arbitrary positive number and the necessary smallness of η depends partly upon K.

Proof. We introduce the notation $\partial\theta_{jm}/\partial\theta_{k0} = v_{jk}(m)$, $\partial u_{jm}/\partial\theta_{k0} = w_{jk}(m)$. $[m, k]$ will be used as a symbol to denote any linear homogeneous function of $v_{1k}(m), v_{2k}(m), \cdots v_{nk}(m), w_{1k}(m), w_{2k}(m), \cdots w_{nk}(m)$, whose coefficients, depending upon m and $(u_0\, \theta_0)$, are infinitesimals of at least the $(\mu - 1)$ th order in ζ_0 for $m \leqq K\zeta_0^{-\mu+1}$, uniformly for (u_0, θ_0) in $R(\eta, 2\alpha)$. From (2.1) we have

$$(3.3) \quad \begin{cases} v_{jk}(1) = \partial\theta_{j1}/\partial\theta_{k0} = \delta_{jk} + \partial\Theta_j(u_0, \theta_0)/\partial\theta_{k0} = \delta_{jk} + a_{jk}(\text{say}) \\ \text{Also } v_{jk}(0) = \delta_{jk},\ w_{jk}(0) = 0. \end{cases}$$

By the elementary rules for partial differentiation we find

$$v_{jk}(m+1) = \sum_{l=1}^{n} (\partial\theta_{jm+1}/\partial\theta_{lm})v_{lk}(m) + \sum_{l=1}^{n} (\partial\theta_{jm+1}/\partial u_{lm})w_{lk}(m)$$

$$w_{jk}(m+1) = \sum_{l=1}^{n} (\partial u_{jm+1}/\partial\theta_{lm})v_{lk}(m) + \sum_{l=1}^{n} (\partial u_{jm+1}/\partial u_{lm})w_{lk}(m).$$

But from (2.1) and the notation $c_{jk}(u) = \partial M_j/\partial u_k$, we obtain

$$(3.4) \quad \begin{cases} \partial\theta_{jm+1}/\partial\theta_{lm} = \delta_{jl} + \partial\Theta_j(u_m, \theta_m)/\partial\theta_{lm} \\ \partial\theta_{jm+1}/\partial u_{lm} = c_{jl}(u_m) + \partial\Theta_j(u_m, \theta_m)/\partial u_{lm} \\ \partial u_{jm+1}/\partial\theta_{lm} = \partial U_j(u_m, \theta_m)/\partial\theta_{lm} \\ \partial u_{jm+1}/\partial u_{lm} = \delta_{jl} + \partial U_j(u_m, \theta_m)/\partial u_{lm}. \end{cases}$$

We now use the inequalities (3.1) with reference to the point (u_m, θ_m). These inequalities are here applicable, because from Theorem I, $u_{jm}/u_{km} \geqq \alpha$ for all pairs of indices j and k, m being restricted in such a way that $m \leqq A\zeta_0^{-\mu}$. Noticing also from Theorem I that $\zeta_m \leqq n\zeta_0$ and introducing the symbols $[m, k]$, we get from (3.4),

$$(3.5) \quad \begin{aligned} v_{jk}(m+1) &= v_{jk}(m) + \sum_{l=1}^{n} c_{jl}(u_m)w_{lk}(m) + [m, k] \\ w_{jk}(m+1) &= w_{jk}(m) + [m, k]. \end{aligned}$$

Since the $c_{ij}(u)$ are polynomials in the u's such that the determinant $|c_{ij}(0)| \neq 0$ (cf. the original definitions of P_μ, the M_i and c_{ij}, remembering that we are dealing with an equilibrium point of general stable type), it is easy to show that the w's can be eliminated from equations (3.5) by sufficient restriction of η. The result is

$$(3.6) \quad \Delta^2 v_{jk}(m) = [m+1, k] + [m, k]$$

where $\Delta^2 v_{jk}(m) = v_{jk}(m+2) - 2v_{jk}(m+1) + v_{jk}(m)$ and $[m+1, k] + [m, k]$ does not contain any w's.

We first wish to find out how large $\sum_{j=1}^{n} | v_{jk}(m) |$ can become. Denoting this quantity by $y_k(m)$, it is seen from (3.6) that it can not increase as rapidly as it would if

$$\Delta^2 y_k(m) = 2\rho y_k(m+1) \qquad [\rho = A\zeta_0^{\mu-1}]$$

under the initial conditions,

$$(3.7) \qquad \begin{cases} y_k(0) = 1, \quad y_k(1) = 1 + \sum_{j=1}^{n} | \partial\Theta_j(u_0, \theta_0) / \partial\theta_{k0} |. \quad \text{cf. } (3.3) \\ \text{Let } \sum_{j=1}^{n} | \partial\Theta_j(u_0, \theta_0) / \partial\theta_{k0} | = b < \rho. \end{cases}$$

The solution of the above difference equation which takes on these initial conditions is found to be

$$y_k(m) = \frac{(\rho - b + \sqrt{\rho^2 + 2\rho})(1 + \rho - \sqrt{\rho^2 + 2\rho})^m + (b - \rho + \sqrt{\rho^2 + 2\rho})(1 + \rho + \sqrt{\rho^2 + 2\rho})^m}{2\sqrt{\rho^2 + 2\rho}}.$$

This may be written in the form $1 + mb + \rho^{-1/2}\,\Omega(m\rho^{1/2}, \rho^{1/2})$, where $\Omega(m\rho^{1/2}, \rho^{1/2})$ is a convergent power series in $m\rho^{1/2}$ and $\rho^{1/2}$ without constant and linear terms.

Let us now restrict m by means of the inequality

$$(3.8) \qquad 0 < m \leqq 2K\zeta_0^{1-\mu/4} = 2A\rho^{\omega-1/4}$$

where $\omega = 3/(4\mu - 4) > 0$. It follows from (3.7) and the above expression for $y_k(m)$ that with m so restricted it is possible to choose ζ_0 (and consequently ρ) so small that $0 \leqq y_k(m) < 2$. If we still further restrict m, so that

$$(3.8') \qquad 0 < m \leqq K\zeta_0^{1-\mu/4}$$

the choice may be made so that we have also $0 \leqq y_k(m+1) < 2$. Hence it appears from (3.6) that the slope of $v_{jk}(m)$ can not increase as rapidly as it would if $\Delta^2 v_{jk}(m) = 2A\zeta_0^{\mu-1}$ and can not decrease as rapidly as if $\Delta^2 v_{jk}(m) = -2A\zeta_0^{\mu-1}$. But the first of these difference relations, under the initial conditions (3.3), yields

$$v_{jk}(m) = \delta_{jk} + a_{jk}m + A\zeta_0^{\mu-1}\, m(m-1)$$

while the second yields

$$v_{jk}(m) = \delta_{jk} + a_{jk}m - A\zeta_0^{\mu-1}\, m(m-1).$$

Since $|a_{jk}| < A\zeta_0^{\mu-1}$ it follows, if m is restricted as in (3.8′), that the true value of $v_{jk}(m)$ differs from δ_{jk} by an infinitesimal in ζ_0 of order $\frac{1}{2}\mu+1$. i. e. $|\partial\theta_{jm}/\partial\theta_{k0} - \delta_{jk}| < A\zeta_0^{1+\mu/2}$. Referring back to (3.2) we see that the last of the four required inequalities has been proved.

Now we go back to equations (3.5) and appraise the $w_{jk}(m)$. We first find out how rapidly $\sum_{j=1}^{n} |w_{jk}(m)|$ can increase. Denoting this expression by $z_k(m)$, we see that it can not increase as rapidly as it would if $dz_k/dm = \rho z_k + \rho$, where, as before, $\rho = A\zeta_0^{\mu-1}$. In order to obtain this result, we, of course, make use of the appraisal already found for the $v_{jk}(m)$. Solving this differential equation with the initial conditions $z_k(0) = 0$, we obtain $z_k(m) = e^{\rho m} - 1$. Therefore, if m is restricted as in (3.8) or (3.8′), $z_k(m)$ certainly remains bounded for sufficiently small values of ζ_0. Hence from (3.5) we find that $w_{jk}(m)$ can not increase as rapidly as it would if $w_{jk}(m+1) - w_{jk}(m) = +A\zeta_0^{\mu-1}$ nor can it decrease as rapidly as if $w_{jk}(m+1) - w_{jk}(m) = -A\zeta_0^{\mu-1}$. But the first of these difference relations, under the initial conditions (3.3) yields $w_{jk}(m) = A\zeta_0^{\mu-1}m$ while the second yields $w_{jk}(m) = -A\zeta_0^{\mu-1}m$. It follows that the true value of $w_{jk}(m)$ is an infinitesimal in ζ_0 of order $3\mu/4$.

i. e. $$|w_{jk}(m)| = |\partial u_{jm}/\partial\theta_{k0}| = |\partial U_{jm}/\partial\theta_{k0}| \leqq A\zeta_0^{3\mu/4},$$

which is the second of the required inequalities.

The proofs for the third and first inequalities can be carried out in a similar way and are hereby left to the reader. Besides, many of the more difficult details of this part of the proof have already been published in a slightly different form in BL.

4. *The determinantal equation for the characteristic exponents.* We shall be concerned with square matrices of order $2n$. Such a matrix will frequently be written in the form

$$\begin{pmatrix} (b_{ij}) & (a_{ij}) \\ (c_{ij}) & (d_{ij}) \end{pmatrix}$$

to indicate that the n^2 elements in the upper right hand corner (which we shall call the first quadrant) are precisely the elements in the square matrix (a_{ij}) of order n; the n^2 elements in the upper left hand corner (the second quadrant) are the elements of the square matrix (b_{ij}) of order n; etc. The matrix (c_{ij}) occupies the 3rd quadrant and (d_{ij}) the 4th.

Suppose that $(\bar{u}_0, \bar{\theta}_0)$ represents a point which is invariant under T^m. Then the characteristic exponents $s_1, s_2, \cdots s_{2n}$, of the corresponding periodic motion satisfy the equation:

$$\det.\begin{bmatrix}\left(\dfrac{\partial u_{im}}{\partial u_{j0}}\right)-Ee^{s} & \left(\dfrac{\partial u_{im}}{\partial \theta_{j0}}\right)\\ \left(\dfrac{\partial \theta_{im}}{\partial u_{j0}}\right) & \left(\dfrac{\partial \theta_{im}}{\partial \theta_{j0}}\right)-Ee^{s}\end{bmatrix}=0.$$

Here E represents the unit matrix of order n, and all the partial derivatives are evaluated at $(\bar{u}_0, \bar{\theta}_0)$. If we set $\xi = e^s - 1$ and substitute from (3.2), this equation appears in the form,

$$\det.\begin{bmatrix}\left(\dfrac{\partial U_{im}}{\partial u_{j0}}\right)-E\xi & \left(\dfrac{\partial U_{im}}{\partial \theta_{j0}}\right)\\ \left(m\dfrac{\partial M_i}{\partial u_{j0}}+\dfrac{\partial \Theta_{im}}{\partial u_{j0}}\right) & \left(\dfrac{\partial \Theta_{im}}{\partial \theta_{j0}}\right)-E\xi\end{bmatrix}=0. \tag{4.1}$$

Taking the point $(\bar{u}_0, \bar{\theta}_0)$ in accordance with § 2, we know that $\delta J/\delta\theta_{i0} = 0$, $i = 1, 2, \cdots n$, at the point $(\bar{u}_0, \bar{\theta}_0)$ in question. Let us compute the second derivatives of this function,

$$J(\theta_{10}, \cdots \theta_{n0}) = \int \sum_{j=1}^{n} (u_{jm} - u_{j0})\, d\theta_{j0} = \int \sum_{j=1}^{n} U_{jm}(u_0, \theta_0)\, d\theta_{j0},$$

where the integral is extended over a path on the manifold B. That is, the u_0's are thought of as functions of the θ_0's according to equations (2.2).

$$\therefore\ \frac{\delta^2 J}{\delta\theta_{i0}\delta\theta_{k0}} = \frac{\delta U_{im}}{\delta\theta_{k0}} = \frac{\partial U_{im}}{\partial\theta_{k0}} + \sum_{j=1}^{n}\frac{\partial U_{im}}{\partial u_{j0}}\,\frac{\delta u_{j0}}{\delta\theta_{k0}}. \tag{4.2}$$

We now obtain an appraisal for the $\delta u_{j0}/\delta\theta_{k0}$, using explicitly not equations (2.2) but the equations $\theta_{im} - \theta_{i0} = 2k_i\pi$ which define the same manifold B. From (3.2) these equations appear in the form $2k_i\pi = mM_i(u_0) + \Theta_{im}(u_0, \theta_0)$. Differentiating, we get

$$\sum_{k=1}^{n}\left(\frac{\partial M_i}{\partial u_{k0}} + \frac{1}{m}\,\frac{\partial\Theta_{im}}{\partial u_{k0}}\right)\frac{\delta u_{k0}}{\delta\theta_{j0}} + \frac{1}{m}\,\frac{\partial\Theta_{im}}{\partial\theta_{j0}} = 0.$$

According to Theorem II, the coefficient of $\delta u_{k0}/\delta\theta_{j0}$ may be considered to differ by an arbitrarily small amount from $c_{kj}(0)$ by taking the invariant point (u_0, θ_0) sufficiently close to the origin. But since the determinant $|\, c_{ij}(0)\,|$ is not zero, we may evidently solve for the $\delta u_{k0}/\delta\theta_{j0}$, obtaining by Theorem II the result that $|\,\delta u_{k0}/\delta\theta_{j0}\,| \leqq (1/m)A\zeta_0^{1+\mu/2}$. And this shows us that

$$\left|\sum_{k=1}^{n}\frac{\partial U_{jm}}{\partial u_{k0}}\,\frac{\delta u_{k0}}{\delta\theta_{i0}}\right| \leqq (1/m)A\zeta_0^{\mu+2}. \tag{4.3}$$

It should be noted that for our present point of view we may always consider m to satisfy the double inequality

$$(4.4)\qquad \tfrac{1}{2}K\zeta_0^{1-\mu/4} \leqq m \leqq K\zeta_0^{1-\mu/4}.$$

For if $m < \frac{1}{2}K\zeta_0^{1-\mu/4}$, we can always find an integer k such that $\frac{1}{2}K\zeta_0^{1-\mu/4} \leqq mk \leqq K\zeta_0^{1-\mu/4}$; and instead of T^m we shall consider T^{mk}. This has the effect merely of multiplying the characteristic exponents by k and does not therefore affect their character of being pure real numbers, pure imaginary, or complex numbers. Hence, with this understanding we combine (4.2), (4.3), and (4.4); and write

$$(4.5)\qquad \partial U_{im}/\partial\theta_{k0} = \delta^2 J/\delta\theta_{i0}\delta\theta_{k0} + r_{ik}, \quad \text{where} \quad |r_{ik}| \leqq A\zeta_0^{1+5\mu/4}.$$

We shall now neglect certain terms in the determinantal equation (4.1) and determine an upper limit for the error which we so commit in the coefficient of ξ^q $(q = 0, 1, 2, 3, \cdots 2n)$. We neglect the r_{ik} in the first quadrant, the $\partial U_{im}/\partial u_{j0}$ in the second, the $\partial\Theta_{im}/\partial u_{j0}$ in the third, and the $\partial\Theta_{im}/\partial\theta_{j0}$ in the fourth. (4.1) then appears in the mutilated form,

$$(4.6)\qquad \text{det.} \begin{bmatrix} (-\delta_{ij}\xi) & \left(\dfrac{\delta^2 J}{\delta\theta_{i0}\delta\theta_{j0}}\right) \\ \left(m\dfrac{\partial M_i}{\partial u_{j0}}\right) & (-\delta_{ij}\xi) \end{bmatrix} \equiv |q_{ij} - \delta_{ij}\xi^2| = 0,$$

where the matrix (q_{ij}) is the product of the matrices $(m\partial M_i/\partial u_j)$ and $(\delta^2 J/\delta\theta_{i0}\delta\theta_{j0})$.

The errors in the coefficients of ξ^{2n-2p} and $\xi^{2n-2p-1}$ are each in absolute value less than $A\zeta_0^{(p+1)(1+\mu/2)}$, $p = 0, 1, 2, 3, \cdots n$. The proof of this result, which rests on (4.5) and Theorem II is left to the reader.

The roots $\xi_1, \xi_2, \cdots \xi_{2n}$ *must satisfy an inequality of the form,*

$$(4.7)\qquad |\xi| < A\zeta_0^{(1/2)+(\mu/4)}.$$

For, if just one of the roots did not satisfy this inequality, we should find that $S_1(\xi) > A_1\zeta_0^{(1/2)+(\mu/4)}$, where we use the notation $S_k(\xi)$ to denote the symmetric function of the ξ_i obtained by forming the sum of all possible products of the ξ_i taken k at a time. Actually, however, we know from (4.1) that $|S_1(\xi)| = |\text{coefficient of } \xi^{2n-1}| < A_2\zeta_0^{1+\mu/2} < A_1\zeta_0^{(1/2)+(\mu/4)}$ for sufficiently small ζ_0. Similarly, if just two of the roots did not satisfy (4.7) we should have $|S_2(\xi)| > A\zeta_0^{2[(1/2)+(\mu/4)]}$ for all A, provided that ζ_0 is sufficiently small, while actually we know that $|S_2(\xi)| = |\text{coefficient of } \xi^{2n-2}| < A\zeta_0^{1+\mu/2}$.

for a certain A. If just three of the roots fail to satisfy (4.7), we should find $|S_3(\xi)| > A\zeta_0^{3[(1/2)+(\mu/4)]}$, when actually we know from (4.1) that $|\text{coefficient of } \xi^{2n-3}| < A\zeta_0^{4[(1/2)+(\mu/4)]}$. In this way we can evidently eliminate all possible failures of the italicized statement.

It would now be desirable to define the "general case" as the case where the roots satisfy the double inequalities,

$$A_1\zeta_0^{(1/2)+(\mu/4)} < |\xi| < A_2\zeta_0^{(1/2)+(\mu/4)},$$

at least for an infinite sequence of invariant points in the neighborhood of the origin. This, however, is never the case, simply because μ is arbitrary. We therefore devote the next paragraph to a more elaborate definition.

5. *Definition of the "General Case."* So far we are sure of the existence of infinitely many points, (u, θ), which are invariant under the iterates of T and which possess the following two properties:

I. At these points the quadratic form, $Q(u) = \sum_{i,j=1}^{n} c_{ij}(u)x_ix_j$, is non-singular and has the same signature as the non-singular form $Q(0) = \sum_{i,j=1}^{n} c_{ij}(0)x_ix_j$.

II. At these points the following inequalities hold:

$$(5.1) \qquad |E_p|^{1/(2p+2)}, \quad |E'_p|^{1/(2p+2)}, \quad |\xi| < A\zeta^{\beta}, \qquad \beta = (1/2) + (\mu/4),$$

where μ, ξ, and ζ have the same meanings as in the previous paragraph,* while E_p and E'_p are the errors in the coefficients of ξ^{2n-2p} and $\xi^{2n-2p-1}$ respectively, which are incurred by assuming equation (4.6) instead of (4.1).

By property II, we mean it to be implied that a manifold B may be defined in the neighborhood (at least) of the invariant point in question. Otherwise the symbols $\delta^2 J/\delta\theta_{i0}\delta\theta_{j0}$ which appear in the definition of E_p and E'_p would fail to have a meaning.

As we normalize our equations to higher and higher terms, thus taking higher and higher values for μ, we reduce the regions of convergence of the power series which define the transformation T and which define the reciprocal relation between the original variables and the normalizing variables. In the limit the point set common to these regions of convergence certainly reduces to the origin, at least, if we exclude the exceptional case of local integrability.†

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* From now on we omit the subscript 0 from ζ_0.

† Local integrability is defined as occurring if and only if the "formal series" converge. Cf. Birkhoff, "Dynamical systems," *American Mathematical Society Colloquium Publications*, vol. 9, pp. 255-259.

Let us start with coördinates normalizing to a certain degree, $\mu = \mu(0)$. We shall call these coördinates the coördinates C^0. In some region $R(\eta, 2\alpha)$, we choose an infinite sequence of points $P^1, P^2, P^3, \cdots$ converging monotonically to the origin, such that P^i is invariant under some iterate T^{m_i} of T and such that Properties I and II hold in the coördinates C^0. We can always do this on account of the existence theorem of BL and the further developments already given in the present paper.

We now define the coördinates C^i. These coördinates are those normalizing to such a degree, $\mu = \mu(i)$, that Properties I and II, hold for the points $P^i, P^{i+1}, P^{i+2}, \cdots$ but such that at least one of these properties does not hold (or actually ceases to have a meaning) in coördinates normalizing to any higher degree, $\mu > \mu(i)$. In this way we get a definite $\mu(i)$. For evidently, in the case of non-integrability, with increasing μ the point P^i would eventually find itself outside of the region of convergence of the above power series; and certainly at this stage, if at no earlier stage, the inequalities (5.1) would cease to have a meaning in the new coördinates. Also it may easily be proved by the methods of BL that $\lim_{i=\infty} \mu(i) = \infty$ for any *fixed* sequence of the type considered.

Let us denote by $\zeta^{(i)}$ the function $(\sum_{j=1}^{n} u_j^2)^{1/2}$ formed for the coördinates C^i, while we omit the superscript in the case of C^0. Then evidently $\zeta^{(i)} \leqq A^{(i)}\zeta$. Hence inequalities (5.1) in the coördinates C^i can be written in the form

$$(5.2) \qquad |E_p|^{1/(2p+2)}, \quad |E'_p|^{1/(2p+2)}, \quad |\xi| < A^{(i)}\zeta^{\beta(i)}.^{*}$$

These inequalities hold, of course, for $P^i, P^{i+1}, P^{i+2}, \cdots$. It is emphasized that, in computing the E_p, E'_p, and ξ's, the coördinates C^i are used, although ζ is expressed in the coördinates C^0. In general, as $i \rightarrow \infty$, we presumably have $A^{(i)} \rightarrow \infty$ as well as $\beta(i) \rightarrow \infty$.

Evidently we may assume that $\mu(0) < \mu(1) < \mu(2) < \mu(3) \cdots$, and $\zeta < 1$ at all the points P^i. If this condition is not fulfilled for the original sequence, we can deal instead with a suitable subsequence.

We now assign the points $P^1, P^2, P^3, \cdots$ to families $F^0, F^1, F^2, \cdots$ in the following way:

Let P^{j_1} be the first point for which $A^{(0)}\zeta^{\beta(0)} > A^{(1)}\zeta^{\beta(1)}$. Such a point must certainly exist, since $\beta(1) > \beta(0)$ and the P^i converge monotonically to the origin. Furthermore $j_1 \geqq 1$. All points P^i for which $i < j_1$ are assigned to F^0.

Let P^{j_2} be the first point after P^{j_1} for which $A^{(1)}\zeta^{\beta(1)} > A^{(2)}\zeta^{\beta(2)}$. Such a point exists since $\beta(2) > \beta(1)$. Furthermore $j_2 \geqq 2$. All points P^i for which $j_1 \leqq i < j_2$ are assigned to F^1.

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* The meaning of the notation is obvious: viz. $\beta(i) = \mu(i)/4 + 1/2$.

The process may be continued by induction. Suppose that the family F^{k-1} has been defined and that P^{j_k} is the next point in the sequence and is such that $j_k \geqq k$. Let $P^{j_{k+1}}$ be the first point after P^{j_k} for which $A^{(k)}\zeta^{\beta(k)} > A^{(k+1)}\zeta^{\beta(k+1)}$. Such a point exists since $\beta(k+1) > \beta(k)$. Furthermore $j_{k+1} \geqq j_k + 1 \geqq k + 1$. All points P^i for which $j_k \leqq i < j_{k+1}$ are assigned to F^k.

With the possible exception of F^0, none of the families F^i is empty. Every P^i belongs to just one family F^{k_i} and the following inequalities always hold at P^i, if $l < j < k_i$:

$$A^{(l)}\zeta^{\beta(l)} > A^{(j)}\zeta^{\beta(j)} > A^{(k_i)}\zeta^{\beta(k_i)}. \tag{5.3}$$

Furthermore since $i \geqq j_{k_i} \geqq k_i$, we know from (5.2) that

$$| E_p |^{1/(2p+2)}, \quad | E'_p |^{1/(2p+2)}, \quad | \xi | < A^{(k_i)}\zeta^{\beta(k_i)} \tag{5.4}$$

for points P^i in F^{k_i}. We agree once and for all that, for such a point P^i, E_p, E'_p, ξ will always hereafter be computed in the coördinates C^{k_i}.

Let σ_i be defined by the equation $A^{(0)}\zeta^{\sigma_i\beta(0)} = A^{(k_i)}\zeta^{\beta(k_i)}$, where ζ has its value at the point P^i. Thus from (5.4), at any point P^i, we have

$$| E_p |^{1/(2p+2)}, \quad | E'_p |^{1/(2p+2)}, \quad | \xi | < A\zeta^{\sigma_i\beta(0)}, \tag{5.5}$$

where $A = A^{(0)}$ is independent of i $(i = 1, 2, \cdots \infty)$.

With the help of (5.3) it is easily shown that

$$\lim_{P_i \to 0} A\zeta^{\sigma_i\beta(0)-N} = 0,$$

no matter how large a number, N, may be preassigned. Thus the inequalities (5.5) put into evidence the fact that the ξ's are infinitesimals of infinite order in ζ.

To simplify the notation, we shall set $z = \zeta^{\sigma_i\beta(0)}$, where it is understood that z is defined only at the points P^i (the only points where we have any use for it). In this notation the inequalities (5.5) may be written

$$| E_p |, \quad | E'_p | < Az^{2(p+1)}, \quad | \xi | < Az, \qquad (p = 0, 1, 2, \cdots n). \tag{5.6}$$

The capital letter A is still used to denote a suitably chosen positive number but in a more restricted sense than formerly. For previously the quantity denoted by A frequently depended upon μ. Since we are now dealing with a discrete sequence of points P^i, the essential thing is settled when we say that from now on A shall be independent of i.

The general case is defined as occurring if there exists at least one sequence of invariant points, $P^1, P^2, \cdots$ of the type described above, such that the following two requirements be fulfilled:

I. *The roots $\xi_1, \xi_2, \cdots$ of the determinantal equation* (4.6) *are non-zero and distinct; each root, and the difference between any two of them, exceeding in absolute value an infinitesimal of the first order in z,*

$$(5.7) \qquad \text{i.e.}\ |\xi_i|, \quad |\xi_j - \xi_i| > A_1 z \qquad (i, j = 1, 2, \cdots 2n,\ i \neq j).$$

II. *Among the corresponding critical values of J* (which are necessarily non-singular on account of I), *there are infinitely many of each signature,* $0, 1, 2, \cdots n$.*

The plausibility of requirement II is made clear, when one remembers from Morse's critical point relations that a function defined on an n-dimensional torus and admitting only non-singular critical values must have at least one critical value of each signature.

The characteristic exponents s_i are of course related to the corresponding roots ξ_i of (4.1) by means of the equation

$$s = \log(1 + \xi) = \xi - \tfrac{1}{2}\xi^2 + \tfrac{1}{3}\xi^3 - \cdots.$$

On the other hand the roots ξ_i of (4.1) differ from the corresponding roots ξ_i of (4.6) by less than Az^2 as follows from (5.6) and (5.7).

Hence the roots of (4.6) give the characteristic exponents correct to within an infinitesimal of the second order in z. The s_i are therefore also non-zero and distinct; each s_i and the difference between any two of them exceeding in absolute value an infinitesimal of the first order in z:

$$(5.8) \qquad |s_i|, \quad |s_i - s_j| > A_2 z.$$

Furthermore *the characteristic exponents are complex, real, or pure imaginary according as the roots of* (4.6) *are complex, real, or pure imaginary; and conversely—at least for z sufficiently small.*

For suppose that s is complex; $s = a + bi$. Then since the Hamiltonian exponents occur in pairs $(s, -s)$, it is seen that $-a - bi$, $a - bi$ and $-a + bi$ must also be characteristic exponents. It follows from (5.8) that both a and b must exceed infinitesimals of the first order in z. Hence s can not be approximated to within an infinitesimal of the second order by a real or pure imaginary ξ. Hence the corresponding ξ is complex. The converse proposition follows similarly from the fact that the roots of (4.6) also occur in pairs $(\xi, -\xi)$. Hence, if s is real, ξ cannot be complex. It must therefore be either real or pure imaginary. It can not be pure imaginary and give the approximation to within an infinitesimal of second order. The rest of the italicized statement can be proved in a similar manner.

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* A critical value of J is non-singular if the Hessian $|\delta^2 J/\delta\theta_{i0}\delta\theta_{j0}|$ evaluated at the critical point in question is not zero. If this Hessian were zero, (4.6) would have to have at least one zero root. The signature of a critical value of J is defined as the signature of the quadratic form $\sum_{i,j} (\delta^2 J/\delta\theta_{i0}/\delta\theta_{j0}) x_i x_j$.

6. *The nature of the periodic motions in the neighborhood of the given periodic motion.*

LEMMA. *Let* $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ *and* $\sum_{i,j=1}^{n} b_{ij}x_ix_j$ *be two non-singular quadratic forms, of which the first is positive definite. Let* $q_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. *Then the roots of the determinantal equation* $| q_{ij} - \delta_{ij}\xi^2 | = 0$ *are either real or pure imaginary. The difference between the number of real roots and the number of imaginary roots is equal to twice the signature of the quadratic form* $\sum_{i,j=1}^{n} b_{ij}x_ix_j$.

The germ of this algebraic theorem is found at least as far back as Cauchy, Jacobi, and Weierstrass; but since I have been unable to find in the existing literature a proof of the lemma in the form stated above, I offer a simple dynamical proof in the footnote.*

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* Consider the homogeneous linear Hamiltonian system with constant coefficients for which

$$H = \tfrac{1}{2}\left[\sum_{j,k=1}^{n} a_{jk}y_jy_k - \sum_{j,k=1}^{n} b_{jk}x_jx_k\right]: \qquad dy_j/dt = -\partial H/\partial x_j, \quad dx_j/dt = \partial H/\partial y_j.$$

The characteristic exponents ξ_i for the equilibrium problem of this system are easily seen to satisfy $| q_{jk} - \delta_{jk}\xi^2 | = 0$. On the other hand, this dynamical problem, formulated by Hamilton's variational principle, can be given the form,

$$\delta \int \Big(\sum_{j,k} a^{jk}\dot{x}_j\dot{x}_k + \sum_{j,k} b_{jk}x_jx_k\Big)\, dt = 0, \qquad (\dot{x} = dx/dt)$$

where (a^{jk}) is the matrix reciprocal to (a_{jk}). Hence $\Sigma\, a^{jk}x_jx_k$ is also positive definite. By a suitable real linear transformation on the x's the above variational equation can be given the form,

$$\delta \int \Big(\sum_{j=1}^{n} \dot{x}_j^{2} + \sum_{j=1}^{p} \lambda_j^{2}x_j^{2} - \sum_{j=p+1}^{n} \lambda_j^{2}x_j^{2}\Big)\, dt = 0$$

where $2p - n$ is the signature of $\Sigma\, b_{jk}x_jx_k$. Hence the characteristic exponents of the system are seen to be $\pm\lambda_j$ $(j = 1, 2, \cdots p)$ and $\pm\lambda_j\sqrt{-1}$ $(j = p+1, p+2, \cdots n)$, where the λ_j are real.

A similar theorem can also be proved if $\sum a_{jk}x_jx_k$ is negative definite. The lemma leads directly to the following theorems:

THEOREM III. *In the neighborhood of a periodic motion of general stable type there are "in general" infinitely many periodic motions of stable type and infinitely many periodic motions of unstable type for which the characteristic exponents are real, provided that the symmetric matrix* $\| c_{jk}(0) \|$ *is the matrix of coefficients of a definite quadratic form.*

These periodic motions correspond to the points on the manifolds B for which J has a maximum or minimum. The proof is carried out with the help of (4. 6) and the above lemma. The matrix $(m\, \partial M_j/\partial u_k)$ plays the rôle of (a_{ij}).

We define the index of an unstable periodic motion which possesses only real and pure imaginary characteristic exponents as the number of pairs of real exponents. Since there are infinitely many critical values of J of each signature the same proof yields the following result:

THEOREM IV. *Under the hypotheses of Theorem III there are also infinitely many unstable periodic motions* (whose characteristic exponents are all real or pure imaginary) *of each index* j, $j = 1, 2, \cdots n$.

Now let us drop the requirement that $(c_{ij}(0))$ be the matrix of a definite quadratic form.

THEOREM V. *In the neighborhood of a periodic motion of general stable type there are "in general" infinitely many periodic motions of unstable type whose characteristic exponents are real and pure imaginary, of index* $\frac{1}{2}(n \pm s)$, *where* s *is the signature of* $(c_{ij}(0))$.

These periodic motions correspond to points on the manifolds B where J has a maximum or minimum. To prove this theorem the matrix $(\delta^2 J/\delta\theta_i\delta\theta_j)$ plays the rôle of the (a_{ij}) of the lemma.

It is not possible on the strength of Theorem III to make any deductions as to the probable existence of quasi-periodic motions as Birkoff has done in the case of two degrees of freedom. This is because we know nothing about the signatures of the matrices $(c_{ij}(0))$ pertaining to the infinitely many periodic motions of stable type, the mere existence of which is assured by Theorem III.

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ALTERNANT SURFACES.*

By Clifford Bell.

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1. *Introduction.* The purpose of this paper is to make a study of the surfaces whose equations are of the form

$$(1)\qquad z = (x^q - 1)(y^r - 1) - (x^r - 1)(y^q - 1),$$

where q and r are integers such that $q < r$. These surfaces are called alternant surfaces because of their relationship to the general third order alternant.†

Suppose that the alternant, $A = \begin{vmatrix} \alpha^m & \alpha^n & \alpha^p \\ \beta^m & \beta^n & \beta^p \\ \gamma^m & \gamma^n & \gamma^p \end{vmatrix}$, has been arranged so that $m < n < p$. Now if neither α, β, nor γ is zero, the first, second, and third rows may be divided by α^m, β^m, and γ^m, and the second and third columns by α^{n-m} and α^{p-m}, respectively. Upon substituting z for $A/[(\alpha\beta\gamma)^m\alpha^{n+p-2m}]$, x for β/α, y for γ/α, q for $n - m$, and r for $p - m$, the result, $z = \begin{vmatrix} 1 & 1 & 1 \\ 1 & x^q & x^r \\ 1 & y^q & y^r \end{vmatrix}$, is obtained which, upon expanding, gives equation (1).

It is shown that a study of these surfaces yields more information in regard to the real values of α, β, γ that make A zero, than can be obtained by the usual factorization of this alternant into factors, one of which is the well known difference product determinant.‡ In this study the surface (1) is said to belong to one of four distinct types according as q and r are both even, q and r are both odd, q is even and r is odd, or q is odd and r is even.

In discussing the surfaces, the points $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 0)$ are designated by the letters A, B, C, D, and O, respectively. The letters P, Q, R, S, T, U, are placed on the lines BA, CA, DA, CB, DB, CD, respectively, external to these segments and in the directions indicated. The corresponding primed letters are placed on the same lines, but external to the above mentioned segments in the opposite directions.

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* Presented to the American Mathematical Society, August 30–September 2, 1932.

† See Burnside and Panton, *Theory of Equations*, London, 1928, vol. 2, pp. 61 and 110, for the definition of an alternant.

‡ See Brioschi, *La Teorica dei Determinanti*, Pavia, 1854, pp. 73-84.

2. *Type I, q and r both even.* By direct substitution in (1) it is found that $z = 0$ when $x = \pm 1$, $y = \pm 1$, $x = \pm y$. Hence these are lines of intersection of the surface (1) with the XY-plane and furthermore it will be shown that all the real points, common to the surface and $z = 0$, lie on these lines.

The partial derivative of z with respect to x vanishes only when $x = 0$, $\pm [(qy^r - q)/(ry^q - r)]^{1/(r-q)}$ and each of these values gives a turning point for the curve of intersection of the surface with the plane $y = c$. Now for each of these curves z vanishes at the points $x = -1, -|y|, +|y|, +1$, and in the order given if $|y| < 1$. But Rolle's Theorem * states that there must be at least one zero of the derivative between each of these values and, as the derivative has only three zeros, there must be exactly one between each of the above z zeros. Hence the zeros of the derivative,

$$x = -[(qy^r - q)/(ry^q - r)]^{1/(r-q)}, \quad 0, \quad +[(qy^r - q)/(ry^q - r)]^{1/(r-q)},$$

fall respectively, in the intervals $(-1, -|y|)$, $(-|y|, +|y|)$, $(+|y|, +1)$. However if $|y| > 1$, z vanishes in the order $x = -|y|, -1, +1, +|y|$ and the zeros of the derivative in the above order fall, respectively, in the intervals $(-|y|, -1)$, $(-1, +1)$, $(+1, +|y|)$.

When $|y| < 1$, $x = 0$, gives a maximum point and

$$x = \pm [(qy^r - q)/(ry^q - r)]^{1/(r-q)}$$

give minimum points. When $|y| > 1$, $x = 0$ gives a minimum and the other two zeros of the derivative give maximum points.

A similar study may be made of the surface by taking the partial derivative of z with respect to y, keeping x constant. Indeed the results may be obtained directly from the preceding ones by making the transformation $x' = y$, $y' = x$, $z' = -z$. Thus when $|x| < 1$, $y = 0$, gives a minimum point and $y = \pm [(qx^r - q)/(rx^q - r)]^{1/(r-q)}$ give maximum points, and if $|x| > 1$ the opposite is true in each case.

It can now be shown that all the real points common to the surface and $z = 0$, lie on the lines $x = \pm 1$, $y = \pm 1$, $x = \pm y$ in that plane. For assume that the surface has a point common with $z = 0$ that is not on one of the above lines. Let the coördinates of this point be $(x_1, y_1, 0)$ and take a section of the surface $y = y_1$. The resulting curve has three turning points, as has already been shown. But this curve has five zeros counting the one at $(x_1, y_1, 0)$. Hence, by Rolle's Theorem, the derivative must vanish at least once between each pair of consecutive zeros or in all at least four times. But the derivative vanishes only three times. Thus, the assumption that the surface has real points common to the $z=0$ plane, other than those on the above mentioned lines, leads to an absurdity.

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* See Fine, *Calculus*, New York, 1927, p. 104.

It is possible now to describe the surface. A maximum for z exists in each of the regions DAO and BCO, also a minimum in each of the regions ABO and CDO. Furthermore it is seen that z is positive in each of the regions DAO, BCO, PAQ, $RABS$, TBP', $U'CQ'$, $S'CDR'$, $T'DU$, and negative in all the other regions.

Both partial derivatives vanish at each of the points O, A, B, C, D. At the origin the section of the surface by $z=0$ consists of the lines $x=\pm y$ and as already pointed out, these lines divide the plane about O into the regions DOA and BCO in which z is positive, and the regions ABO and CDO, for which z is negative. Therefore the origin is a minimax point. Likewise a minimax exists at each of the points A, B, C and D, but these are of a different type than that at the origin because the section of the surface by $z=0$ has A, B, C and D as triple points. Thus at A the lines $x=y$, $x=1$, $y=1$, in the plane $z=0$, form the triple point and divide the plane about A into six regions in which z is alternately positive and negative.

3. *Type II, q and r both odd.* Again by direct substitution, it can be shown that the surface contains the lines $x=y$, $x=1$, $y=1$, in the plane $z=0$. A curve c, through $(-1,0,0)$, $(0,-1,0)$, E, F, Q, is also part of the intersection of the surface with $z=0$, where E is a point on the line $y=1$ between $(-1,1,0)$ and $(-2,1,0)$, F is on the line $x=y$ between the points $(-\frac{1}{2},-\frac{1}{2},0)$ and $(-1,-1,0)$, and G is on the line $x=1$ between $(1,-1,0)$ and $(1,-2,0)$.

The partial derivative of z with respect to x gives two turning points, the x coördinates of which are $\pm\,[(qy^r-q)/(ry^q-r)]^{1/(r-q)}$, for the curve obtained by holding y constant. Hence there can be at most only three real zeros on each of these curves. In the plane $z=0$, $x=1$, $x=y$, and a point on the curve c, account for the three real zeros and there can be no others. Therefore the curve c has only one real value of x for each real y and the surface can intersect the $z=0$ plane in no real points other than those on the above mentioned curves. Likewise, by considering the curves obtained by holding x constant, it can be shown that the curve c has only one real value of y for each x.

If the letter W is placed on the curve to the right of G, and W' is placed to the left of E, the positive regions are PAQ, $RAGW$, $P'GFQ'$, $R'EW'$ and

AFE. A maximum occurs for a point in the region *AFE* and a minimum for a point in the region *AGF*. It is readily seen that *E*, *F*, and *G* are ordinary minimax points,* while *A* is a minimax of the same kind that occurred at the point *A* in the surface of Type *I*.

4. *Type III, q even and r odd.* The lines $x = y$, $x = 1$, $y = 1$, in the plane $z = 0$, together with a curve *c*, are found to be in the surface. The curve *c* consists of two branches. One branch intersects $y = 1$ at *E*, a point between $(0, 1, 0)$ and $(-1, 1, 0)$, $y = x$ at the origin, $x = 1$ at *G*, a point between $(1, 0, 0)$ and $(1, -1, 0)$; and the other branch falls in the region *S′CU′* and intersects $x = y$ at *F*, a point for which $x < -1$. The above facts may be easily verified by substituting $z = 0$ in (1) and dividing out in turn $y - 1$, $y - x$, $x - 1$, from the resulting equation. The three equations so obtained will then yield the above results. Again when the equation, obtained by substituting $z = 0$ in (1), is written first in descending powers of y and then in descending powers of x, it is seen at once that $x + 1 = 0$ and $y + 1 = 0$ are asymptotes of the curve *c*.

A section of the surface, obtained by letting y be constant, has two turning points given by setting $\partial z/\partial x$ equal to zero. Therefore this curve can have at most only three zeros, which are $x = 1$, $x = y$, and one point on the curve *c*. Hence the curve *c* has only one real x for each real y. Furthermore the surface has no real point common to the plane $z = 0$ other than those on the above mentioned curves. In like manner it can be shown that the curve *c* has also only one real y for each real x.

Place the letter *W* on the branch *EOG*, of the curve *c*, to the right of *G*, and *W′* to the left of *E*. Likewise on the other branch place *V* to the right of *F* and *V′* to the left. The regions within which z is positive are *PAQ*, *RAGW*, *P′GOFV*, *Q′FV′*, *R′EW′*, and *AOE*. In all the other regions z is negative. The function z takes on a maximum in *AOE* and in *AGO*. The points *E*, *O*, *G*, and *F* are ordinary minimax points, while *A* is again a minimax of the same kind as that described for the Type *I* surface.

5. *Type IV, q odd and r even.* The lines $x = 1$, $y = 1$, $x = y$, in the plane $z = 0$, are on the surface. A section of the surface, obtained by keeping y constant, has only one turning point. Hence, this curve has the zeros $x = 1$ and $x = y$, and no others. Like results are obtained when a section is taken by letting x be constant.

The function z is positive in the regions *PAQ*, *RAP′*, *Q′AR′*, and negative in all others. No maximum or minimum points exist and only one minimax, this being at A. It is the same kind of minimax point that occurs at A of the first surface described.

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* For convenience a minimax, such as exists at the origin for the surface of Type *I*, is called an ordinary minimax point.

6. *Conclusion.* Obviously the alternant A vanishes when any one of the quantities α, β, or γ is zero, and furthermore all other zeros of A are zeros of z. For all of the above described surfaces, z vanishes when either $x = y$, $x = 1$, or $y = 1$, and for Type I surfaces z also vanishes when either $x = -y$, $x = -1$, or $y = -1$. But for these values of x and y the alternant vanishes on account of two or more rows or columns being identical. Thus A *vanishes if one or more of the following equations hold:* $\alpha = \beta$, $\alpha = \gamma$, $\beta = \gamma$, *for all positive integral values of* m, n *and* p. In addition A *also vanishes when* m, n *and* p *are all even positive integers if one or more of the following equations hold:* $\alpha = -\beta$, $\alpha = -\gamma$, $\beta = -\gamma$.

Information regarding the zeros of A, other than the above mentioned rather obvious ones, may be obtained by further examination of the descriptions of the surfaces (1). The surfaces of Types I and IV have no other real values of x and y for which z vanishes; but for the surfaces of Types II and III, z vanishes whenever x and y are taken on the curves c previously mentioned. As $q = n - m$, $r = p - m$, and q and r are both odd for the Type II surface, and q is even and r odd for the Type III surface, it may be said that other zeros of A are possible *if and only if one of the following holds:* (1) n *and* p *even,* m *odd,* (2) n *and* p *odd,* m *even,* (3) n *and* m *even,* p *odd,* (4) n *and* m *odd,* p *even.*

In addition to what has been said above, about the vanishing of A, other studies may be made of certain regions on the $z = 0$ plane. Thus it is seen that z is different from zero for all positive values of x and y, other than values of x and y that satisfy one or more of the equations $x = 1$, $y = 1$, $x = y$. Hence *the alternant* A *cannot vanish if* α, β, *and* γ *all have like signs and if neither* α, β, *nor* γ *are zero or equal to each other.*

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GROUPS INVOLVING FOUR OPERATORS WHICH ARE SQUARES.

By G. A. Miller.

If a group G involves four and only four operators which are squares of its operators its order g can not be divisible by any odd prime number except 3 and when g is divisible by 3 then G contains only one subgroup of order 3. It must therefore involve an operator of order 4 in all cases since it must also involve such an operator when g is of the form 2^m. When g is of the form $3 \cdot 2^m$ then it must be non-abelian since no two of its operators of orders 3 and 4 can be commutative. All of its operators which are commutative with one of its operators of order 3 constitute a subgroup of index 2 under G. This subgroup is abelian since it cannot involve an operator of order 4 and hence it is the direct product of the group of order 3 and the abelian subgroup of order 2^{m-1} and of type $(1, 1, 1, \cdots)$.

Since the remaining operators of G include operators of order 4 two cases present themselves. In the first case such an operator of order 4 is commutative with every operator in the given abelian subgroup of order 2^{m-1} and hence all the remaining operators of G are of order 4 and the commutator subgroup of G is of order 3. In the second case this operator is commutative with only one-half of the operators of this subgroup and hence $m > 2$ and one-half of the remaining operators are of order 4 and the commutator subgroup of G is of order 6. This proves the following theorem: *If a group involves four and only four operators which are squares of operators contained therein and its order is not of the form 2^m then this order is of the form $3 \cdot 2^m$ and the group is non-abelian. There are two and only two such groups when $m > 2$ while there is only one such group when $m = 2$.*

In what follows it will be assumed that g is of the form 2^m and we shall first consider the case when the four operators which are squares under G include an operator of order 4 and hence G involves operators of order 8. The operators which are squares under G therefore constitute a cyclic subgroup of order 4 which is invariant under G and the corresponding quotient group is abelian since it involves only operators of order 2 besides the identity. If the operators of order 4 which are squares under G are not in the central of G then G involves a subgroup of index 2 which includes all of its operators of order 8 and under which the operators of order 4 which are squares are invariant. Hence we shall first determine the possible G's under which all the operators which are squares are invariant. The commutator subgroup of each of these is therefore either the identity or of order 2, and the operators whose orders are less than 8 constitute a subgroup of index 2 thereunder.

When this subgroup of index 2 is abelian it is of type $(2, 1, 1, \cdots)$ and there are two such groups of order 2^m, $m > 3$, while there is only one such group when $m = 3$. One of these two groups is abelian and the other contains a central of index 4 which includes operators of order 4. When this subgroup of index 2 is non-abelian it belongs to the second of the three categories of such groups whose operators involve exactly two operators which are squares thereunder since its central involves operators of order 4. Cf. *American Journal of Mathematics,* volume 55 (1933), page 419, where the number of such groups is determined but it is not explicitly stated that the groups considered are characterized by the fact that each of them involves two and only two operators which are squares thereunder whenever it involves operators of order 4.

If an operator of order 8 in G transforms the operators of this subgroup of index 2 according to an inner isomorphism it may be assumed that it is commutative with each of its operators and hence there is one and only one such G of order 2^m. If this operator of order 8 transforms this subgroup of index 2 according to an outer isomorphism it is non-commutative with an operator of order 2 in its central and hence there is one and only one such group of order 2^m whenever the central of the given subgroup of index 2 has an order which exceeds 4. This proves the following theorem: *The number of the groups of order 2^m, m odd, which separately have the property that each of them involves exactly four operators which are square thereunder and these operators constitute a cyclic subgroup of the central thereof is one more than twice the number of the different natural numbers which can be substituted for λ in the expression $m \geqq 2\lambda + 3$. When m is even there is one more such group.*

If the operators of order 4 in G which are squares under G are not invariant under G then G can be obtained by extending one of the groups of the preceding theorem by an operator whose square appears therein and under which it is invariant. The order of each of the additional operators divides 4 and hence the given subgroup of index 2 is a characteristic subgroup of G since it is generated by its operators of order 8. The groups obtained by extending one of these subgroups must therefore be distinct from those obtained by thus extending any other such subgroup. Each of the added operators transforms one-fourth of the operators of this subgroup into themselves, another fourth into themselves multiplied by an operator of order 2, and the rest, composed of its operators of order 8, into themselves multiplied by operators of order 4. The first one-fourth of these operators constitute an invariant subgroup giving rise to the cyclic quotient group of order 4 with respect to this subgroup. This invariant subgroup cannot involve all the operators of order 4 which appear in the central of this subgroup of index 2 since such an operator is transformed into its inverse under G.

When the operators of order 4 in this subgroup of index 2 generate an abelian group but the entire subgroup of index 2 is non-abelian there are four such groups whenever $m > 5$ while there are only three such groups when $m = 5$. When this subgroup of index 2 is abelian and $m > 4$ there are again four such groups, but when $m = 4$ there are only three such groups. When the operators of order 4 in this subgroup of index 2 generate a non-abelian group then this group involves a subgroup of index 2 under it in which half the operators are of 4. It also involves subgroups of this index in which more than half of these operators have this property and less than half of these operators have this property since each of these subgroups can be extended by means of an operator of order 4 which is commutative with each of its operators so as to obtain the required group of index 4 under G. This proves the following theorem: *There are five groups of order* 2^m *which separately involve the same group coming under the preceding italicized theorem as a subgroup of index* 2 *whenever the operators of order* 4 *contained therein generate a non-abelian group whose central is of a larger order than* 4. *There are only four such groups when this central is of order* 4 *or when this subgroup of index* 2 *is abelian and its order exceeds* 8 *or when it is non-abelian but its operators of order* 4 *generate an abelian group whose order exceeds* 8. *In the remaining cases there are only three such groups.*

It remains to consider the case when three of the four operators in G which are squares thereunder are of order 2. These three operators must be commutative with each other since they could not all be conjugate under G as there is no operator of order 3 in G. When G is abelian it is obviously of type $(2, 2, 1, 1, \cdots)$. It will therefore be assumed in what follows that G is non-abelian. As the four squares are relatively commutative they generate either the four-group or the group of order 8 and of type $(1, 1, 1)$. The corresponding quotient group involves no operator of order 4 and hence is abelian. The operators which are squares must therefore generate the commutator subgroup of G. To construct such a group in which these squares generate a subgroup of order 8 we may extend the group of order 16 involving three distinct squares which do not include its commutator of order 2 by an operator of order 4 whose square is commutative with each of the operators of this subgroup of order 16 but is not found therein and which transforms each of the operators of order 4 in this subgroup into itself multiplied by the product of the three squares of order 2.

To prove that each of these four squares is in the central of G it may first be noted that if two of these squares were conjugate under G then a co-set of G composed of operators of order 4 which have all their squares equal to one of these two conjugates would not be invariant under G. This is contrary to the observation made above that the quotient group of G with respect to the subgroup generated by its operators which are squares involves no operator whose order exceeds 2. This proves the following theorem: *If a group involves four and only four operators which are squares thereunder and if three of these are of order* 2 *then each of them is in the central of the group and they generate a subgroup whose order is either* 4 *or* 8 *which includes the commutator subgroup of the group.*

From this theorem it results that when the four squares of G generate a group of order 8 then the product of two distinct squares is never a square while such a product is always a square when they generate a group of order 4. In the former case all the operators of G which are commutative with one of its operators of order 4 constitute a subgroup whose index under G cannot exceed 4 since this operator cannot be transformed under G into itself multiplied by the product of the two squares of order 2 which differ from its own square. All the operators of order 4 in this subgroup have a common square and exactly one-half of the operators of this subgroup are of order 4. As the order of this subgroup is at least 16 the order of G is at least 32. When the order of G is 32 its commutator subgroup is of order 2 and is generated by the product of its three operators of order 2 which are squares. This proves the following theorem: *There is one and only one group of order* 32 *which involves four and only four operators which are squares thereunder and generate a group of order* 8.

We proceed to determine all the possible groups which involve exactly four operators which are squares such that these squares generate a group of order 8. If the order of the central of such a group exceeds 8 it is the direct product of such a group whose central is of order 8 and an abelian group of type $(1, 1, 1, \cdots)$. Hence we may confine our attention in what follows to the consideration of such groups in which the central is of order 8. Three operators of order 4 contained in such a group and having three distinct squares generate a group whose order is either 32 or 64. In the former case this group is completely determined according to the theorem closing the preceding paragraph. In the latter case this group of order 32 appears as a subgroup of the group of order 64 and hence it results *that every group of order* 2^m *which contains four and only four operators which are squares thereunder and generate a subgroup of order* 8 *contains invariantly the group of order* 32 *which has these properties.*

By means of this theorem it is possible to determine an upper limit for the order of such a group when the order of its central does not exceed 8. The given invariant subgroup of order 32 has a group of inner isomorphisms whose order is 4. If we can prove that it cannot be transformed according to a group whose order is as large as 16 under such a group it results that the order of this group cannot exceed 64. If we select a set of two independent generators of this invariant subgroup of order 32 it may be noted that when the former is transformed into itself the latter cannot be transformed in more than two different ways by an operator of order 4 since this must have the same square as the operator which is left invariant. As one of these corresponds to an inner isomorphism there is only one such group of order 64. The group of inner isomorphisms of this group of order 64 is of order 8.

When neither of these two independent generators is transformed according to an inner isomorphism there results a group which is conformal to the one just found but in which the non-invariant operators of order 2 have four conjugates while they have only two conjugates under the preceding group. Hence it follows that there are two such groups of order 64 whose group of inner isomorphisms is of order 8. The direct product of the given group of order 32 and the group of order 2 has also the property that it contains exactly four operators which are squares thereunder. This proves the following theorem: *If the four operators which are squares under a given group of order 2^m generate a subgroup of order* 8 *then the order of the group is divisible by* 32 *and there is one and only one such group of order* 32 *while there are exactly three such groups whose order is an arbitrary higher power of* 2.

It remains to consider the cases when the four operators of G which are squares constitute the four group. Just as before it results that each of these four operators appears in the central of G and that the commutator subgroup of G is contained in this four group. When G is abelian it is the direct product of the group of type $(2, 2)$ and a group of type $(1, 1, 1, \cdots)$, where the latter may reduce to the identity. When G is non-abelian its commutator subgroup is either of order 2 or of order 4, and its order is divisible by 32. When its commutator subgroup is of order 2 and not all the operators whose square is the commutator of order 2 are commutative then these operators generate a subgroup of index 2 under G and each of the remaining operators is of order 4. In this case G can be obtained by extending one of the known groups whose operators involve two distinct squares thereunder by an operator of order 4 whose square is contained therein and which transforms this group into itself. This is also the case when all the operators of order 4 in G whose square is the commutator of order 2 are commutative, provided the subgroup they generate is of index 2 under G. When this subgroup is not of index 2 under G it must be of index 4 thereunder and G is obtained by extending a generalized dihedral group in the given manner.

When the commutator subgroup of G is the four group the central of G may be of one of the following three types: $(1, 1, 1, \cdots)$, $(2, 1, 1, \cdots)$, $(2, 2, 1, 1, \cdots)$. In the first case G is the direct product of a group whose commutator subgroup is the four group and an abelian group of type $(1, 1, 1, \cdots)$. In the second case it is the direct product of such an abelian group and a group whose central is of order 8 and of type $(2, 1)$, while in the third case it is the direct product of such an abelian group and a group whose central is of order 16 and of type $(2, 2)$. In this last case it contains a subgroup of index 4 whose central is the four-group, while in the preceding case it contains a subgroup of index 2 whose central is the four group. Hence in all cases G may be constructed by first constructing a group whose central is the four group. There is no upper limit for the order of such a group but the lower limit is obviously 32. The number of these groups of order 2^m increases with m even when it is assumed that their common central is the four group. Cf. *Proceedings of the National Academy of Sciences,* volume 19 (1933), page 848.

ON ISOMORPHISMS OF THE ABELIAN GROUP OF TYPE 1, 1, · · ·.

By H. R. BRAHANA.

It has been shown recently* that if the group of isomorphisms I of an abelian group H of order p^n and type $1, 1, \cdots$ be written as a linear group in the ordinary way, then a necessary and sufficient condition that an operator U in I be of order a power of p is that the characteristic determinant of U be $(-1)^n(\lambda-1)^n$. Moreover, the order of U is exactly p^m where p^m is the smallest power of p greater than or equal to the degree of the n-th invariant factor of U. These facts with some others contained in the paper referred to above give immediately a considerable amount of information concerning the Sylow subgroup I_p of order $p^{n(n-1)/2}$ of I. Much of this information is to be found in a paper by Miller.† It is believed, however, that the present approach throws new light on the subject since it connects so closely with the well-known invariants of a linear transformation.

1. *Classification of transformations.* The invariant factors of U are all powers of $(1-\lambda)$ and consequently the canonical form of U is determined by the degrees of its invariant factors. There exists a canonical form having for invariant factors any set of powers of $(1-\lambda)$ subject to the condition that their product is of degree n, and such a canonical form determines an isomorphism U. Two U's are conjugate if and only if they have the same canonical form. From these facts we obtain the following fundamental theorem:

(1.1) *The operators of order a power of p in the group of isomorphisms of H constitute θ_n conjugate sets, where θ_n is the number of partitions of n. The correspondence between conjugate sets and partitions of n is one to one.*

The identity transformation is the only transformation in one of the conjugate sets and it corresponds to the partition $n = 1 + 1 + \cdots + 1$.

In the following pages we shall designate an operator of I by means of the partition of n to which it corresponds. Let us consider the partition

$$n = n_1 + n_2 + \cdots + n_\gamma + \cdots + n_\delta,$$

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* Proceedings of the National Academy of Sciences, December, 1932, p. 722.

† "Determination of all the groups of order p^m, etc.," *Bulletin of the American Mathematical Society*, ser. 2, vol. 8 (1902), p. 391.

in which we suppose the terms ordered so that $n_i \geq n_{i+1}$, $n_\gamma > 1$, and $n_{\gamma+1} = 1$, if $\gamma < \delta$. Then from the second of the theorems quoted above it follows that the order of U is p^m where m is the smallest integer such that $n_1 \leq p^m$. Obviously n_1 is not greater than n and we have the following theorem:

(1. 2) *The Sylow subgroup I_p of I contains operators of order p^m where m is the smallest number such that $n \leq p^m$, and contains no operator of higher order.**

The operators of highest order p^m of I constitute a single conjugate set only if there is but one partition of n such that $n_1 > p^{m-1}$. In this case $n - 1 = p^{m-1}$. Hence,

(1. 3) *A necessary and sufficient condition that the operators of highest order of I_p constitute a single conjugate set in I is that n be of the form $p^a + 1$.*

If n_1 is not greater than p, the corresponding operator is of order p. Hence,

(1. 4) *The number of conjugate sets of operators of order p in I is one smaller* † *than the number of partitions of n with a greatest term not greater than p.*

In like manner it is obvious that

(1. 5) *The number of conjugate sets of operators of order p^a is equal to the number of partitions of n with a greatest term greater than p^{a-1} and not greater than p^a.*

There are many theorems about non-cyclic abelian subgroups of I_p which are as easily obtained. Statement of them would be so complicated as to obscure their essential simplicity. The method is more important and is sufficiently indicated by an illustration. Let $p = 5$ and $n = 8$. Then I_p is of order 5^{28}. The preceding theorems state: (a) the operators of I_p constitute 22 conjugate sets; (b) I_p contains operators of order 25 and none of higher order; (c) I contains 4 conjugate sets of operators of order 25 and 17 conjugate sets of operators of order 5. Now let us consider an operator U_{ij} in the conjugate set corresponding to the partition $n = 2 + 1 + \cdots + 1$. If the generators of H are $s_1, s_2, \cdots, s_8$, then U_{ij} may be defined as follows:

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* This result is contained implicitly in the paper by Miller cited above.

† This is to account for the partition in which $n_1 = 1$.

$U_{ij}^{-1}s_iU_{ij} = s_is_j$, $U_{ij}^{-1}s_kU_{ij} = s_k$, $k \neq i$. The operators U_{12}, U_{34}, U_{56}, and U_{78} obviously generate an abelian group of order 5^4 and type $1, 1, \cdots$ in I_p. We made certain that the group was abelian by making the U's affect different generators of H. The same considerations show that I_p contains an abelian group of order 5^3 and type $2, 1$ generated by operators corresponding to the partitions $n = 6 + 1 + 1$ and $n = 2 + 1 + \cdots + 1$.

2. *Characteristics of a transformation.* When U is expressed as a linear transformation the degrees of its invariant factors identify the conjugate set to which it belongs. These degrees constitute the terms in a partition of n. Then any characteristic subgroups of the subgroup of the holomorph of H determined by U should be interpretable in terms of the partition of n to which U corresponds.

From the theory of linear transformations* it follows that generators of H may be chosen, as for U_{ij} above, so that U can be written as a set of δ partial transformations each on n_i, $i = 1, 2, \cdots, \delta$, variables distinct from those transformed by the other $\delta - 1$ partial transformations. Each partial transformation U_i transforms one of the operators of H successively into a set of operators which generate a subgroup H_i whose order is p^{n_i}. This group H_i is of the type considered in a recent paper.† Every such group H_i contains one and only one subgroup of order p whose operators are invariant under U_i and hence invariant under U. Hence we have the theorem: ‡

(2.1) *The subgroup G of the holomorph of H determined by U has a central of order p^{δ}.*

Moreover, commutators obtained by transforming operators of H_i by U_i constitute a group K_i invariant under U_i. If the order of H_i is greater than p, i. e., if $n_i > 1$, then the order of K_i is at least p and K_i contains a subgroup of order p whose operators are invariant under U_i. Hence, the theorem:

(2.2) *The commutator subgroup K and the central C of G have a cross-cut of order p^{γ}.*

If the generators $s_1, s_2, \cdots, s_n$ of H are chosen so that U is in canonical form we have

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* Dickson, *Modern Algebraic Theories*, 1930, p. 90, Theorem I.

† "Groups $\{S, T\}$ whose commutator subgroups are Abelian," *Transactions of the American Mathematical Society*, vol. 35 (1933), p. 386.

‡ Cf. the first reference, Theorem 4.

$$U_1^{-1}s_iU_1 = s_{i+1}, \qquad (i = 1, 2, \cdots, n_1 - 1),$$
$$U_1^{-1}s_{n_1}U_1 = s_1^{a_1}s_2^{a_2}\cdots s_{n_1}^{a_{n_1}}.$$

The commutator determined by U_1 and s_i, for $i < n_1$, is $s_i^{-1}s_{i+1}$. These $n_1 - 1$ commutators are independent and generate a group of order p^{n_1-1} in H_1. The n_1-th commutator, arising from U_1 and s_{n_1}, is in this group for otherwise K_1 would be H_1 which is impossible because H_1 contains operators invariant under U_1. The argument is identical for any other partial transformation U_i. It is obvious that K is the direct product of the K_i's. Hence,

(2. 3) *The order of the commutator subgroup K of G is $p^{n-\delta}$.*

It is interesting to apply the above theorems to the question of determining G by means of its order and the order of its characteristic subgroups. There are as many mutually non-conjugate subgroups G of order p^{n+1} in the holomorph of H which contain H invariantly as a maximal abelian subgroup, i. e., $G = \{H, U\}$, as there are partitions of n with $n_1 \leq p$. Of the G's just described there are as many with the same central of order p^{δ} as there are partitions of n into δ terms. Of the second set of G's there are as many with the same commutator subgroup of order $p^{n-\delta}$ as there are partitions of $n - \delta$ into γ terms. The theorems stated above give some obvious interpretations of the partition of n which determines G in terms of some well known and extremely useful characteristics of G, viz., its order and the orders of its central and commutator subgroup. It is obvious that no set of quantities can determine G without determining every one of the terms in the partition of n to which it corresponds.

Another characteristic of G expressible immediately in terms of the partition of n is its class.* We repeat the definition of class. Let G' be the group of inner isomorphisms of G, let G'' be the group of inner isomorphisms of G', and so on. If G is solvable the sequence $G, G', G'', \cdots$ contains $G^{(k)} = 1$ such that $G^{(k-1)} \neq 1$. The group G is then said to be of class k. We recall that the group of inner isomorphisms of a group is simply isomorphic with its central quotient group.

The group G is of order p^{n+m}, where p^m is the order of U. The order of the central of G is p^{δ} and the corresponding quotient group is G' of order $p^{n+m-\delta}$. G' contains an abelian invariant subgroup H' of order $p^{n-\delta}$ corresponding to H. The group G' is determined by H' and an isomorphism of H' which corresponds to the partition

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* Fite, "On Metabelian groups," *Transactions of the American Mathematical Society*, vol. 3 (1902), p. 348.

$$n-\delta=(n_1-1)+(n_2-1)+\cdots+(n_\gamma-1).$$

The remainder of the proof is obvious. We may apply the same argument to G' to obtain G'' and show that G'' is determined by H'' and a transformation of H'' corresponding to the partition

$$n-\delta-\gamma=(n_1-2)+(n_2-2)+\cdots+(n_\epsilon-2).$$

At the j-th step we shall have $G^{(j)}$ determined by $H^{(j)}$ and an isomorphism of $H^{(j)}$ corresponding to the partition of the exponent of p in the order of $H^{(j)}$ into

$$(n_1-j)+(n_2-j)+\cdots.$$

The central quotient group $G^{(j+1)}$ will be identity only if $n_1-j=1$. Hence, we have

(2. 4) *The class of G is n_1.*

The difference between the exponents of p in the orders of the centrals of G and G' is $\delta-\gamma$ which is the number of the n_i's which are equal to 1. The corresponding difference for G' and G'' is the number of the n_i's which are equal to 2. Consequently if the orders of the centrals of $G, G', G'', \cdots$ are given, the n_i's are completely determined. Hence,

(2. 5) *A group $G=\{H, U\}$ in the holomorph of H is completely determined by the orders of its central and of the centrals of the successive groups of inner isomorphisms.*

These numbers are not independent for the sum of the exponents of p in the orders of the centrals of the successive central quotient groups is n. They are, however, independent of any set of generators and are characteristic numbers of the group G. They therefore constitute a partition of n into an ordered set of terms each of which is directly interpretable in terms of a characteristic group determined by G.

On the other hand there is a set of subgroups of G each directly connected with one of the numbers n_i. Each of the groups H_i is invariant under U and hence the group $\{H_i, U\}$ is a non-abelian subgroup of G. Its commutator subgroup is of order p^{n_i-1} and hence determines n_i. This group is not even invariant, much less characteristic, for if s_j is not in H_i and is not permutable with U it will not transform $\{H_i, U\}$ into itself. However, the order of the central of G determines δ and the order of the cross-cut of the central and the commutator subgroup determines γ. Any operator U outside of H which with H generates G will transform K into itself according to an isomorphism which corresponds to a partition of $n-\delta$ into γ terms. Each of the terms of this partition determines one of the numbers n_i.[*]

3. *Groups of order p^{n+1} which contain H.* Every group G of order p^{n+1} which contains H contains H invariantly. Any operator U of G has its p-th power in H. Therefore if U is outside of H it must transform H according to an isomorphism of H of order p. The partition of n to which U corresponds must then have $n_1 \leq p$. On the other hand we have seen in the preceding section that for every partition of n in which $n_1 < p$ there exists a group of order p^{n+1} in the holomorph of H, generated by H and U, of order p. Any other group G of order p^{n+1} which contains H must contain an operator of order greater than p which transforms H according to an isomorphism of order p. The p-th power of U is an operator of H which is invariant under U.[†] In the first place the order of U cannot be greater than p^2. Next, for a given partition of n there are in general two groups G of order p^{n+1} which contain operators of order p^2, one in which the p-th power of U_1 is an invariant commutator, and one in which the p-th power of U_1 is an invariant operator not a commutator. We shall show that the number of groups G of order p^{n+1} corresponding to a given partition is one greater than the number of different values among the n_i's.

If G is in the holomorph and $n_1 < p$ then every operator of G, except identity,[‡] is of order p. The condition that the order of sU be p, where s is an operator of H and U is of order p, is that the product of the set of conjugates of s under powers of U be identity, and this is true if and only if n_1 is less than p.[§] The restriction of G to the holomorph of H required U to be of order p; any group of order p^{n+1} which contains H and whose operators are of order p will obviously be simply isomorphic with a G in the holomorph. Hence we may note the following theorem:

(3.1) *The number of groups of order p^{n+1} which contain H and whose operators are all of order p is equal to the number of partitions of n in which n_1 is not greater than $p-1$.*

The abelian group of type $1, 1, \cdots$ is included above. The theorem also states that when $p = 2$ there is but one group, which is abelian.

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[*] This is the method of determination of U given by Miller in the paper cited above.

[†] "On the groups which contain a given invariant subgroup, etc.," *American Journal of Mathematics*, vol. 52 (1930), p. 915.

[‡] Hereafter when we speak of every operator of G we shall understand that identity is excluded.

[§] Cf. Theorems (3.3) and (4.4) of the second reference of the preceding section.

Since all the operators of the above groups are of order p the one determined by a given partition of n will be distinct from any G corresponding to the same partition which contains operators of order p^2. Let U_1 be of order p^2 and let U_1^p be s_a which, as noted above, must be in the central of G. The p-th power of sU_1, where s is any operator of H, is U_1^p multiplied by the product of the set of conjugates of s under powers of U. This set of conjugates has identity for a product if and only if $n_1 < p$. Therefore when $n_1 < p$, the group of p-th powers in G constitutes a group of order p. Since the group of p-th powers and the commutator subgroup are both characteristic it follows that if two groups G_1 and G_2 correspond to the same partition of n and have their groups of p-th powers respectively in and not in the commutator subgroups, the two groups are not simply isomorphic. The group of isomorphisms I of H contains an operator which is permutable with the isomorphism to which U_1 corresponds and which transforms any operator in the central of G and not in the commutator subgroup into any other such operator. Consequently, any two groups for which U_1^p is not in the commutator subgroup are simply isomorphic.

Now let us consider two groups G_1 and G_2 generated by H and U_1 and U_2 respectively, where U_1 and U_2 correspond to the same isomorphism U of H. Let U_1^p be an invariant operator in H_i and U_2^p be an invariant operator in H_j. If $n_i = n_j$, the correspondence obtained by interchanging H_i and H_j, and letting U_1 and all the remaining generators of H correspond to themselves determines a simple isomorphism between G_1 and G_2. If $n_i \neq n_j$, then no simple isomorphism exists, for, if $i < j$, G contains a subgroup $\{H_i, U_1\}$ whose central is of order p and whose order is p^{n_i}, and G_2 contains no such subgroup. Any subgroup of G_2 of that order will have a central of order at least p^2. Therefore there are as many groups corresponding to a given partition of n as there are different numbers n_i in that partition and each of these groups contains operators of order p^2.

There is left the possibility that U_1^p is in the commutator subgroup but is not in any of the groups H_i. It will, however, be a product of operators from the H_i's and the corresponding product of the operators whose conjugates generate those H_i's will with its conjugates under U generate a group of order $p^{n_{i_1}}$, where i_1 is the subscript of the first of the H_i's that has operators appearing in the expression for U_1^p. If H_{i_1} is replaced by the group just described, all the other H_i's being left unchanged, then G is in the form of the groups considered in the last paragraph. We have proved the theorem:

(3.2) *The number of groups of order p^{n+1} which contain H and transform it according to a given operator U is one greater than the number of different numbers among the n_i's of the partition of n to which U corresponds.*

By means of this theorem we may obtain an explicit expression for the number of metabelian groups of order p^{n+1} which contain H. The metabelian groups are the groups of class 2. If G is metabelian, then $n_1 = 2$. Then n_i is either 2 or 1. The number of partitions of n into 2's and 1's is $n/2$ or $(n-1)/2$ according as n is even or odd. Every partition contains two distinct numbers except the one $n = 2 + 2 + \cdots + 2$, when n is even. Hence,

(3.3) *The number of metabelian groups of order* p^{n+1} *which contain* H *is* $(3n-2)/2$ or $3(n-1)/2$, *according as* n *is even or odd. Of these* $n/2$ *or* $(n-1)/2$ *contain only operators of order* p.

The only groups of order p^{n+1} which contain H and are not counted in (3.2) are the groups of class p. The isomorphism U determined by such a group corresponds to the partition $n = p + n_2 + n_3 + \cdots + n_\delta$. The group H_1 is of order p^p. It has p independent generators which are conjugate under U. Their product is of order p and therefore the order of s_iU is p^2, where s_i is any operator whose conjugates under U generate a group of order p^p. Since $(s_iU)^p$ is in H it is invariant and since it is the product of the conjugates of s_i under U it is in one of the groups H_i whose order p^{n_i} is p^p, if it is not identity. Hence the order of the group of p-th powers in $\{H, U\}$ is p^α, where α is the number of the n_i's which are equal to p. The group just described is in the holomorph of H. If now we take U_1 to be an operator of order p^2 which transforms H according to U, of order p, and such that U_1^p is an invariant operator of H not a p-th power in the group above, $G_1 = \{H, U_1\}$ will have a group of p-th powers of order $p^{\alpha+1}$. G_1 is therefore not simply isomorphic with G. An argument similar to that carried out for groups of class lower than p shows that there are as many ways of choosing U_1 as there are distinct numbers among the n_i's of the partition of n in question. Therefore,

(3.4) *The number of distinct groups of order* p^{n+1} *and class* p *corresponding to a given partition of* n *with* $n_1 = p$ *is equal to the number of distinct numbers among the* n_i*'s.*

There is one group missing for each partition of n in (3.4) as compared with (3.2). This is due to the fact that when U is of order p the invariant operators of H_1 are p-th powers and so no new group is obtained by choosing U_1 differently to make them p-th powers.

The question of the identification of a given group G of order p^{n+1} which contains H involves the determination of the partition of n to which an operator outside of H corresponds. By the argument preceding (2.5) this involves essentially the determination of the orders of the centrals of the successive groups of inner isomorphisms. It is then necessary to determine the location of the group of p-th powers of G with regard to the invariant operators of the commutator subgroup of G. This in turn involves the determination of the order of the largest subgroup of G whose central is of order p.

Another interesting fact, which we have proved incidentally, is that if the class of G is not greater than $p-1$ G is conformal with the abelian group of type $1, 1, \cdots$ or with the abelian group of type $2, 1, 1, \cdots$.

URBANA, ILLINOIS.

ON SUMMABILITY OF MULTIPLE SEQUENCES.*

By RALPH PALMER AGNEW.

1. *Introduction.* The object of this note is to extend to sequences, of multiplicity greater than two, certain theorems on transformations of double sequences obtained by Lösch † and the writer.‡ These theorems have, in the special case of the arithmetic mean transformation, been partially extended to sequences of multiplicity greater than two by Bochner.§

We shall, for brevity, state and prove our results only for triple sequences. The reader will see that our methods can readily be applied also to double sequences and to sequences of any multiplicity greater than three.

2. *Transformations.* Let $\|a_{mi}^{(\lambda)}\|$, $\lambda = 1, 2, 3$, be three triangular matrices of real or complex constants satisfying the conditions

$$\sum_{i=0}^{m} |a_{mi}^{(\lambda)}| < K \qquad (\lambda = 1, 2, 3;\ m = 0, 1, 2, \cdots) \tag{2.1}$$

$$\text{for each } i, \lim_{m\to\infty} a_{mi}^{(\lambda)} = 0 \qquad (\lambda = 1, 2, 3) \tag{2.2}$$

$$\lim_{m,n,p\to\infty} \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p} a_{mi}^{(1)} a_{nj}^{(2)} a_{pk}^{(3)} = 1 \tag{2.3}$$

where K is a constant independent of m. With each triple sequence s_{ijk}, convergent or not, we associate a transform S_{mnp} defined by

$$F: \qquad S_{mnp} = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p} a_{mi}^{(1)} a_{nj}^{(2)} a_{pk}^{(3)} s_{ijk}.$$

The sequence s_{ijk} is said to be summable F to S if $\lim_{m,n,p\to\infty} S_{mnp} = S$ and to be ultimately bounded F if $\limsup_{m,n,p\to\infty} |S_{mnp}| < \infty$. It is easy to show that each bounded convergent triple sequence is summable F to the

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† Lösch, "Über den Permanenzsatz gewisser Limitierungsverfahren für Doppelfolgen," *Mathematische Zeitschrift*, vol. 34 (1931), pp. 281-290; and a second paper bearing the same title, *ibid.*, vol. 37 (1933), pp. 77-84.

‡ Agnew, "On summability of double sequences," *American Journal of Mathematics*, vol. 54 (1932), pp. 648-656.

§ Bochner, "Limitierung mehrfacher Folgen nach dem Verfahren der arithmetischen Mittel," *Mathematische Zeitschrift*, vol. 35 (1932), pp. 122-126.

value to which it converges. However it is not true that each convergent sequence is so summable unless the matrices $\| a^{(\lambda)}_{mi} \|$ satisfy in addition to (2.1), (2.2) and (2.3) a condition which we do not impose, namely

$$\text{(2.4)} \qquad \text{for each } i,\ a^{(\lambda)}_{mi} = 0 \text{ for all sufficiently great } m,$$

when $\lambda = 1, 2, 3$. An extent to which unbounded convergent triple sequences are summable F is given by the extension to triple sequences of work of C. R. Adams * on double sequences.

3. *Theorems on triple sequences.* The principal result we prove is the following extension to triple sequences of Theorem 1 of the author's previous paper.

THEOREM 3.1. *If s_{ijk} converges to s and if there exist an index Q and three sequences α_m, β_n and γ_p of constants such that*

$$\text{(3.11)} \qquad \begin{cases} \text{for each } m > Q, & |S_{mnp}| < \alpha_m \qquad n, p > Q \\ \text{for each } n > Q, & |S_{mnp}| < \beta_n \qquad m, p > Q \\ \text{for each } p > Q, & |S_{mnp}| < \gamma_p \qquad m, n > Q \end{cases}$$

then s_{ijk} is summable F to s.

The condition (3.11) is obviously necessary in order that s_{ijk} may be ultimately bounded F or summable F; hence we obtain from Theorem 3.1 the following extensions to triple sequences of theorems given by Lösch and the writer.

THEOREM 3.2. *If s_{ijk} converges to s and is ultimately bounded F, then s_{ijk} is summable F to s.*

THEOREM 3.3. *If s_{ijk} converges to s and is summable F to S, then $S = s$.*

The last of these theorems has been proved by Bochner, *loc. cit.*, for the case in which F is the arithmetic mean transformation, i. e. $a^{(\lambda)}_{mi} = 1/(m+1)$.

4. *Proof of Theorem* 3.1. To prove Theorem 3.1, let s_{ijk} satisfy its hypotheses. Since $s_{ijk} \to s$, we can choose an index $R > Q$ such that s_{ijk} is uniformly bounded for all $i, j, k > R$. Then for $m, n, p > R$,

$$\text{(4.1)} \qquad \sum_{i=R+1}^{m} \sum_{j=R+1}^{n} \sum_{k=R+1}^{p} a^{(1)}_{mi}\, a^{(2)}_{nj}\, a^{(3)}_{pk}\, s_{ijk} = s + o_{mnp}$$

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* Adams, "Transformations of double sequences with application to Cesàro summability of double series," *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 741-748.

where o_{mnp} is a uniformly bounded sequence converging to 0 as $m, n, p \to \infty$. Using 4.1 we obtain for $m, n, p > R$

$$S_{mnp} - s - o_{mnp} = \{\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p} - \sum_{i=R+1}^{m}\sum_{j=R+1}^{n}\sum_{k=R+1}^{p}\} a^{(1)}_{mi}\, a^{(2)}_{nj}\, a^{(3)}_{pk}\, s_{ijk}$$

$$= \{\sum_{i=0}^{R}\sum_{j=0}^{n}\sum_{k=0}^{p} + \sum_{i=R+1}^{m}\sum_{j=0}^{R}\sum_{k=0}^{p} + \sum_{i=R+1}^{m}\sum_{j=R+1}^{n}\sum_{k=0}^{R}\} a^{(1)}_{mi}\, a^{(2)}_{nj}\, a^{(3)}_{pk}\, s_{ijk}.$$

This assumes for $m, n, p > R$ the form

$$S_{mnp} - s - o_{mnp} = \sum_{i=0}^{R} a^{(1)}_{mi} A^{(1)}_{inp} + \sum_{j=0}^{R} a^{(2)}_{nj} A^{(2)}_{jmp} + \sum_{k=0}^{R} a^{(3)}_{pk} A^{(3)}_{kmn} \tag{4.2}$$

when we introduce notation whose definition is obvious.

Using (4.2), we verify easily that Theorem 3.1 is a consequence of the following lemma.

LEMMA 4.3. *Let R be a non-negative integer and let*

$$H_{mnp} = \sum_{i=0}^{R} a_{mi}A_{inp} + \sum_{j=0}^{R} b_{nj}B_{jmp} + \sum_{k=0}^{R} c_{pk}C_{kmn} \tag{4.4}$$

where

$$\lim_{m\to\infty} a_{mi} = 0, \quad \lim_{n\to\infty} b_{nj} = 0, \quad \lim_{p\to\infty} c_{pk} = 0 \quad \text{for each } i, j, k. \tag{4.5}$$

If there exist an index N and three sequences α'_m, β'_n and γ'_p such that

$$\begin{cases} \text{for each } m > N & |H_{mnp}| < \alpha'_m \qquad n, p > N \\ \text{for each } n > N & |H_{mnp}| < \beta'_n \qquad m, p > N \\ \text{for each } p > N & |H_{mnp}| < \gamma'_p \qquad m, n > N \end{cases} \tag{4.6}$$

then $H_{mnp} \to 0$ as $m, n, p \to \infty$.

5. *Proof of Lemma* 4.3. We prove Lemma 4.3 by induction on R, proving it first for the case $R = 0$.

When $R = 0$, we may omit the subscript 0 from a_{m0}, A_{0np}, etc., writing (4.4) in the form

$$H_{mnp} = a_m A_{np} + b_n B_{mp} + c_p C_{mn}. \tag{5.1}$$

If $a_m = b_n = c_p = 0$ for all sufficiently great m, n, p, then obviously $H_{mnp} \to 0$ as $m, n, p \to \infty$.

Otherwise, because of symmetry in m, n, p of our conditions, we may assume that $c_{p'} \neq 0$ for some fixed $p' > N$. Then we obtain from (5.1)

$$c_{p'}H_{mnp} - c_p H_{mnp'} = a_m(c_{p'}A_{np} - c_p A_{np'}) + b_n(c_{p'}B_{mp} - c_p B_{mp'})$$

which has the form

$$(5.2)\qquad c_{p'}H_{mnp} - c_pH_{mnp'} = a_mA'_{np} + b_nB'_{mp}.$$

If $a_m = b_n = 0$ for all sufficiently great m and n, then obviously $c_{p'}H_{mnp} - c_pH_{mnp'} \to 0$; but $c_p \to 0$ and $| H_{mnp'} | < \alpha'_{p'}$ so that $c_pH_{mnp'} \to 0$ and therefore in this case $H_{mnp} \to 0$.

If we do not have $a_m = b_m = 0$ for all sufficiently great m and n, we may assume that $b_{m'} \neq 0$ for some fixed $n' > N$. In this case we obtain from (5. 2) a relation having the form

$$(5.3)\qquad b_{n'}(c_{p'}H_{mnp} - c_pH_{mnp'}) - b_n(c_{p'}H_{mn'p} - c_pH_{mn'p'}) = a_mA''_{np}.$$

If $a_m = 0$ for all sufficiently great m, then (4. 6) together with the facts that $b_n \to 0$, $c_p \to 0$ and $b_{n'}c_{p'} \neq 0$ imply that $H_{mnp} \to 0$.

If finally we do not have $a_m = 0$ for all sufficiently great m, then we may assume $a_{m'} \neq 0$ for some fixed $m' > N$ and obtain from (5. 3) the relation

$$(5.4)\qquad \begin{aligned} &a_{m'}\{b_{n'}(c_{p'}H_{mnp} - c_pH_{mnp'}) - b_n(c_{p'}H_{mn'p} - c_pH_{mn'p'})\} \\ &\quad - a_m\{b_{n'}(c_{p'}H_{m'np} - c_pH_{m'np'}) - b_n(c_pH_{m'n'p} - c_pH_{m'n'p'})\} = 0. \end{aligned}$$

Hence (4. 6) together with the facts that $a_m \to 0$, $b_n \to 0$, $c_p \to 0$ while $a_{m'}b_{n'}c_{p'} \neq 0$ implies that $H_{mnp} \to 0$ and the proof of the lemma for the case $R = 0$ is complete.

Now let R be a positive integer, and assume that the lemma holds when we replace R by $R - 1$. Leaving consideration of simpler cases to the reader, we suppose we can choose fixed indices $m', n', p' > N$ such that $a_{m'R}b_{n'R}c_{p'R} \neq 0$. From (4. 4) we obtain

$$\begin{aligned} c_{p'R}H_{mnp} - c_{pR}H_{mnp'} &= \sum_{i=0}^{R} a_{mi}(c_{p'R}A_{inp} - c_{pR}A_{inp'}) \\ &\quad + \sum_{j=0}^{R} b_{nj}(c_{p'R}B_{jmp} - c_{pR}B_{jmp'}) + \sum_{k=0}^{R-1}(c_{p'R}c_{pk} - c_{pR}c_{p'k})C_{kmn} \end{aligned}$$

which, through introduction of new notation, we write in the form

$$(5.5)\qquad H'_{mnp} = \sum_{i=0}^{R} a_{mi}A'_{inp} + \sum_{j=0}^{R} b_{nj}B'_{jmp} + \sum_{k=0}^{R-1} c'_{pk}C_{kmn}.$$

From (5. 5) we obtain

$$\begin{aligned} b_{n'R}H'_{mnp} - b_{nR}H'_{mn'p} &= \sum_{i=0}^{R} a_{mi}(b_{n'R}A'_{inp} - b_{nR}A_{in'p}) \\ &\quad + \sum_{j=0}^{R-1}(b_{n'R}b_{nj} - b_{nR}b_{n'j})B'_{jmp} + \sum_{k=0}^{R-1} c'_{pk}(b_{n'R}C_{kmn} - b_{nR}C_{km'n}) \end{aligned}$$

which we write in the form

$$(5.6)\qquad H''_{mnp} = \sum_{i=0}^{R} a_{mi}A''_{inp} + \sum_{j=0}^{R-1} b'_{nj}B'_{jmp} + \sum_{k=0}^{R-1} c'_{pk}C'_{kmn}.$$

From (5. 6) we obtain

$$a_{m'R}H''_{mnp} - a_{mR}H''_{m'np} = \sum_{i=0}^{R-1} (a_{m'R}a_{mi} - a_{mR}a_{m'i})A''_{inp}$$
$$+ \sum_{j=0}^{R-1} b'_{nj}(a_{m'R}B'_{jmp} - a_{mR}B'_{jm'p}) + \sum_{k=0}^{R-1} c'_{pk}(a_{m'R}C'_{kmn} - a_{mR}C'_{km'n})$$

which we write in the form

$$(5.7)\qquad H'''_{mnp} = \sum_{i=0}^{R-1} a'_{mi}A''_{inp} + \sum_{j=0}^{R-1} b'_{nj}B''_{jmp} + \sum_{k=0}^{R-1} c'_{pk}C''_{kmn}.$$

Since

$$(5.8)\qquad H'''_{mnp} = a_{m'R}b_{n'R}c_{p'R}H_{mnp} - a_{m'R}b_{n'R}c_{pR}H_{mnp'}$$
$$- a_{m'R}b_{nR}c_{p'R}H_{mn'p} - a_{mR}b_{n'R}c_{p'R}H_{m'np} + a_{m'R}b_{nR}c_{pR}H_{mn'p'}$$
$$+ a_{mR}b_{n'R}c_{pR}H_{m'np'} + a_{mR}b_{nR}c_{p'R}H_{m'n'p} - a_{mR}b_{nR}c_{pR}H_{m'n'p'},$$

we see from (4. 5) and (4. 6) that the sequence H'''_{mnp} has the essential property which (4. 6) imposes on H_{mnp}. Also a'_{mi}, b'_{nj} and c'_{pk} have the property which (4. 5) imposes on a_{mi}, b_{nj}, and c_{pk}. Therefore, on using (5. 7) and applying the lemma when R is replaced by $R - 1$, we find that $H'''_{mnp} \to 0$ as $m, n, p \to \infty$. Furthermore (4. 5) and (4. 6) ensure that each term of the right member of (5. 8) except the first converges to zero as $m, n, p \to \infty$. Therefore, since $a_{m'R}b_{n'R}c_{p'R} \neq 0$, it follows from (5. 8) that $H_{mnp} \to 0$ as $m, n, p \to \infty$ and the lemma is proved by induction.

6. *The arithmetic mean transformation.* For the special case of the arithmetic mean transformation, (4. 2) becomes

$$S_{mnp} - s - o_{mnp}$$
$$= (R+1)\{\sum_{i=0}^{R} A^{(1)}_{inp}/(m+1) + \sum_{j=0}^{R} A^{(2)}_{jmp}/(n+1) + \sum_{k=0}^{R} A^{(3)}_{kmn}/(p+1)\}$$

which may be written

$$S_{mnp} - s - o_{mnp} = A_{np}/(m+1) + B_{mp}/(n+1) + C_{mn}/(p+1).$$

Thus the proof of Theorem 3. 1 reduces simply to the proof of Lemma 4. 3 for the case where $R = 0$ and none of the coefficients a_{mi}, b_{nj}, c_{pk} vanish.

7. *Applications.* After having proved Theorems 3. 1, 3. 2, 3. 3 and their analogues for sequences of arbitrary multiplicity, we can, without meeting new difficulties, extend to sequences of arbitrary multiplicity many of the results given by Adams * and the writer, *loc. cit.*

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* Adams, "On summability of double series," *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 215-230.

8. *Non-factorable transformations.* The transformation F is of the type called factorable by C. R. Adams. An example has been given by Adams* which shows that the double sequence analogues of Theorems 3.1, 3.2, and 3.3 cannot be extended from factorable to non-factorable transformations.

We give here the triple sequence analogue of a very simple example the writer constructed after having seen the one of Adams. The non-factorable transformation

$$(\mathrm{N}) \qquad S_{mnp} = \begin{cases} s_{000} & p = 0 \\ s_{mnp} + [1/(m+1)] s_{m00} & p > 0 \end{cases}$$

obviously evaluates each bounded convergent sequence to the value to which it converges. The sequence

$$(8.1) \qquad s_{mnp} = \begin{cases} m+1 & p = 0 \\ 0 & p > 0 \end{cases}$$

converges to 0, and has for its N transform the sequence $S_{mnp} = 1$, for all m, n, p, which converges to 1. The sequence

$$(8.2) \qquad s_{mnp} = \begin{cases} (-1)^m (m+1) & p = 0 \\ 0 & p > 0 \end{cases}$$

has for its N transform a bounded divergent sequence. This example shows that there exist non-factorable transformations, regular for bounded convergent sequences, for which analogues of Theorems 3.1, 3.2, and 3.3 do not hold. It is worth noting specifically that the transformation N does not share with the factorable transformations F the property of being consistent with convergence.

W. A. Hurwitz pointed out to the writer that our methods, with only a slight change in detail, suffice to prove Theorems 3.1, 3.2, and 3.3 for the following set of transformations G which includes a large class of non-factorable transformations. Let Δ be a positive integer and let

$$G: \begin{cases} S_{mnp} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{p} a_{mnpijk} s_{ijk} \\ a_{mnpijk} = \sum_{\delta=1}^{\Delta} a_{mi}^{(1,\delta)} a_{nj}^{(2,\delta)} a_{pk}^{(3,\delta)} \end{cases}$$

where $\| a_{mi}^{(\lambda,\delta)} \|$ satisfies, for each $\lambda = 1, 2, 3$; $\delta = 1, 2, \cdots, \Delta$, conditions corresponding to (2.1) and (2.2) and, instead of (2.3), the condition

$$(8.3) \qquad \lim_{m,n,p \to \infty} \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{p} a_{mnpijk} = 1.$$

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* Adams, "On non-factorable transformations of double sequences," *Proceedings of the National Academy of Sciences*, vol. 19 (1933), pp. 564-567.

In proving Theorem 3.1 for G, Lemma 4.3 is applied with R replaced by $\Delta(R+1)-1$.

The transformations which fall in the class G merit some discussion. If we put

$$(8.4) \qquad A_q^{(\lambda,\delta)} = \sum_{t=1}^{q} a_{qt}^{(\lambda,\delta)}$$

then (8.3) may be written

$$(8.5) \qquad \lim_{m,n,p\to\infty} \sum_{\delta=1}^{\Delta} A_m^{(1,\delta)} A_n^{(2,\delta)} A_p^{(3,\delta)} = 1.$$

If $\Delta = 1$, in which case G reduces to F, we have

$$(8.6) \qquad \lim_{m,n,p\to\infty} A_m^{(1,1)} A_n^{(2,1)} A_p^{(3,1)} = 1$$

which implies that each of the three limits

$$\lim_{m\to\infty} A_m^{(1,1)}, \quad \lim_{n\to\infty} A_n^{(2,1)}, \quad \lim_{p\to\infty} A_p^{(3,1)}$$

exists and is different from 0, and that the product of the three limits is 1. Thus each of the three matrices $\| a_{mt}^{(\lambda,1)} \|$ corresponds to a simple-sequence transformation which is convergence preserving and regular for null sequences.

In case $\Delta > 1$, however, (8.5) does not imply that all of the limits

$$\lim_{q\to\infty} A_q^{(\lambda,q)} \qquad (\lambda = 1, 2, 3;\ \delta = 1, 2, \cdots, \Delta)$$

exist. For example, (8.5) holds if

$$\lim_{m,n,p\to\infty} A_m^{(1,1)} A_n^{(2,1)} A_p^{(3,1)} = 1; \quad \lim_{m\to\infty} A_m^{(1,\delta)} = 0 \text{ for each } \delta > 1,$$

and each of the sequences $A_m^{(\lambda,\delta)}$, $\lambda = 2, 3$, $\delta > 1$ is a bounded divergent sequence. Thus when $\Delta > 1$, it is not necessary that all of the matrices $\| a_{mt}^{(\lambda,\delta)} \|$ entering into the definition of G correspond to convergence-preserving simple-sequence transformations.

The double-sequence analogues of the results that Theorems 3.1, 3.2, and 3.3 hold for the transformations G have not been previously pointed out. Thus, for example, our results considerably enlarge the class of double sequence transformations which are known to be consistent with convergence.

CORNELL UNIVERSITY.

ON THE UNITS IN A CYCLIC FIELD.

By Claiborne G. Latimer.

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1. *Introduction.* Let F be an algebraic field which is cyclic with respect to the rational field. The purpose of this paper is to prove three theorems on the existence of certain types of fundamental systems of units in F.

If F is the field defined by a primitive m-th root of unity, m a power of an odd prime, it is well-known that there is a fundamental system of units in F which are all real. This will be extended to any imaginary cyclic field.

Let F be of degree E and let $e \equiv E$ or $e \equiv E/2$ according as F is real or imaginary. F is identical with its conjugates. It follows, from Dirichlet's theorem, that there are exactly $n \equiv e - 1$ units in a fundamental system. A unit of F will be said to be a fundamental unit if it and $n - 1$ of its conjugates form a fundamental system. It has been shown in another paper * that the class number of F, or the "second factor" of the class number, could be expressed in terms of an ideal $\mathfrak{K}$ in a certain commutative ring. It will be shown that F contains a fundamental unit if and only if $\mathfrak{K}$ is a principal ideal. A special case of this is equivalent to a result of Eisenstein's.

Let F be obtained by composition of two fields, F_1 and F_2, the degrees of which are relatively prime. It will be shown that there is a fundamental system of units in F such that certain of them form a fundamental system in F_1 and certain others form such a system in F_2.

2. *A fundamental system of real units.* Let s be a generating substitution of the Galois group of F. If θ is a number of F, it will be understood that $\theta^{(i)} = s^i(\theta)$, $(i = 0, 1, 2, \cdots, E)$, $\theta^{(0)} = \theta^{(E)} = \theta$. $\theta^{(e)}$ is the conjugate imaginary of θ. All the roots of unity in F are powers of one of them ρ, which is of even degree $2m$, $m \geqq 1$. Let $\rho' = \rho^h$, $h^e \equiv H$. Then $\rho^{(e)} = \rho^H = \rho^{-1}$, $H \equiv -1 \pmod{2m}$ and h is odd.

Let $\eta_1, \eta_2, \cdots, \eta_n$ be a fundamental system of units in F. Every η'_i belongs to F and therefore

$$(1) \qquad \eta'_i = \rho^{\alpha_i} \cdot \eta_1^{\alpha_{i1}} \cdot \eta_2^{\alpha_{i2}} \ldots \eta_n^{\alpha_{in}} \qquad (i = 1, 2, \cdots, n),$$

where the α's are rational integers. By Lemma 1, Tr., the matrix $A \equiv (\alpha_{ij})$ $(i, j = 1, 2, \cdots, n)$ is a root of

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* Latimer, "On the class number of a cyclic field," *Transactions of the American Mathematical Society*, vol. 35 (1933), p. 411. This paper will be referred to hereafter as Tr.

$$(2) \qquad f(x) \equiv x^n + x^{n-1} + \cdots + x + 1 = 0,$$

and is not a root of an equation, with rational coefficients, of degree $< n$.

Let F be imaginary. Employing (1) and $\rho' = \rho^h$, we may express successively the η_i'', η_i''', $\cdots$, $\eta_i^{(e)}$ as products of powers of the η's and ρ. Since A is a root of (2), $A^e = I$, the identity matrix of order n. Hence we find

$$\eta_i^{(e)} = \rho^{t_i}\eta_i \qquad (i = 1, 2, \cdots, n),$$

where, if δ_{ij} is Kronecker's δ and $A^k \equiv (\alpha_{ij}^{(k)})$, $(k = 2, 3, \cdots, n)$,

$$t_i = \sum_{j=1}^{n} \alpha_j(\delta_{ij}h^n + \alpha_{ij}h^{n-1} + \alpha_{ij}^{(2)}h^{n-2} + \cdots + \alpha_{ij}^{(n)}), \qquad (i = 1, 2, \cdots, n).$$

But A is a root of (2) and h is odd. It follows that every term in parentheses above is even and therefore every t_i is even. Let $t_i = 2T_i$. Then the units $\theta_i \equiv \rho^{T_i}\eta_i$ belong to F and

$$\theta_i^{(e)} = \rho^{-T_i}\eta_i^{(e)} = \rho^{T_i}\eta_i = \theta_i \qquad (i = 1, 2, \cdots, n).$$

Hence the θ's are real and we have

THEOREM 1. *Let F be an imaginary algebraic field which is cyclic with respect to the rational field and let $\eta_1, \eta_2, \cdots, \eta_n$ be a fundamental system of units in F. By multiplying each η_i by a properly chosen root of unity in F, we obtain a fundamental system of units in F which are all real.*

We shall assume hereafter, without loss of generality, that all the η's are real and positive.

3. *The ring $\mathfrak{G}$.* Let $\mathfrak{A}$ be the set of all polynomials in A with rational coefficients and let $\mathfrak{G}$ be the set of all such polynomials with integral coefficients. If $\phi(x)$ is a polynomial in the indeterminate x, with rational coefficients, the constant term being c, it will be understood that $\phi(A)$ is obtained from $\phi(x)$ by replacing x by A and c by cI. $\mathfrak{A}$ and $\mathfrak{G}$ are commutative rings. $I, A, A^2, \cdots, A^{n-1}$ are linearly independent and form a basis of $\mathfrak{A}$ and of $\mathfrak{G}$.

The ideal $\mathfrak{K}$ referred to in § 1 is a non-singular ideal [*] in $\mathfrak{G}$ and it was seen in Tr. that $\mathfrak{K}$ has a basis $\omega_1, \omega_2, \cdots, \omega_n$ such that

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[*] It will be understood throughout this paper that we employ the same definitions of an ideal and of all terms referring to ideals as given by MacDuffee in his "An introduction to the theory of ideals etc.," *Transactions of the American Mathematical Society*, vol. 31 (1929), pp. 71-90. The results of this paper are valid for ideals in $\mathfrak{G}$ by § 2, Latimer and MacDuffee, "A correspondence between classes of matrices and classes of ideals," *Annals of Mathematics*, series 2, vol. 34 (1933), pp. 313-316.

$$A\omega_i = \sum_{j=1}^{n} \alpha_{ij}\omega_j \qquad (i = 1, 2, \cdots, n).^{*} \tag{3}$$

If

$$\xi_i = z_{i1}\omega_1 + z_{i2}\omega_2 + \cdots + z_{in}\omega_n \qquad (i = 1, 2, \cdots, n), \tag{4}$$

are elements of $\mathfrak{R}$, they form a basis of $\mathfrak{R}$ if and only if the determinant $|Z|$, of the matrix $Z \equiv (z_{ij})$ is ± 1.[†] Similarly, it is well-known that the units of F,

$$\zeta_i = \eta_1^{z_{i1}} \cdot \eta_2^{z_{i2}} \cdots \eta_n^{z_{in}} \qquad (i = 1, 2, \cdots, n), \tag{5}$$

form a fundamental system if and only if $|Z| = \pm 1$.

If $\xi_1, \xi_2, \cdots, \xi_n$ and $\zeta_1, \zeta_2, \cdots, \zeta_n$ are given by (4) and (5) respectively, using the same matrix Z, $|Z| = \pm 1$, they will be said to be corresponding systems. We thus have a one-to-one correspondence between the systems of basal elements of $\mathfrak{R}$ and those fundamental systems of units in F in which every unit is real and positive. For corresponding systems, we have by (1) and (3),

$$\begin{aligned} \zeta'_i &= \pm\, \zeta_1^{\beta_{i1}} \cdot \zeta_2^{\beta_{i2}} \cdots \zeta_n^{\beta_{in}} \\ A\xi_i &= \beta_{i1}\xi_1 + \beta_{i2}\xi_2 + \cdots + \beta_{in}\xi_n, \end{aligned} \qquad (i = 1, 2, \cdots, n) \tag{6}$$

where $B \equiv (\beta_{ij}) = ZAZ^{-1}$.

Theorems 2 and 3 are proved by employing (6) to show that certain properties of a system of basal elements of $\mathfrak{R}$ imply certain properties of the corresponding fundamental system of units in F, and conversely.

4. *On the existence of a fundamental unit.* Suppose $\mathfrak{R}$ is a principal ideal $\{\xi\}$. Then $\xi_i \equiv A^{i-1}\xi$ $(i = 1, 2, \cdots, n)$ form a basis of $\mathfrak{R}$. For this system of basal numbers, the matrix B in (6_2) is

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

For the corresponding fundamental system of units ζ_i, we have by (6_1), $\zeta_{i+1} = \pm\, \zeta'_i = \pm\, \zeta_1^{(i)}$ $(i = 1, 2, \cdots, n-1)$. Hence ζ_1 is a fundamental

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[*] Neither the conditions (3) nor the particular fundamental system of units in F which is employed determine $\mathfrak{R}$ uniquely. However, it may be shown that when the generating substitution s, of the Galois group of F is fixed, the class of ideals to which $\mathfrak{R}$ belongs is uniquely determined.

[†] MacDuffee, *loc. cit.*, p. 74.

unit. Conversely, if F contains a fundamental unit ζ; we may assume that it is real and that $\zeta_i = \pm \zeta^{(i-1)} > 0$ $(i = 1, 2, \cdots, n)$ form a fundamental system. Then by (6), for the corresponding system of basal numbers we have $\xi_{i+1} = A\xi_i = A^i\xi_1$ $(i = 1, 2, \cdots, n-1)$, $\mathfrak{K} = \{\xi_1\}$. We have then

THEOREM 2. *F contains a fundamental unit if and only if $\mathfrak{K}$ is a principal ideal.*

Suppose $\zeta = \eta_1^{z_{11}} \cdot \eta_2^{z_{12}} \cdots \eta_n^{z_{1n}}$ is a positive real fundamental unit in F. Then $\mathfrak{K} = \{\xi\}$, $\xi = z_{11}\omega_1 + z_{12}\omega_2 + \cdots + z_{1n}\omega_n$. If ζ' is another such unit, we find similarly that $\mathfrak{K} = \{\xi'\}$ where ξ' is uniquely determined by ζ'. Since $\mathfrak{K}$ is non-singular, ξ is not a divisor of zero.* It follows that $\xi' = U\xi$, where U is a unit; i. e. an element of $\mathfrak{G}$ such that for a properly chosen element U' of $\mathfrak{G}$, $UU' = I$. Thus if ζ is fixed, every real positive fundamental unit in F determines uniquely a unit in $\mathfrak{G}$. Conversely, every unit in $\mathfrak{G}$ determines uniquely such a unit in F. We have therefore the

COROLLARY. *If $\mathfrak{K}$ is a principal ideal, there is a one-to-one correspondence between the units in $\mathfrak{G}$ and the positive real fundamental units in F.*

If e is a prime, $\mathfrak{G}$ is equivalent to the set of all integral algebraic numbers in the field defined by a primitive e-th root of unity. Hence if $e = 3$ or $e = 5$, all the ideals in $\mathfrak{G}$ are principal. It follows that in the first case there are exactly six real positive fundamental units in F; in the second case, there is an infinitude. For the case where $E = e = 3$ and the discriminant of F is the square of a prime, this is equivalent to a result of Eisenstein's.†

5. *Fundamental systems of units in sub-fields of F.* Let F be obtained by composition of two fields F_1 and F_2, of degrees E_1 and E_2 respectively, where E_1 and E_2 are relatively prime. We shall assume that E_2 is odd and hence F_2 is real. Let $e_1 \equiv E_1$ or $e_1 \equiv E_1/2$ according as F_1 is real or imaginary, and let $e_2 \equiv E_2$, $n_1 \equiv e_1 - 1$, $n_2 \equiv e_2 - 1$, $n_3 \equiv n - n_1 - n_2$. There are exactly n_i units in a fundamental system of F_i $(i = 1, 2)$.

We shall show that $\mathfrak{A}$ is the direct sum of three invariant sub-rings $\mathfrak{A}_i$, of order n_i; also that $\mathfrak{K}$ has a basis $\xi_1, \xi_2, \cdots, \xi_n$ such that n_1 of the ξ's belong to $\mathfrak{A}_1$ and n_2 belong to $\mathfrak{A}_2$. It is then shown that in the corresponding system of units, n_1 form a fundamental system in F_1 and n_2 form such a system in F_2.

Let

$$f_1(x) \equiv (x^{e_1} - 1)/(x - 1); \quad f_2(x) \equiv (x^{e_2} - 1)/(x - 1).$$

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* MacDuffee, *loc. cit.*, Theorem 3, p. 74.

† *Journal für Mathematik*, vol. 28 (1844), p. 315.

The zeros of these f_i are zeroes of $f(x)$. Furthermore, they are distinct since e_1 and e_2 are relatively prime. Hence $f_3(x) \equiv f(x)/f_1(x)f_2(x)$ is a polynomial with rational integral coefficients. Let $\phi_i(x) = f(x)/f_i(x)$ $(i = 1, 2, 3)$ and let

$$I_1 \equiv e_2^{-1}\phi_1(A), \quad I_2 \equiv e_1^{-1}\phi_2(A), \quad I_3 \equiv I - I_1 - I_2.$$

It may be shown that $I_i^2 = I_i$, $I_iI_j = 0$ $(i \neq j)$, $I_if_i(A) = 0$. It may also be shown that $I_i, I_iA, \cdots, I_iA^{n_i-1}$ are linearly independent and form a basis of the ring $\mathfrak{A}_i \equiv \mathfrak{A}I_i$. Each I_i is the modulus of the corresponding $\mathfrak{A}_i$ and $\mathfrak{A}$ is the direct sum of the three $\mathfrak{A}_i$.

Let $\mathfrak{G}'$ be the ring consisting of all elements in the form $\psi_1 + \psi_2 + \psi_3$ where each ψ_i is a linear homogeneous function in $I_i, I_iA, \cdots, I_iA^{n_i-1}$ with rational integral coefficients. $\mathfrak{G}'$ is a pure infinite group with respect to addition and the above $n_1 + n_2 + n_3 = n$ elements form a basis. $\mathfrak{R}$ is a sub-group of $\mathfrak{G}'$ and hence by a well-known theorem, it has a basis in the form

$$\begin{aligned}
&\xi_i = \sum_{j=1}^{i} t_{ij} I_1 A^{j-1} && (i = 1, 2, \cdots, n_1),\\
(7) \qquad &\xi_i = \xi'_i + \sum_{j=n_1+1}^{i} t_{ij} I_2 A^{j-(n_1+1)} && (i = n_1 + 1, \cdots, n_1 + n_2),\\
&\xi_i = \xi'_i + \xi''_i + \sum_{j=n_1+n_2+1}^{i} t_{ij} I_3 A^{j-(n_1+n_2+1)} && (i = n_1 + n_2 + 1, \cdots, n),
\end{aligned}$$

where the t's are rational integers, the ξ'_i are linear homogeneous functions, with rational integral coefficients, in $I_1, I_1A, \cdots, I_1A^{n_1-1}$, and the ξ_i'' are similar functions in $I_2, I_2A, \cdots, I_2A^{n_2-1}$. $\mathfrak{R}$ is non-singular and hence the ξ's are linearly independent.* It follows that every $t_{ii} \neq 0$ and $\xi_1, \xi_2, \cdots, \xi_{n_1}$ form a basis of those elements of $\mathfrak{R}$ in $\mathfrak{A}_1$. e_2I_1 and e_1I_2 belong to $\mathfrak{G}$. Therefore

$$\left.\begin{aligned}
&e_2I_1\xi_i = e_2\xi'_i\\
&e_1(I - I_2)\xi_i = e_1(I_1 + I_3)\xi_i = e_1\xi'_i
\end{aligned}\right| \quad (i = n_1 + 1, \cdots, n_1 + n_2)$$

belong to $\mathfrak{R}$. Since e_1 and e_2 are relatively prime, it follows that each of the above ξ'_i belongs to $\mathfrak{R}$. We may then assume that if $i \leqq n_1 + n_2$, $\xi'_i = 0$; or that $\xi_{n_1+1}, \xi_{n_1+2}, \cdots, \xi_{n_1+n_2}$ belong to $\mathfrak{A}_2$. It follows that for the basis of $\mathfrak{R}$ given in (7), the matrix B, in (6_2) is in the form

$$B = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

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* MacDuffee, *loc. cit.*

where each 0 represents a rectangular block of zeroes and each B_{ij} is a matrix with n_i rows and n_j columns. By (6_2) and the form of B, we have

$$(8) \qquad A^k \xi_i = \sum_{j=1}^{n_1} \beta_{ij}^{(k)} \xi_j \qquad (i, k = 1, 2, \cdots, n_1),$$

it being understood that $(\beta_{ij}^{(k)}) \equiv B_{11}^k$. From $I_1 f_1(A) = 0$ and (8) we have

$$0 = I_1 f_1(A) \xi_i = f_1(A) \xi_i = \sum_{j=1}^{n_1} (\delta_{ij} + \beta_{ij} + \beta_{ij}^{(2)} + \cdots + \beta_{ij}^{(n_1)}) \xi_j \qquad (i = 1, 2, \cdots, n_1).$$

Since the ξ's are linearly independent, every expression on the right is zero. Hence, if I'_1 and I'_2 are the identity matrices of order n_1 and n_2 respectively, we have $B_{11}^{n_1} + B_{11}^{n_1-1} + \cdots + B_{11} + I'_1 = 0$, $B_{11}^{e_1} = I'_1$. Similarly, $B_{22}^{e_2} = I'_2$.

Consider the fundamental system of units ζ_i, corresponding to the basis (7) of $\mathfrak{R}$. From $B_{11}^{e_1} = I'_1$ and (6_1) it follows that

$$\zeta_i^{(e_1)} = (-1)^{t_i} \zeta_i \qquad (i = 1, 2, \cdots, n_1),$$

where the t's are certain integers. It may be shown, as in § 2, that the t's are even. Hence the first n_1 of the ζ's belong to F_1. Similarly, the next n_2 belong to F_2. Since the n ζ's form a fundamental system in F, it follows that the first n_1 and the next n_2 of them form such systems in F_1 and F_2 respectively. We have therefore

THEOREM 3. *Let F be an algebraic field which is cyclic with respect to the rational field and let F be obtained by composition of two fields,* F_1 and F_2, *the degrees of which are relatively prime. Then F contains a fundamental system of units such that certain of them form such a system in F_1 and certain others form such a system in F_2.*

Every cyclic field is obtained by composition of irreducible cyclic fields, the degrees of which are powers of distinct primes. By repeated application of the above theorem, it follows that there is a fundamental system of units in F which contains a fundamental system in each of the above mentioned irreducible fields.

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NORMAL DIVISION ALGEBRAS OF DEGREE 4 OVER *F* OF CHARACTERISTIC 2.[*]

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1. *Introduction.* I have determined[†] all normal division algebras of degree 4 over any non-modular field F and it is evident that the determination is valid when F is any infinite[‡] modular field of characteristic $p \neq 2$. There remains the case $p = 2$.

In the non-modular case there are but two types of algebras, the cyclic algebras and a *second type* containing algebras which may or may not be cyclic. I have proved[§] the existence of both primary and non-primary *non-cyclic* algebras over a non-modular F.

In the present paper I shall assume that F is any infinite field of characteristic $p = 2$ and shall determine all normal division algebras D of degree 4 (order 16) over F. As in the non-modular case the above two types of algebras appear. However I shall prove that every non-primary D is cyclic. I shall also prove that a necessary and sufficient condition that D be cyclic is that D contain a quantity t not in F such that t^2 is in F.

Finally I shall give a construction of the non-cyclic *type* of algebra D and shall prove that D is a non-cyclic primary normal division algebra if and only if a certain quadratic form in nine variables (and with coefficients determined by the multiplication table of D) is not a zero form.

2. *Presupposed results.* Let F be any infinite field and let D be a normal division algebra of order m over F. Then it is well known that $m = n^2$, D has rank (or degree) n, and every sub-field of D has degree a divisor of n. In particular D has sub-fields of degree n generated by a root x of an irreducible equation of degree n over F. Moreover such fields are maximal so that if $dx = xd$ for d in D then d is in $F(x)$.

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[*] Presented to the American Mathematical Society, December 1, 1933. Received in October, 1933.

[†] Cf. *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 703-706. I refer to this paper as A.

[‡] It is well known that the only division algebras over finite fields P are finite fields and hence that there exist no normal division algebras of degree $n > 1$ over P.

[§] See my paper in the *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 449-456 for non-primary algebras and in the *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 112-121, for primary algebras. A normal division algebra D is said to be primary if it is not expressible as a direct product of normal division algebras of lower degree.

A field $F(x)$ is called separable if the minimum equation of x, which has coefficients in F and is irreducible in F, has no multiple roots. If F has characteristic $p = 2$ then $F(x)$ is inseparable if and only if the equation has even powers only. In particular a quadratic field

$$F(x), \qquad x^2 = \gamma \text{ in } F, \tag{1}$$

is inseparable. But if $x^2 = \gamma x + \delta$, $\gamma \neq 0$ then $F(x)$ is separable and, if we write $u = \gamma^{-1}x$ then

$$u^2 = u + \alpha, \qquad \alpha \text{ in } F. \tag{2}$$

By applying certain results of Artin-Schreier I have proved *

THEOREM 1. *Every cyclic field of degree four over F of characteristic 2 is generated by a root of*

$$x^2 = x + (\alpha u + \beta), \qquad u^2 = u + \alpha, \tag{3}$$

where α, β are in F, $F(u)$ is a separable quadratic field. Conversely if x satisfies (3) then $F(x)$ is cyclic of degree four over F and contains $F(u)$.

THEOREM † 2. *Let $u^2 = u + \alpha$, $v^2 = v + \beta$. Then every non-cyclic normal quartic field K over F is generated by u and v, $K = F(u, v)$, such that neither $\alpha, \beta, \alpha + \beta$ has the form*

$$\delta(\delta + 1) \tag{4}$$

for any δ of F. Conversely if the above condition on $\alpha, \beta, \alpha + \beta$ is satisfied then $F(u, v)$ is a normal non-cyclic quartic field.

I have also proved

THEOREM ‡ 3. *Let D be a normal division algebra of degree four over any infinite field F and let $F(u)$ be a quadratic sub-field of D. Then the algebra of all quantities of D commutative with u is a normal division algebra of degree two over $F(u)$.*

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* In a paper "On non-primitive fields of degree p^2 over P of characteristic p," which has been offered for publication in the *Annals of Mathematics.*

† We assume henceforth that F has characteristic 2. Then, if a and b are commutative, $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2$, $-1 = 1$. We use these properties repeatedly without mention.

‡ In a paper, "On normal simple algebras," *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 620-625, for F non-modular. The proof holds for any infinite field.

THEOREM[*] 4. *Every D of degree 4 over F contains a separable quartic sub-field.*

THEOREM 5. *Let $\phi(\omega) = 0$ be the minimum equation of x in D. Then if y is in D and $\phi(y) = 0$ there exists a t in D such that $y = txt^{-1}$.*

We may now apply Theorem 4. By precisely the proof for the non-modular case in A we have immediately

THEOREM 6. *Every normal division algebra of degree four over an infinite field F contains a quadratic sub-field.*

We do not know, however, whether or not the above field is separable or inseparable.

3. *The existence of a separable quadratic sub-field of D.* Let $K = F(u)$ where K is quadratic over F. By Theorem 3 algebra D contains a normal division sub-algebra Q of degree 2 over K and it is known[†] that

$$Q = (1, i, j, ij), \quad i^2 = i + \lambda,\ j^2 = \mu,\ ji = (i + 1)j, \tag{5}$$

where λ and μ are in K. Suppose now that $u^2 = \sigma$ in F so that K is inseparable. Then we may prove

THEOREM 7. *The field $K(i)$ contains a separable quadratic sub-field.*

For let $\lambda = \lambda_1 + \lambda_2 u$, $v = 1 + \lambda_2 u + i$ where λ_1 and λ_2 are in F. Then

$$\begin{aligned} v^2 + v &= 1 + \lambda_2^2 u^2 + i^2 + 1 + \lambda_2 u + i \\ &= \lambda_2^2 \sigma + i + \lambda_1 + \lambda_2 u + \lambda_2 u + i = \lambda_1^2 + \lambda_2^2 \sigma = \rho \end{aligned}$$

in F. Since v is evidently not in F, the equation $v^2 + v = \rho$ defines a separable quadratic sub-field of $K(i)$.

We apply Theorem 6 to obtain

THEOREM 8. *Every D of degree 4 over F of characteristic 2 contains a separable quadratic sub-field.*

4. *On cyclic algebras.* Let K be an infinite field of characteristic two and let Q be a normal division algebra of degree two over K. As before Q is given by (5). Let ξ be a scalar root of $\omega^2 = \omega + \lambda$ so that the matric algebra with basis

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[*] Theorems 4 and 5 are very well known.

[†] A trivial consequence of the general case of Theorem 4 and of the fact that every separable quadratic field is cyclic.

$$(6)\qquad I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad i_0=\begin{pmatrix}\xi&0\\0&\xi+1\end{pmatrix},\quad j_0=\begin{pmatrix}0&\mu\\1&0\end{pmatrix},\quad i_0j_0=\begin{pmatrix}0&\mu\xi\\\xi+1&0\end{pmatrix}$$

is equivalent to Q. If $x=\alpha_1+\alpha_2 i+\alpha_3 j+\alpha_4 ij$ then x corresponds to

$$(7)\qquad x_0=\begin{pmatrix}\alpha_1+\alpha_2\xi & \mu(\alpha_3+\alpha_4\xi)\\ \alpha_3+\alpha_4(\xi+1) & \alpha_1+\alpha_2(\xi+1)\end{pmatrix},$$

whose characteristic equation,

$$(8)\qquad \phi(\omega;\alpha_1,\alpha_2,\alpha_3,\alpha_4)$$
$$\equiv\omega^2-\omega\alpha_2+[\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2\lambda-(\alpha_3^2+\alpha_3\alpha_4+\alpha_4^2\lambda)\mu]=0,$$

is the rank equation of Q.

Suppose that $K=F(u)$, $u^2=u+\alpha$, so that K is a separable quadratic extension of F. Let μ be in F, $\lambda=\lambda_1+\lambda_2 u$ with λ_1,λ_2 in F. Take

$$(9)\qquad \alpha_1=0,\ \alpha_2=1,\ \alpha_3=\pi,\ \alpha_4=u;\qquad x=i+\pi j+uij,$$

where

$$(10)\qquad \pi=\mu^{-1}(\lambda_2-\alpha)-(\lambda_1+\lambda_2+\lambda_2\alpha)$$

is in F. Then (8) becomes

$$(11)\quad \omega^2-\omega+\lambda_1+\lambda_2 u-\mu[\pi^2+(\lambda_1+\lambda_2)\alpha+u(\pi+\lambda_1+\lambda_2+\lambda_2\alpha)]=0$$

since $\quad \alpha_1^2+\alpha_1\alpha_2=0,\ \alpha_2^2\lambda=\lambda_1+\lambda_2 u=\lambda,\ \alpha_3^2=\pi^2,\ \alpha_3\alpha_4=\pi u,$

and $\quad \alpha_4^2\lambda=u^2(\lambda_1+\lambda_2 u)=(u+\alpha)(\lambda_1+\lambda_2 u)$
$$=\lambda_2\alpha+\lambda_1\alpha+\lambda_2 u+\lambda_1 u+\lambda_2\alpha u.$$

But by (10) we have

$$-\mu(\pi+\lambda_1+\lambda_2+\lambda_2\alpha)=-\lambda_2+\alpha,$$

so that (11) becomes *

$$(12)\qquad \omega^2-\omega+\alpha u+\delta=0,\qquad \omega^2=\omega+(\alpha u+\delta)$$

where δ is in F. By Theorem 1 the field $F(x)$ defined by a root x of (12) is a cyclic quartic field over F. We have

LEMMA 1. *If K is a separable quadratic extension of F and if $j^2=\mu$ in F then Q contains a cyclic quartic field over F.*

We next prove

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* Since $-1=1$, $-(\alpha u+\delta)=\alpha u+\delta$.

LEMMA 2. *Let Q contain y not in K such that $y^2 = \mu_0$ in K, $K = F(u)$, $u^2 = u + \alpha$. Then Q has a basis*

$$(1, k, y, ky), \quad k^2 = k + \nu, \quad y^2 = \mu_0, \quad yk = (k + 1)y. \tag{13}$$

We take the basis (5) for Q. Evidently $\mu_0 \neq \eta^2$ for any η of K since Q is a division algebra. Write

$$y = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 ij$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are in F. By (8) we have $\alpha_2 = 0$. Then

$$yi = \alpha_1 i + \alpha_3 (i + 1) j + \alpha_4 (i + 1) ij,$$
$$(i + 1) y = (i + 1)(\alpha_1 + \alpha_3 j + \alpha_4 ij) = yi + \alpha_1.$$

We let

$$\lambda_1 = (1 + \alpha_1^2 \mu_0^{-1})^{-1}, \qquad \lambda_2 = \alpha_1 \mu_0^{-1} \lambda_1$$

so that

$$\lambda_2 \mu_0 + \alpha_1 \lambda_1 = 0, \qquad 1 + \lambda_1 + \alpha_1 \lambda_2 = 0$$

and write

$$k = (\lambda_1 + \lambda_2 y) i.$$

Then

$$\begin{aligned} yk &= (\lambda_1 + \lambda_2 y) yi = (\lambda_1 + \lambda_2 y)[(i + 1) y - \alpha_1] \\ &= (k + 1) y - y + (\lambda_1 + \lambda_2 y) y - \alpha_1 (\lambda_1 + \lambda_2 y) \\ &= (k + 1) y + (\lambda_1 - \alpha_1 \lambda_2 - 1) y + (\lambda_2 \mu_0 - \alpha_1 \lambda_1) = (k + 1) y \end{aligned}$$

as desired. Moreover $k^2 = k + \nu$, ν in F. For evidently k is not in F so that $k^2 = \xi_1 k + \xi_2$, ξ_1 and ξ_2 in F. Then

$$\begin{aligned} yk^2 &= (k + 1)^2 y = (k^2 + 1) y \\ &= (\xi_1 k + \xi_2 + 1) y = y(\xi_1 k + \xi_2) = [\xi_1 (k + 1) + \xi_2] y. \end{aligned}$$

Now $\xi_1 k + \xi_2 + 1 = \xi_1 k + \xi_1 + \xi_2$, so that $\xi_1 = 1$ as desired. Finally it is evident that $Q = (1, k, y, ky)$.

We are now in a position to prove

THEOREM 9. *A normal division algebra D of degree 4 over F of characteristic 2 is cyclic if and only if D contains an inseparable quadratic sub-field.*

For it is well known that if D is cyclic it contains a quartic field $F(y)$, $y^4 = \gamma$ in F. But then $F(y^2)$ is an inseparable quadratic sub-field of D.

Conversely let D contain j not in F such that $j^2 = \mu$ in F. By Theorem 7 algebra D contains a separable quadratic sub-field $F(u)$, $u^2 = u + \alpha$ such that u is commutative with j. By Theorem 3 the algebra of all quantities of D

commutative with u is a normal division algebra Q of degree 2 over $F(u)$. Evidently j is in Q. By Lemma 2 algebra Q has a basis (5) with $j^2 = \mu$ in F. By Lemma 1 algebra Q contains a cyclic quartic field $F(x)$. Hence $D > Q > F(x)$ is cyclic.

We have immediately.

THEOREM 10. *Every non-primary normal division algebra of degree two over F of characteristic 2 is cyclic.*

For if D is not primary it has a sub-algebra Q of degree 2 over F. But then $Q > F(j)$, $j^2 = \mu$ in F and D is cyclic by Theorem 9. We also have

THEOREM 11. *Let D be a normal simple algebra of degree four over F of characteristic 2. Then if D contains no t not in F such that t^2 is in F the algebra D is a primary non-cyclic division algebra.*

For evidently D is not a cyclic division algebra and hence is either as desired or is not a division algebra. But in the latter case $D = M \times Q$ where M is a total matric algebra of degree two and Q is normal simple. Moreover if

$$e_{11},\ e_{12},\ e_{21},\ e_{22}$$

is an ordinary total matric basis of M the quantity $t = \lambda e_{12} + e_{21}$ has the property $t^2 = \lambda I$ where I is the modulus of D, a contradiction.

5. *The determination.* Let D be a normal division algebra of degree four over F of characteristic two. Then either D is cyclic or D contains no inseparable quadratic sub-field. Let D be non-cyclic.

By Theorem 6 algebra D contains a separable quadratic sub-field $F(u)$, $u^2 = u + \alpha$. Since also $(u+1)^2 = (u+1) + \alpha$ there exists a quantity j_2 in D such that $j_2 u = (u+1) j_2$ by Theorem 5. If $j_2{}^2$ is in F then D is cyclic, a contradiction. But $j_2{}^2 = g_2$, $g_2 u = u g_2$ and g_2 is in the algebra Σ of all quantities of D commutative with u. Evidently $F(g_2) < F(j_2)$ since j_2 is not in Σ. Hence $F(g_2)$ is a separable quadratic field, $g_2{}^2 = \sigma(g_2 + \sigma\beta)$ and, if $v = \sigma^{-1} g_2$ then $v^2 = v + \beta$, $g_2 = \sigma v$. The algebra $F(u, v)$ is then a quartic sub-field of D with G_4 group and we have proved

LEMMA. *Every D of degree 4 is either cyclic or a crossed product defined by a quartic field $F(u, v)$.*

It remains to prove every cyclic algebra is also of the second type. Let then D be cyclic so that D has a sub algebra

$$Q = (1, x, v, xv), \quad v^2 = \sigma, \quad x^2 = x + a, \quad vx = (x+1)v$$

where σ is in F, $a = \alpha u + \beta$ with α and β in F and Q is a normal division algebra of degree two over $F(u)$, a separable quadratic field with $u^2 = u + \alpha$. In particular $F(x)$ is the cyclic quartic field defining D and it is sufficient to prove that Q contains a separable quadratic field $F(w)$, $w^2 = w + \eta$, η in F where w is not in $F(u)$. For then D is the crossed product defined by $F(u, w)$. We take

$$w = x + \sigma^{-1}[\alpha\sigma + a + (u + 1)x]v$$

and utilize (8). Evidently

$$\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = \sigma^{-1}(\alpha\sigma + a),\ \alpha_4 = \sigma^{-1}(u + 1),\ \lambda = a,\ \mu = \sigma,$$

so that

$$\begin{aligned} w^2 &= w - a + \sigma[\sigma^{-2}(\alpha\sigma + a)^2 + \sigma^{-2}(\alpha\sigma + a)(u + 1) + \sigma^{-2}(u + 1)^2 a] \\ &= w + \eta \end{aligned}$$

where we wish to prove that η is in F. But,

$$\eta = \sigma^{-1}[\alpha^2\sigma^2 + a^2 + \alpha\sigma + \alpha\sigma u + au + a + (u + 1 + \alpha)a] + \alpha u + \beta.$$

However

$$a^2 + \alpha a = \alpha^2(u + \alpha) + \beta^2 + \alpha^2 u + \alpha\beta = \alpha^3 + \beta^2 + \alpha\beta = \nu \text{ is in } F,$$
$$au + a + (u + 1)a = 0,\ \eta = \sigma^{-1}\alpha\sigma u + \sigma^{-1}\nu + \alpha u + \beta = \sigma^{-1}\nu + \beta \text{ is in } F$$

where ν is in F. We have proved

THEOREM 12. *Every normal division algebra of degree four over F of characteristic two contains a non-cyclic normal quartic field $F(u, v)$ and hence is a crossed product.*[*]

6. *The structure of non-cyclic algebras.* Let D be a normal simple algebra of degree four over F of characteristic 2, and let D contain a quartic field $F(u, v)$ where

$$u^2 = u + \alpha, \qquad v^2 = v + \beta \tag{14}$$

with α and β in F and such that neither α, β nor $\alpha + \beta$ has the form $\delta^2 + \delta$ for any δ in F. The only quadratic sub-field of $F(u, v)$ distinct from $F(u)$,

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[*] If D is a normal division algebra of degree n over F and D contains a normal (Galois) field of degree n then D has known structure, satisfies the postulates for the algebras of L. E. Dickson (*Algebren*, p. 51), is what I have called an algebra of type R, is what E. Noether has called a crossed product.

$F(v)$ is the field $F(u+v)$, $(u+v)^2 = (u+v)+\alpha+\beta$. Moreover $F(u,v) = F(i)$ where the minimum equation of

$$i = i(u,v) \tag{15}$$

with respect to F is a quartic with G_4 group and roots

$$\begin{gathered} i, \quad \theta_1(i) = i(u, v+1), \quad \theta_2(i) = i(u+1, v), \\ \theta_3(i) = \theta_1[\theta_2(i)] = \theta_2[\theta_1(i)] = i(u+1, v+1). \end{gathered} \tag{16}$$

By Theorem[*] 5, D contains quantities j_1, j_2, $j_3 = j_1 j_2$ such that $j_1 i = \theta_1(i) j_1$, $j_2 i = \theta_2(i) j_2$, $j_3 i = \theta_3(i) j_3$ and

$$j_1^2 = g_1, \quad j_2^2 = g_2, \quad j_3^2 = g_3 \tag{17}$$

where g_1, g_2, g_3 are in $F(i)$. Evidently D has a basis

$$j_i, \quad u j_i, \quad v j_i, \quad u v j_i \qquad (i = 0, 1, 2, 3;\ j_0 = 1). \tag{18}$$

Moreover D has a sub-algebra Σ of order eight over F, order four and degree two over $F(u)$ and in fact Σ is the normal simple algebra

$$\Sigma = (1, v, j_1, v j_1), \quad j_1 v = (v+1) j_1, \quad v^2 = v + \beta, \quad j_1^2 = g, \tag{19}$$

over $F(u)$. Every quantity of D has the form

$$X_1 + X_2 j_2 \tag{20}$$

where X_1 and X_2 range independently over all quantities of Σ and the multiplication table of D is completely defined by the fact that

$$j_2 u = (u+1) j_2, \quad j_2 v = v j_2, \quad j_2 j_1 = \Gamma j_1 j_2 \tag{21}$$

with Γ in $F(u,v)$.

Since j_1 is commutative with $j_1^2 = g_1$, j_2 with $j_2^2 = g_2$, j_3 with $j_3^2 = g_3$, we evidently have g_1 in $F(u)$, g_2 in $F(v)$, g_3 in $F(u+v)$. If $g_1 = \gamma_1 + \gamma_2 u$ and D is non-cyclic then necessarily

$$\begin{aligned} &g_1 = \gamma_2(\gamma_0 + u) = \gamma_2 u_0, \\ &u_0^2 = u^2 + \gamma_0^2 = u + \gamma_0 + \alpha + \gamma_0^2 - \gamma_0 = u_0 + \alpha_0, \quad \alpha_0 \text{ in } F. \end{aligned}$$

Hence without loss of generality we may write $g_1 = \rho u$, ρ in F. Similarly we may write $g_2 = \sigma v$, $g_3 = \tau w$. But here $w = u + v + \epsilon$, ϵ in F and we have proved

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[*] Theorem 5 holds if D is merely normal simple.

THEOREM 13. *If D is a non-cyclic algebra defined by* (14)-(21) *we may take*

$$(22)\qquad g_1 = \rho u, \quad g_2 = \sigma v, \quad g_3 = \tau w, \qquad (\rho, \sigma, \tau \text{ in } F),$$

where ϵ is in F,

$$(23)\qquad w = u + v + \epsilon, \quad w^2 = w + \gamma, \quad \gamma = \alpha + \beta + \epsilon^2.$$

Cecioni has proved that if we have taken D in the above form and if D is associative then

$$(24)\qquad \Gamma = \frac{g_3(\theta_1)}{g_1(\theta_1)g_2(\theta_1)} = \frac{g_3(\theta_1)}{g_1g_2(\theta_1)}, \quad g_3g_3(\theta_1) = g_1g_1(\theta_2)g_2g_2(\theta_1),$$

while conversely if (24) is satisfied then D is associative. Hence we have

THEOREM 14. *Algebra D of Theorem* 13 *is associative if and only if*

$$(25)\qquad \Gamma = \frac{\tau(w+1)}{\rho u\sigma(v+1)} = [\tau(\rho\sigma\alpha\beta)^{-1}]v(u+1)(w+1),$$

and

$$(26)\qquad \tau^2\gamma = \rho^2\alpha\sigma^2\beta.$$

Consider now a quantity bj_1 of Σ where

$$b = \beta_1 + \beta_2 u + (\beta_3 + \beta_4 u)v = b_1 b_2 v.$$

We then have $(bj_1)^2 = f_1 + f_2 u$ where f_1 is in F and we shall compute f_2 in F. Evidently

$$\begin{aligned}(bj_1)^2 &= [(b_1 + b_2 v)j_1]^2\\ &= (b_1 + b_2 v)[b_1 + b_2(v+1)]g_1 = (b_1^2 + b_2^2\beta + b_1b_2)g_1\\ &= [\beta_1^2 + \beta_2^2(u+\alpha) + \beta_1\beta_3 + \beta_2\beta_4\alpha\\ &\qquad + (\beta_1\beta_4 + \beta_2\beta_3 + \beta_2\beta_4)u + \beta(\beta_3^2 + \beta_4^2\alpha) + \beta\beta_4^2u]\rho u\end{aligned}$$

so that since $u^2 = u + \alpha$,

$$\begin{aligned}(27)\qquad f_2 = \rho[&\beta_1^2 + \beta_2^2 + \beta_2^2\alpha + \beta_1\beta_3 + \beta_2\beta_4\\ &+ \beta_1\beta_4 + \beta_2\beta_3 + \beta_2\beta_4 + \beta(\beta_3^2 + \beta_4^2 + \beta_4^2\alpha)].\end{aligned}$$

Similarly if

$$c = \gamma_1 + \gamma_2 v + (\gamma_3 + \gamma_4 v)u$$

then $(cj_2)^2 = f_3 + f_4 v$ where

$$\begin{aligned}(28)\qquad f_4 = \sigma[&\gamma_1^2 + \gamma_2^2 + \gamma_2^2\beta + \gamma_1\gamma_3 + \gamma_2\gamma_4 + \gamma_1\gamma_4\\ &+ \gamma_2\gamma_3 + \gamma_2\gamma_4 + \alpha(\gamma_3^2 + \gamma_4^2 + \gamma_4^2\beta)].\end{aligned}$$

Finally if

$$a = (\delta_1 + \delta_2 w) + (\delta_3 + \delta_4 w) v \tag{29}$$

then $(dj_3)^2 = f_5 + f_6 w$ where

$$f_6 = \tau[\delta_1^2 + \delta_2^2 + \delta_2^2\gamma + \delta_1\delta_3 + \delta_2\delta_4 + \delta_1\delta_4 + \delta_2\delta_3 + \delta_2\delta_4 + \beta(\delta_3^2 + \delta_4^2 + \delta_4^2\gamma)]. \tag{30}$$

We shall utilize the above formulas in obtaining a necessary and sufficient condition on $\alpha, \beta, \gamma, \rho, \sigma, \tau$ that D shall be a non-cyclic normal division algebra.

We have proved that D is non-cyclic if and only if it contains no quantity t not in F but such that t^2 is in F. Let then

$$t = X_1 + X_2 j_2, \quad t^2 = \lambda, \qquad (\lambda \text{ in } F,\ X_1 \text{ and } X_2 \text{ in } \Sigma), \tag{31}$$

where t is not in F. We first prove

LEMMA 3. *If t_1 is in Σ and not in F and $t_1^2 = \lambda$ in F then there exists a quantity t_2 not in Σ such that $t^2 = \lambda$.*

We let $t_2 = (1 + j_2) t_1 (1 + j_2)^{-1}$ so that evidently $t_2^2 = t_1^2 = \lambda$. Also $(1 + j_2)^2 = 1 + g_2 = r \neq 0$ in $F(v)$, and hence $(1 + j_2)^{-1} = r^{-1}(1 + j_2)$,

$$t_2 = (t_0 + t'_0 j_2)(1 + j_2) = (t_0 + t'_0 g_2) + (t_0 + t'_0) j_2 \tag{32}$$

where

$$t_0 = t_1 r^{-1}, \quad t'_0 = j_2 t_0 j_2^{-1} = j_2 t_1 j_2^{-1} r^{-1} = t'_1 r^{-1} \text{ in } \Sigma. \tag{33}$$

If t_2 is in Σ then $t_0 + t'_0 = 0$, $t'_1 r^{-1} + t_1 r^{-1} = (t'_1 + t_1) r^{-1} = 0$ so that $t_1 + t'_1 = 0$. But F has characteristic 2 so that $t_1 = t'_1 = j_2 t_1 j_2^{-1}$, $t_1 j_2 = j_2 t_1$. But the only quantities of Σ commutative with j_2 are quantities* of $F(v)$ a separable quadratic field which cannot contain t_1.

Hence if D is cyclic it contains a quantity $t = X_1 + X_2 j_2$ such that $X_2 \neq 0$, $t^2 = \lambda$ in F. But if we let $X_2 j_2 = j$ then

$$t^2 = (X_1^2 + j^2) + (X_1 X_2 + X_2 X'_1) j_2 = \lambda, \tag{34}$$

and $(X_1 X_2 + X_2 X'_1) = X_1 j + j X_1 = 0$ since t^2 is in Σ. Hence

$$j^2 = g \text{ in } \Sigma, \quad X_1^2 + g = \lambda, \quad X_1 j = j X_1. \tag{35}$$

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* For if t_1 is commutative with j_2 then, by the first paragraph of § 2, t_1 is in $F(j_2)$ and hence has the form $\lambda_1 + \lambda_2 v + (\lambda_3 + \lambda_4 v) j_2$. If t_1 is also in Σ we have $\lambda_3 = \lambda_4 = 0$, t_1 in $F(v)$.

Suppose first that g is not in F. Then $F(g)$ is a sub-field of Σ and j is not in Σ so that $F(g) < F(j)$. Hence $F(j)$ is a quartic field and $X_1 j = j X_1$ implies that X_1 is in $F(j)$. Since X_1 is in Σ we must have X_1 in $F(g)$.

Evidently $g^2 = \nu_1 g + \nu_2$. If $\nu_1 = 0$ then $X_1 = \delta_1 + \delta_2 g$

$$X_1^2 + g = \delta_1^2 + \delta_2^2 \nu_2 + g \neq \lambda \text{ in } F.$$

Hence $\nu_1 \neq 0$ and we may write

$$(36) \qquad y = \nu_1^{-1} g, \quad y^2 = y + \nu, \quad \nu = \nu_2 \nu_1^{-2} \text{ in } F.$$

But then $j = \nu_1 y = g$, $X_1 = \delta_1 + \delta_2 y$ and

$$(37) \qquad \delta_1^2 + \delta_2^2 (y + \nu) + \nu_1 y = \lambda \text{ in } F,$$

whence

$$(38) \qquad \nu_1 = \delta_2^2.$$

We now compute the quantity ν_1. We write $X_2 = c + d j_1$ so that $j = c j_2 + d j_3$ and $j^2 = g = (c j_2)^2 + (d j_3)^2 + e j_1$ where we shall not need the form of e, a quantity of $F(u, v)$. But we know that

$$(39) \qquad (c j_2)^2 + (d j_3)^2 = f_3 + f_4 v + f_5 + f_6 w = f$$

where f_3 and f_5 are in F and f_4 and f_6 are given respectively by (28), (30).

Hence $g^2 = f^2 + (e j_1)^2 + (fe + e\bar{f}) j_1$

$$= f^2 + (e j_1)^2 + (f + \bar{f})(g - f) = (f + \bar{f}) g + (e j_1)^2 - f\bar{f}$$

where

$$(40) \qquad \bar{f} = j_1 f j_1^{-1} = f_3 + f_4 (v + 1) + f_5 + f_6 (w + 1),$$

so that, since evidently $\nu_1 = f + f$, we have

$$(41) \qquad \nu_1 = f_4 + f_6 = \delta_2^2.$$

There remains the case where g is in F. We use the above computation which shows that then $f + e j_1$ is in F so that evidently $f = (f_3 + f_5) + f_4 + f_6 w$ is in F. But then $f_4 = f_6 = 0$ for values of the variables not all zero.

We now assume that the quadratic form $f_4 + f_6 - \xi^2$ is not a zero form. Then our above proof shows that D is a non-cyclic normal division algebra. For otherwise D contains a quantity t not in F such that $t^2 = \lambda$ in F. But then either g is in F whence $f_4 + f_6 - \xi^2 = 0$ for $\xi = 0$ and for either $f_6 \neq 0$ or $f_4 \neq 0$ or g is not in F and $f_4 + f_6 - \xi^2$ is a zero form contrary to hypothesis.

Conversely if D is a non-cyclic normal division algebra then $f_4 + f_6 - \xi^2$

is not a zero form. For otherwise let $f_4 + f_6 = \beta^2$ for values of the variables not all zero. If all the variables in f_4, f_6 are zero then $\beta^2 = 0$, $\beta = 0$, a contradiction. Hence $j = cj_2 + dj_3 \neq 0$, $j^2 = g = f + ej_1$, $g^2 = \nu_1 g + \nu_2$. If $\nu_1 \neq 0$ then we write $X_1 = \beta y$, $y = \nu_1^{-1} g$ and have

$$t = \beta y + j, \qquad t^2 = \beta^2(y + \nu) + \nu_1 y = \beta^2 \nu \text{ in } F,$$

since we have proved that $\nu_1 = f_4 + f_6 = \beta^2$. But then D is cyclic. If $\nu_1 = 0$ and g is not in F then $g^2 = \nu_2$ in F. If g is in F then $f_4 = f_6 = 0$ so that if $f_4 \neq 0$ then $(bj_2)^2 = f_3$ in F, $b \neq 0$, while if $f_6 \neq 0$ then $(cj_3)^2 = f_5$ in F. In any case D contains a t such that t is not in F, $t^2 = \lambda$ in F, contrary to our hypothesis that D is non-cyclic. We have proved

THEOREM 15. *Let D be the linear associative normal simple algebra defined as in Theorems 13, 14. Then D is a primary non-cyclic normal division algebra over F of characteristic two if and only if the quadratic form in the variables $\xi, \gamma_1, \cdots, \gamma_4, \delta_1, \cdots, \delta_4$ in F,*

$$f_4 + f_6 + \xi^2, \tag{42}$$

where f_4 and f_6 are given by (28), (30), *is not a zero form.*

We shall not attempt here to discuss the existence of $\rho, \sigma, \tau, \alpha, \beta, \gamma$ satisfying $\tau^2\gamma = \rho^2\alpha\sigma^2\beta$, $\gamma = \alpha + \beta + \epsilon^2$ and such that (42) is not a zero form. It seems likely however that a proof may be made by the methods of my previous papers for the non-modular case.

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THE REPRESENTATION OF STIRLING'S NUMBERS AND STIRLING'S POLYNOMIALS AS SUMS OF FACTORIALS.

By MORGAN WARD.

1. I give here a new representation of the Stirling numbers and the associated Stirling polynomials [*] as sums of factorials, and use the formulas to deduce various arithmetical and algebraic properties of the numbers. My fundamental formula for the Stirling polynomial [†] $\psi_{p-1}(x)$ reads as follows:

$$(3.31)\qquad \psi_{p-1}(x) = \frac{(-1)^{p-1}}{(p+1)!}\left[H_p^{p-1} - \frac{x+2}{p+2}H_p^{p-2} + \frac{(x+2)(x+3)}{(p+2)(p+3)}H_p^{p-3} - \cdots + (-1)^{p-1}\frac{(x+2)(x+3)\cdots(x+p)}{(p+2)(p+3)\cdots 2p}H_p^{0}\right].$$

The constants H_p^r appearing here are positive integers defined recursively by

$$(4.1)\qquad H^r_{p+1} = (2p+1-r)H_p^r + (p-r+1)H_p^{r-1},$$

with the initial values

$$(4.11)\qquad H_0^0 = 1,\quad H^0_{p+1} = 1\cdot 3\cdot 5\cdots(2p+1),\quad H^p_{p+1} = 1.$$

Nielsen [‡] has expressed the Stirling polynomial $\psi_{p-1}(x)$ in the form

$$\psi_{p-1}(x) = \sigma_{p-1,0}x^{p-1} + \sigma_{p-1,1}x^{p-2} + \cdots + \sigma_{p-1,p-2}x + \sigma_{p-1,p-1}.$$

Unfortunately, the numbers $\sigma_{p,r}$ are not integers, and the recursion formulas for them are very complicated, so that it is difficult both to ascertain their form,[§] and to obtain properties of the Stirling polynomial from such a representation. In contrast, the numbers H_p^r are integers of comparatively simple form, while (3.31) leads directly to interesting congruential properties of the Stirling polynomials and Stirling numbers.

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[*] We use here freely the notation and formulas for the Stirling numbers given by Nielsen in his well known *Handbuch der Theorie der Gammafunction* (Leipzig, 1906), Chapter V. We shall refer to this source as Nielsen, *Handbuch*, giving page reference. A recent paper by C. Tweedie, *Proceedings of the Edinburgh Mathematical Society*, vol. 37 (1918), pp. 2-25, contains many interesting new results on these numbers. Since this paper was in press, C. Jordan (*Tohoku Journal*, vol. 37 (1933), pp. 254-278) has given an expression for the Stirling number as a sum of factorials. See especially pp. 264-265 of his paper, where his numbers $\bar{C}_{mi}$ are my $H_m{}^i$.

[†] Nielsen, *Handbuch*, p. 72.

[‡] *Handbuch*, pp. 72-73. See also *Annali di Matematica, III*, vol. 10, pp. 309-316.

[§] Nielsen, *Annali di Matematica, III*, vol. 10, p. 313; Tweedie, paper cited, Section 11.

To give an example of such congruences, let P be any prime greater than $2p$, and r any positive integer. Then if $C_n{}^p$ denotes the Stirling number,

$$C^p{}_{n+1} \equiv 1 \pmod{P^r} \qquad \text{if } n+2 \equiv 0 \pmod{P^r},$$
$$C^p{}_{n+1} \equiv 2^{p+1} - 1 \pmod{P^r} \qquad \text{if } n+3 \equiv 0 \pmod{P^r}.$$

As a numerical illustration, take $p=3$, $P=7$, $r=1$. Then from Glaisher's table * of $C_m{}^p$, $C_5{}^3 = 225$, $C_4{}^3 = 50$, and

$$225 \equiv 1 \pmod 7, \qquad 50 \equiv 2^4 - 1 \pmod 7.$$

2. We begin with the Stirling numbers of the first kind defined by

$$x(x+1)\cdots(x+n-1) = C_n{}^0x^n + C_n{}^1x^{n-1} + \cdots + C_n{}^px^{n-p} + \cdots + C_n{}^{n-1}x.$$

We call n the rank of $C_n{}^p$ and p its order. We have the immediate relations

$$(2.1) \qquad C^p{}_{n+1} = C_n{}^p + nC_n{}^{p-1},$$

$$(2.2) \qquad C_n{}^0 = 1, \quad C_n{}^{n-1} = (n-1)\,!, \quad (n=1,2,\cdots;\ p=0,1,\cdots,n-1).$$

We now define $C_n{}^p$ for all integral values of n and p, positive or negative, by the recursion formula (2.1) with the initial values (2.2). Then it is readily shown that

$$(2.3) \qquad C_n{}^{n+r-1} = 0, \qquad (n=0,1,\cdots;\ r=1,2,\cdots).$$

Furthermore, if

$$F_p(z) = \sum_{n=0}^{\infty} C_n{}^pz^n$$

is the generating function for the Stirling numbers

$$C_0{}^p,\ C_1{}^p,\ C_2{}^p,\ \cdots$$

of fixed order p, then an easy induction shows us that

$$(2.4) \qquad F_p(z) = [z^{p+1}/(1-z)^{2p+1}]H_p(z), \qquad (p=0,1,2,\cdots)$$

where $H_p(z)$ is a polynomial in z of degree $p-1$ with positive integral coefficients, and, by convention, we take

$$(2.41) \qquad H_0(z) = 1.$$

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* *Quarterly Journal*, vol. 31 (1900), pp. 26-28. This Table extends as far as $n=20$. $C_n{}^p$ is denoted in Glaisher's notation by $S_p(1,2,\cdots,n-1)$.

The polynomials $H_p(z)$ appearing in (2.4) satisfy the recursion relation

$$(2.5)\qquad H_{p+1}(z) = (pz + p + 1)H_p(z) + (1 - z)z(d/dz)H_p(z)$$

which with (2.41) determines them completely.

3. We next put the polynomial $H_p(z)$ in the form

$$(3.1)\qquad H_p(z) = H_p{}^0 - H_p{}^1(1 - z) + H_p{}^2(1 - z)^2 - \cdots + (-1)^{p-1} H_p{}^{p-1}(1 - z)^{p-1}.$$

Before studying the constants $H_p{}^r$, we shall deduce our main formulas. On substituting (3.1) into (2.4) and then expanding in ascending powers of z, we find that

$$F_p(z) = z^{p+1} \sum_{r=0}^{p-1} \sum_{s=0}^{\infty} (-1)^r H_p{}^r \frac{(s+1)(s+2)\cdots(s+2p-r)}{1\cdot 2\cdot 3\cdots(2p-r)} z^s.$$

Therefore by comparing the coefficient of z^n on both sides of this expression, we find that

$$C_n{}^p = \sum_{r=0}^{p-1} (-1)^r H_p{}^r (n-p)(n+1-p)\cdots(n+p-r-1)/(2p-r)\,!.$$

On replacing n by $n+1$ in this expression and removing the common factor $(-1)^{p-1}(n+1)n\cdots(n+1-p)/(p+1)\,!$ from the right side, we obtain finally the formula

$$(3.2)\qquad C^p{}_{n+1} = \frac{(n+1)\,!\,(-1)^{p-1}}{(n-p)\,!\,(p+1)\,!} \times \Big[H_p{}^{p-1} - \frac{n+2}{p+2} H_p{}^{p-2} + \frac{(n+2)(n+3)}{(p+2)(p+3)} H_p{}^{p-3} - \cdots + (-1)^{p-1} \frac{(n+2)(n+3)\cdots(n+p)}{(p+2)(p+3)\cdots(2p)} H_p{}^0\Big].$$

Now *

$$C^p{}_{n+1} = [(n+1)\,!/(n-p)\,!\,]\,\psi_{p-1}(n)$$

where $\psi_{p-1}(x)$ is the Stirling polynomial of order $p-1$. Hence

$$\psi_{p-1}(n) = \frac{(-1)^{p-1}}{(p+1)\,!}\Big[H_p{}^{p-1} - \cdots + (-1)^{p-1}\frac{(n+2)\cdots(n+p)}{(p+2)\cdots 2p} H_p{}^0\Big].$$

Since this formula holds for all positive integral values of n, we deduce that

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* Nielsen, *Handbuch*, p. 14, formula (15).

$$(3.21)\quad \psi_{p-1}(x) = \frac{(-1)^{p-1}}{(p+1)!}\Big[H_p^{p-1} - \frac{x+2}{p+2}H_p^{p-2} + \frac{(x+2)(x+3)}{(p+2)(p+3)}H_p^{p-3} - \cdots + (-1)^{p-1}\frac{(x+2)(x+3)\cdots(x+p)}{(p+2)(p'+3)\cdots 2p}H_p^{0}\Big]$$

for all values of the variable x.

We may use this result to obtain a formula similar to (3.2) for the Stirling numbers $\mathfrak{S}_n^s$ of the second kind [*] defined by the expansion

$$1/x(x+1)\cdots(x+n-1) = \sum_{s=0}^{\infty}(-1)^s\,\mathfrak{S}_n^s/x^{n+s}.$$

For [†]

$$\mathfrak{S}_n^p = [(-1)^{p-1}(n+p-1)/(n-2)!]\,\psi_{p-1}(-n),$$

so that by (3.21),

$$(3.3)\quad \mathfrak{S}_n^p = \frac{(n+p-1)!}{(n-2)!\,(p+1)!}\Big[H_p^{p-1} + \frac{n-2}{p+2}H_p^{p-2} + \frac{(n-2)(n-3)}{(p+2)(p+3)}H_p^{p-3} + \cdots + \frac{(n-2)(n-3)\cdots(n-p)}{(p+2)(p+3)\cdots 2p}H_p^{0}\Big].$$

These formulas have immediate arithmetical consequences. For suppose that P denotes a fixed prime greater than $2p$, and r any positive integer. Then we deduce from (3.21) that

$$\psi_{p-1}(n) \equiv [(-1)^{p-1}/(p+1)!]H_p^{p-1} \pmod{P^r} \quad \text{if } n+2 \equiv 0 \pmod{P^r},$$
$$\psi_{p-1}(n) \equiv [(-1)^{p-1}/(p+1)!]\{H_p^{p-1} + [1/(p+2)]H_p^{p-2}\} \pmod{P^r} \quad \text{if } n+3 \equiv 0 \pmod{P^r},$$

and so on. There are analogous congruences for the Stirling numbers deducible from (3.2) and (3.3); namely,

$$(3.4)\quad \begin{aligned} C^p_{n+1} &\equiv H_p^{p-1} \pmod{P^r} && \text{if } n+2 \equiv 0 \pmod{P^r},\\ C^p_{n+1} &\equiv (p+2)H_p^{p-1} + H_p^{p-2} \pmod{P^r} && \text{if } n+3 \equiv 0 \pmod{P^r},\\ \mathfrak{S}_n^p &\equiv H_p^{p-1} \pmod{P^r} && \text{if } n-2 \equiv 0 \pmod{P^r},\\ \mathfrak{S}_n^p &\equiv (p+2)H_p^{p-1} + H_p^{p-2} \pmod{P^r} && \text{if } n-3 \equiv 0 \pmod{P^r}, \end{aligned}$$

and so on.

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[*] Nielsen, *Handbuch*, p. 68.

[†] Nielsen, *Handbuch*, p. 74.

We may note in passing an interesting consequence of the form of the generating function $F_p(z)$ given in (2.4). For if

$$\Delta C_n{}^p = C_n{}^p - C^p{}_{n-1} \quad \Sigma C_n{}^p = C_0{}^p + C_1{}^p + \cdots + C_n{}^p$$

denote the usual operations of the calculus of finite differences applied to the rank of the Stirling number $C_n{}^p$, the generating functions of the numbers $\Delta C_n{}^p$ and $\Sigma C_n{}^p$ are $(1-z)F_p(z)$ and $(1-z)^{-1}F_p(z)$ respectively. But with $H_p(z)$ in the form (3.1), each of these functions may be immediately expanded in ascending powers of z. We obtain in this manner the formulas

$$\begin{aligned} \Delta C^p{}_{n+1} &= (-1)^{p-1}\binom{n}{p} \\ &\quad \times \left[H_p{}^{p-1} - \frac{n+1}{p+1} H_p{}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p{}^{p-3} - \cdots \right], \\ (3.5) \qquad & \\ \Sigma C^p{}_{n+1} &= (-1)^{p-1}\binom{n+2}{p+2} \\ &\quad \times \left[H_p{}^{p-1} - \frac{n+3}{p+3} H_p{}^{p-2} + \frac{(n+3)(n+4)}{(p+3)(p+4)} H_p{}^{p-3} - \cdots \right], \end{aligned}$$

and it is easy to write down analogous formulas for the higher differences and summations of $C^p{}_{n+1}$. The method by which we obtained the congruences (3.4) yields then an unlimited number of congruences involving sums and differences of Stirling numbers of the same order.

If we compare (3.5) (i) with (2.1), we see that

$$\begin{aligned} nC_n{}^{p-1} &= (-1)^{p-1}\binom{n}{p} \\ &\quad \times \left[H_p{}^{p-1} - \frac{n+1}{p+1} H_p{}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p{}^{p-3} - \cdots \right]. \end{aligned}$$

On the other hand, if we put $n = n-1$, $p = p-1$ in (3.2), we find that

$$\begin{aligned} C_n{}^{p-1} &= (-1)^{p-2}\binom{n}{p} \\ &\quad \times \left[H_{p-1}^{p-2} - \frac{n+1}{p+1} H_{p-1}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_{p-1}^{p-3} - \cdots \right]. \end{aligned}$$

Therefore, for all integral values of n, we have the fundamental formula

$$\begin{aligned} &\Bigg[H_p{}^{p-1} - \frac{n+1}{p+1} H_p{}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_p{}^{p-3} \\ (3.6)\ &- \cdots + (-1)^{p-1} \frac{(n+1)(n+2)\cdots(n+p-2)}{(p+1)(p+2)\cdots(2p-2)} H_p{}^{p-1} \Bigg] = -n \\ &\Bigg[H_{p-1}^{p-2} - \frac{n+1}{p+1} H_{p-1}^{p-2} + \frac{(n+1)(n+2)}{(p+1)(p+2)} H_{p-1}^{p-3} \\ &\qquad - \cdots + (-1)^{p-2} \frac{(n+1)(n+2)\cdots(n+p-3)}{(p+1)(p+2)\cdots(2p-3)} H_{p-1}^{p-2} \Bigg]. \end{aligned}$$

We may if we please replace n here by a continuous variable x as in formula (3.21).

In particular, if we let $n = p$, we have

$$H_p^{p-1} - H_p^{p-2} + H_p^{p-3} - \cdots = -(H_{p-1}^{p-2} - H_{p-1}^{p-3} + H_{p-2}^{p-3} - \cdots).$$

From this result and the fact that $H_1^0 = 1$, we deduce that

$$(3.7) \qquad H_p^0 - H_p^1 + H_p^2 - \cdots + (-1)^{p-1} H_p^{p-1} = p\,!,$$

a formula which affords a convenient check when computing the numerical values of the integers H_p^r.

4. We now proceed with the study of the numbers H_p^r. If we assume that (3.1) holds for all positive integral values of p, we obtain by substituting in (2.5) and comparing the coefficients of the various powers of $1 - z$, the recursion relations

$$(4.1) \qquad \begin{array}{l} H^0_{p+1} = (2p+1)H_p^0, \quad H^p_{p+1} = H_p^{p-1} \quad \text{and} \\ H^r_{p+1} = (2p+1-r)H_p^r + (p-r+1)H_p^{r-1}. \end{array}$$

Since $H_0^0 = 1$, we deduce from the first two relations that

$$(4.12) \qquad H^0_{p+1} = 1 \cdot 3 \cdot 5 \cdots 2p+1, \quad H^p_{p+1} = 1, \qquad (p = 0, 1, \cdots).$$

The first few numbers H_p^r are given in the following table:*

p	$r=0$	1	2	3	4	5	6	7	8	9
1	1									
2	3	1								
3	15	10	1							
4	105	105	25	1						
5	945	1260	490	56	1					
6	10395	17325	9450	1918	119	1				
7	135135	270270	190575	56980	6825	246	1			
8	2027025	4729725	4099095	1636635	302995	22935	501	1		
9	34459425	91891800	94594500	47507460	12122110	1487200	74316	1012	1	
10	654729075	1964187225	2343240900	1422280360	466876410	81431350	6914908	235092	2035	1

Here the number in the p-th row and r-th column is H_p^r; thus $H_4^2 = 25$.

We next extend the definition of H_p^r to all integral values of p and r by (4.1) and (4.12). By a brief induction, we find that

$$(4.13) \qquad H_p^{-r} = 0 \quad (r \geqq 1;\ p = 0, 1, 2, \cdots)$$

$$(4.14) \qquad H_p^{p+r} = 0 \quad (r = 0,\ p = 1, 2, 3, \cdots;\ r \geqq 1,\ p = r, r+1, \cdots).$$

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* The table has been checked by the use of formula (3.7).

Now replace r by $p - r$ in (4. 1). We obtain

$$(4.2)\qquad H_{p+1}^{p-r} = (p + r + 1)H_p^{p-r} + (r + 1)H_p^{p-r-1}.$$

Let the generating function of the numbers

$$H_0^{-r}, H_1^{1-r}, H_2^{2-r}, \cdots, H_p^{p-r}, \cdots$$

be denoted by $\mathcal{H}_r(x)$, so that

$$(4.3)\qquad \mathcal{H}_r(x) = \sum_{p=0}^{\infty} H_p^{p-r}x^p = \sum_{p=r}^{\infty} H_p^{p-r}x^p$$

since by (4. 13), $H_0^{-r} = H_1^{1-r} = \cdots = H_{r-1}^{-1} = 0$.

On replacing r by $r + 1$ in (4. 3), changing the summation variable from p to $p + 1$, and reducing by (4. 2), we obtain the formula

$$(4.4)\quad (1 - (r + 1)x)\mathcal{H}_{r+1}(x) = (r + 1)x\mathcal{H}_r(x) + x^2(d/dx)\mathcal{H}_r(x), \qquad (r = 0, 1, 2, \cdots).$$

Since by (4. 14), $H_p^p = 0$ and $H_0^0 = 1$, we have

$$(4.41)\qquad \mathcal{H}_0(x) = 1.$$

These two formulas serve then to define the functions $\mathcal{H}_r(x)$ completely, and $\mathcal{H}_r(x)$ is seen to be a rational function of x. It is easy to determine its form. For by direct calculation from (4. 4), we find that

$$\mathcal{H}_1(x) = \frac{x}{1 - x}; \quad \mathcal{H}_2(x) = \frac{x^2[3 - 2x]}{(1 - x)^2(1 - 2x)},$$

$$(4.5)\quad \mathcal{H}_3(x) = \frac{x^3[15 - 45x + 40x^2 - 12x^3]}{(1 - x)^3(1 - 2x)^2(1 - 3x)},$$

$$\mathcal{H}_4(x) = \frac{x^4[105 - 840x + 2625x^2 - 4130x^3 + 3500x^4 - 1560x^5 + 288x^6]}{(1 - x)^4(1 - 2x)^3(1 - 3x)^2(1 - 4x)}.$$

We are therefore led to infer that the generating function $\mathcal{H}_r(x)$ is of the form

$$(4.51)\qquad \mathcal{H}_r(x) = x^r\Phi_r(x)/(1 - x)^r(1 - 2x)^{r-1}\cdots(1 - rx)$$

where $\Phi_r(x)$ is a polynomial in x with integral coefficients of degree $r(r - 1)/2$. The proof is a straightforward induction from (4. 41) and (4. 4) and will be omitted here.*

If we put the right-hand side of (4. 51) into partial fractions, we see that $\mathcal{H}_r(x)$ may be written as

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* The relationship between $\Phi_{r+1}(x)$ and $\Phi_r(x)$ deduced in the course of the induction is unfortunately too complicated to be of much service.

$$\mathcal{H}_r(x) = A_0 + \frac{B_1}{1-rx} + \frac{C_1}{1-(r-1)x} + \frac{C_2}{(1-(r-1)x)^2} + \cdots + \frac{U_1}{1-x} + \cdots + \frac{U_r}{(1-x)^r},$$

where the numbers $A_0, \cdots, U_r$ are all rational. If we now expand the right-hand side of this expression in ascending powers of x and collect the coefficient of x^p, we find that $H_p{}^{p-r}$ is of the form

$$H_p{}^{p-r} = b_0 r^p + (c_0 + c_1 p)(r-1)^p + \cdots + (u_0 + u_1 p + u_2 p^2 + \cdots + u_{r-1} p^{r-1})$$

where the numbers $b_0, \cdots, u_{r-1}$ are again all rational. We can however assert more than this. For if we apply the process just described to the expressions in (4.5), we find that*

$$\begin{aligned} & H_p{}^{p-1} = 1, \quad H_p{}^{p-2} = [2^{p+1} - (p+3)], \\ (4.6) \quad & H_p{}^{p-3} = \tfrac{1}{2}[3^{p+2} - (2p+10)2^{p+1} + (p^2 + 7p + 13)], \\ & H_p{}^{p-4} = \tfrac{1}{6}[4^{p+3} - (3p+21)3^{p+2} \\ & \qquad + (3p^2 + 33p + 96)2^{p+1} - (p^3 + 12p^2 + 50p + 73)]. \end{aligned}$$

We infer therefore that $H_p{}^{p-r}$ is actually of the form

$$(4.61) \qquad H_p{}^{p-r} = [1/(r-1)!] \sum_{l=0}^{r-1} (-1)^l \theta_l(p)(r-l)^{p+r-1-l}$$

where $\theta_l(p)$ is a polynomial in p of degree l with positive integral coefficients, and $\theta_0(p) = 1$. I cannot however prove this statement.†

5. We conclude by giving a method for calculating the polynomials $\theta_l(p)$ in (4.61) recursively. We begin by assuming that

$$(5.1) \qquad H_p{}^{p-(r+1)} = (1/r!) \sum_{l=0}^{r} (-1)^l \Theta_l(p)(r+1-l)^{p+r-l},$$

where $\Theta_l(p)$ is a polynomial of the same form as $\theta_l(p)$. On setting $p = p+1$ in (5.1), we find that

$$H_{p+1}^{p-r} = (1/r!) \sum_{l=0}^{r} (-1)^l \Theta_l(p+1)(r+1-l)^{p+r-1-l}.$$

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* All of these formulas have been checked numerically, and are believed to be correct. The two congruences mentioned in the introduction are obtained by substituting for $H_p{}^{p-1}$ and $H_p{}^{p-2}$ from (4.6) into (3.4).

† As additional support for it, I have found that

$$H_p{}^{p-5} = (1/24)[5^{p+4} - (4p+36)4^{p+3} + (6p^2 + 90p + 354)3^{p+1} - (4p^3 + 72p^2 + 452p + 992)2^p + (p^4 + 18p^3 + 125p^2 + 400p + 501)].$$

If we now substitute these expressions for H_{p+1}^{p-r}, H_p^{p-r}, $H_p^{p-(r+1)}$ into (4.2) and express the fact that the resulting expression must be an identity in p, we obtain the formula

$$(5.2) \qquad l\Theta_l(p+1) - (r+1)(\Theta_l(p+1) - \Theta_l(p)) = r(p+r+1)\theta_{l-1}(p), \qquad (r \geqq l \geqq 1),$$

which determines $\Theta_l(p)$ if $\theta_{l-1}(p)$ is known.

If we attempt to determine $\Theta_l(p)$ by writing it as a polynomial in p with undetermined coefficients, we find that we can express the coefficients only as determinants in the coefficients of $\theta_{l-1}(p)$. We therefore assume instead that $\theta_{l-1}(p)$ and $\Theta_l(p)$ are expressed as sums of factorials:

$$(5.3) \quad \begin{aligned} \theta_{l-1}(p) &= y_0 + y_1 p + y_2 p(p+1) + \cdots + y_{l-1} p(p+1)\cdots(p+l-2), \\ \Theta_l(p) &= x_0 + x_1 p + x_2 p(p+1) + \cdots + x_l p(p+1)\cdots(p+l-1), \end{aligned}$$

and seek to determine the x in terms of the y. Needless to say, the x and y are all integers, when and only when all the coefficients in the ordinary polynomial expressions for $\theta_{l-1}(p)$ and $\Theta_l(p)$ are integers.

If for convenience we set

$$x_{l+1} = y_{l+1} = y_l = y_{-1} = 0,$$

we obtain on substituting from (5.3) into (5.2) the difference relation

$$(5.4) \quad lx_s - (s+1)(r+1)x_{s+1} = r(y_{s-1} + (r-2s)y_s - (s+1)(r-s)y_{s+1}), \qquad (s = 0, 1, \cdots, l).$$

As a numerical verification of this formula, take $r = 3$ and $l = 2$ so that we have to do with H_p^{p-4} and H_p^{p-3}. From the formulas (4.6), we have $\Theta_2(p) = 3p^2 + 33p + 96$, $\theta_1(p) = 2p + 10$, so that $x_0 = 96$, $x_1 = 30$, $x_2 = 3$, $y_0 = 10$, $y_1 = 2$. The formula (5.6) with $s = 0, 1, 2$ then gives

$$2x_0 - 4x_1 = 3(3y_0 - 3y_1); \quad 2x_1 - 8x_2 = 3(y_0 + y_1); \quad 2x_2 = 3y_1;$$

or

$$192 - 120 = 3(30 - 6); \quad 60 - 24 = 3(10 + 2); \quad 6 = 6.$$

Since (5.4) is effectively a linear difference equation of the first order for x_s, the explicit form of x_s may be written down, but the general result is too complicated to be of interest.

CREMONA INVOLUTIONS DEFINED BY A PENCIL OF CUBIC SURFACES.

By Evelyn Carroll-Rusk.

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1. *Introduction.* In an earlier paper [*] a $(k, 1)$ correspondence between the surfaces of a pencil of surfaces of order n and the points of an $n-2$ fold line on each of the surfaces was discussed. If for $k=1$, $n=3$, the line $x_1=0$, $x_2=0$ is taken for d, the pencil of cubic surfaces is represented by

$$z_4F_3(x)-z_3F'_3(x)=0, \tag{1}$$

where $F_3(x)\equiv ux_1+vx_2$ and $F'_3(x)\equiv wx_1+tx_2$, (u, v, w, t being quadratic forms in x_1, x_2, x_3, x_4). If $(z)\equiv(0, 0, z_3, z_4)$ is a variable point on the line d, then a point (y) in space determines a cubic surface of the pencil $z_4F_3(y)-z_3F'_3(y)=0$. The residual point (y') in which the line joining (y) to (z) meets (1) again, has coördinates of the form $\rho x_1=\tau y_1$, $\rho x_2=\tau y_2$, $\rho x_3=\tau y_3+\sigma F_3(y)$, $\rho x_4=\tau y_4+\sigma F'_3(y)$, where

$$\begin{aligned}
\tau &\equiv F'_3(y)\,[y_1u(F)+y_2v(F)]-F_3(y)\,[y_1w(F)+y_2t(F)]\\
\sigma &\equiv -2\{F'_3(y)\,[y_1u(y,F)+y_2v(y,F)]-F_3(y)\,[y_1w(y,F)+y_2t(y,F)]\},
\end{aligned}$$

in which $u(F)$, $v(F)$, etc. are quadratic forms in $[0, 0, F_3(y), F'_3(y)]$, and $u(y,F)$, $v(y,F)$, etc. are the polar forms of $F\equiv[0, 0, F_3(y), F'_3(y)]$ with respect to $u(y)$, $v(y)$, etc., respectively. This involutorial transformation is of order eleven and the table of characteristics has the representation

$$\begin{array}{rcl}
S_1\sim S_{11} & : & d^4\,\bar{d}^2\,\gamma_8{}^3\,18g\\
d\sim \tau_{10} & : & d^4\,\bar{d}\ \ \gamma_8{}^3\,18g\\
\sigma_8\sim \sigma_8 & : & d^3\,\bar{d}\ \ \gamma_8{}^2\,18g\\
\gamma_8\sim \Gamma_{20} & : & d^8\,\bar{d}^4\,\gamma_8{}^5\,18g^2\\
J_{40} & : & \Gamma_{20}\,\tau_{10}{}^2
\end{array}$$

in which the $18g$ straight lines are parasitic, and d and γ_8 constitute the complete base of the pencil.

2. *Problem.* The problem considered in the present paper is essentially concerned with those cases in which the residual γ_8 is composite. Each component contributes a principal surface and a number of parasitic lines. The interest lies in the distribution of the parasitic lines, the reductions in the

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[*] E. T. Carroll, "Systems of involutorial birational transformations contained multiply in special linear line complexes," *American Journal of Mathematics*, vol. 54 (1932), pp. 707-717.

order of the transformation, and the contact of the various principal surfaces along the line d.

If γ_8 includes a conic, either proper or composite, which lies in a plane through d, the conic is a fundamental curve of the second kind. A point in this plane lying on neither d nor the conic determines a composite surface of the pencil, the plane consisting of a pencil of fundamental lines of the second kind with its vertex at the point Z associated with that surface. In case the conic is composite, each component is parasitic in a new sense, every point of each having itself and the other entire line for images. The order of the transformation is reduced by unity for each such conic, proper or composite, and the number of parasitic lines is reduced by three.

If a component of γ_8 is a proper or composite plane cubic curve γ_3 lying in a plane $u = 0$, then one surface of the pencil $u\phi = 0$ is composite, ϕ being a quadric form $vx_1 + wx_2$. If $u\phi = 0$ is associated with a point Z in the plane $u = 0$, every line of the pencil Z, u is parasitic and the order of the transformation is reduced by unity, while the number of parasitic lines is reduced by five. In every case the generator of $\phi = 0$ through Z is parasitic in the proper transformation. In general, if $|F|$ includes $k_m\phi_{3-m}$, k_m being a cone of order m with vertex at Z associated with the composite surface, the order of the transformation is reduced by m and the number of parasitic lines by $4 + m$, $m \geqq 1$.

A combination of several of these planes of either type or of both does not reduce the number of parasitic lines by the sum of the reductions computed for each plane individually. Thus, when γ_8 is composed of eight lines, four of them meet d, and the four others form a skew quadrilateral, no side of which meets d. The number of parasitic lines is four instead of two according to the formula $18 - 2 \times 3 - 2 \times 5$; two pairs of planes intersect in base lines and are counted twice.

In addition to the above cases two types appear in which the order of the transformation is lowered without γ_8 becoming composite. If $|F|$ includes a cubic cone with its vertex on d, the transformation is of order eleven unless the vertex is at the associated point of the cone. In the latter case every generator of the cone is parasitic and the order of the transformation is reduced to eight, the number of parasitic lines to eleven. If the pencil of cubic surfaces is determined by two cubic cones with vertices at their respective associated points, the order of the transformation is five and the number of parasitic lines is four.

3. *Method.* It is necessary to determine the genus of each component of γ_8 and the number of intersections of the components in pairs. By mapping

one cubic surface of the pencil on a plane by the usual six point method, the complete basis curve γ_9 may be represented by $c_9 : 1^3 2^3 3^3 \cdots 6^3$. The genus of c_9, and hence of γ_9, is ten.

If the base line d is represented by $d_1 : 12$ on the map, the residual γ_8 may be represented by $c_8 : 1^2 2^2 3^3 4^3 5^3 6^3$. Since the line 12 meets c_8 in four points apart from basis points, d meets γ_8 in four points. The genus of c_8, and likewise of γ_8, is seven. From a point on d there are eight bisecants apart from d, which counts for six.

There are twenty-seven lines on each cubic surface, one of which is d. The ten others meeting d are those having for images the base points 1 and 2; the lines 34, 35, 36, 45, 46, 56; and the conics 13456, 23456.

A point P in space determines the surface of the pencil to which it belongs. Each of the twenty-seven base lines on a cubic surface is met by ten others; if P is chosen on one of the ten lines intersecting d and meeting γ_8 twice, the line lies on that surface determined by P. If the line meets the cubic surface F in its associated point Z, this line is parasitic in the transformation. It has been shown * that there are eighteen such lines. This may also be demonstrated as follows: Given Z, there are ten lines meeting d in points P, while given any point P on d, there are eight bisecants of γ_8 through it; any point on any one of these bisecants determines a surface, hence will fix Z; therefore there exists between Z and P an $(8, 10)$ correspondence; since d is rational, there are eighteen coincidences, hence eighteen parasitic lines.

4. *Classification of γ_8 components.* When γ_8 is composite, the components may be classified as follows:

I. k curves γ_n of order n and a non-composite curve γ_{8-kn} of order $8 - kn$, $n = 1, 2, 3, 4$, $kn \leqq 8$.

II. k curves γ_n of order n and $8 - kn$ straight lines, $k > 1$, $n > 1$, $kn < 8$.

III. k_1 curves of order n_1, k_2 curves of order n_2, and $8 - k_1 n_1 - k_2 n_2$ straight lines, $k_1 \neq 0$, $k_2 \neq 0$, $n_2 > n_1 > 1$, and $k_1 n_1 + k_2 n_2 < 8$.

5. *Case I, A.* $\gamma_8 : k\gamma_1\gamma_{8-k}$. There exist the following configurations of d and the k straight lines which do not necessitate the existence of other base lines. It is impossible to have four base lines mutually skew or two pairs of intersecting lines skew to each other, unless further base lines exist. In the former case there are the transversals meeting the four given lines; in the

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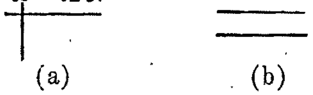
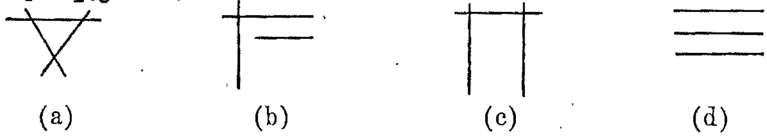
* Carroll, *op. cit.*, p. 714.

latter case the intersection of the planes of the intersecting lines is an additional base line. The configurations may be represented by the following schemes, in which *d* may be any one of the lines, so that various subdivisions have to be considered for each configuration.

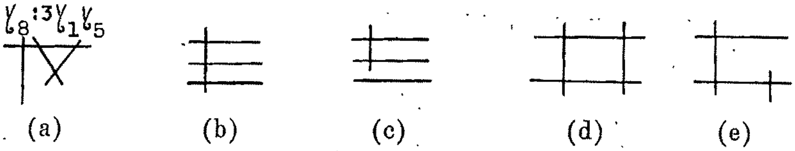
1. $\gamma_8: \gamma_1\gamma_7$


(a) (b)

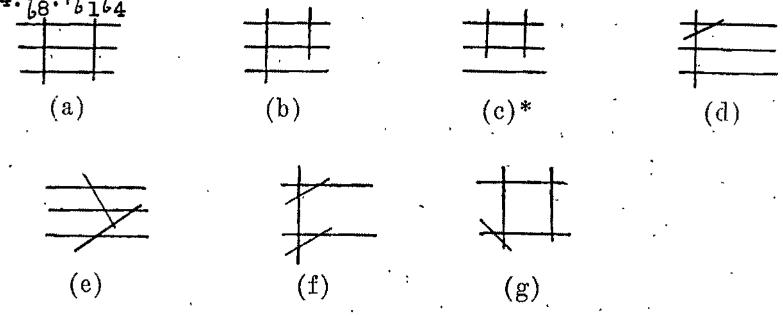
2. $\gamma_8: 2\gamma_1\gamma_6$


(a) (b) (c) (d)

3. $\gamma_8: 3\gamma_1\gamma_5$

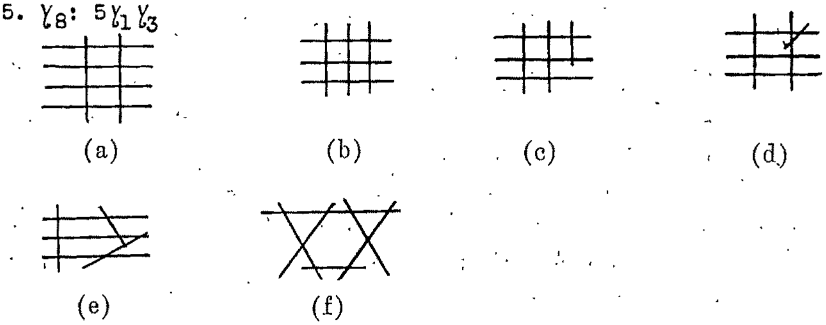

(a) (b) (c) (d) (e)

4. $\gamma_8: 4\gamma_1\gamma_4$


(a) (b) (c)* (d)

(e) (f) (g)

5. $\gamma_8: 5\gamma_1\gamma_3$


(a) (b) (c) (d)

(e) (f)

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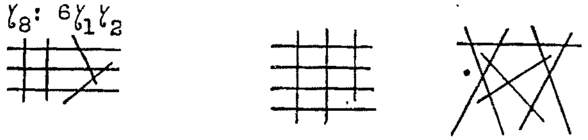
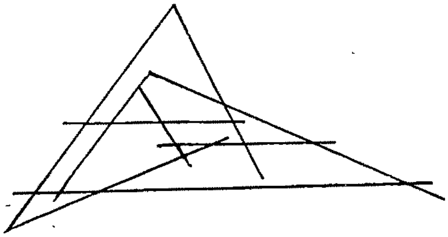
* In 4 (c) the residual γ_4 consists of two conics in planes through the isolated line.

6. γ_8: $6\gamma_1\gamma_2$



(a) (b) (c)

7. γ_8: $8\gamma_1$



6. *Discussion of Case I, A.* 1. If γ_8 is composed of a straight line γ_1 and γ_7, there are two possible cases — γ_1 either meets or is skew to d. In the first case let $d \sim d_1 : 12$, $\gamma_1 \sim c_1 : 34$; then $\gamma_7 \sim c_7 : 1^2 2^2 3^2 4^2 5^3 6^3$, $p = 5$. γ_1 and d each meet γ_7 in three points. There are two parasitic lines (d, γ_1, γ_7), that is, lines meeting d, γ_1, and γ_7 each once, for, given Z, there is just one point P on γ_7, but on neither γ_1 nor d, and lying in the plane determined by γ_1 and d; given P, there is just one point Z, hence there are two coincidences.

There remain eight lines meeting d, but not γ_1, and intersecting γ_7 twice, thus determining eight points P for each Z. Given P on d, there are ten bisecants of γ_7, d itself counting for three. Each one of the seven remaining bisecants determines a Z; this $(8, 7)$ correspondence has fifteen coincidences accounting for fifteen parasitic lines of the type (d, γ_7^2). γ_1 is a parasitic line and also a fundamental line of the first kind with the plane determined by γ_1 and d as its principal image surface.

If γ_1 is skew to d, it may be represented by $c_1 : 13$ and γ_7 by $c_7 : 1^1 2^2 3^2 4^3 5^3 6^3$, $p = 4$. Each of the lines γ_1 and d meets γ_7 in four points. Five lines on a cubic surface—those having for images the point 1, the lines 45, 46, 56, and the conic 23456, respectively,—meet d, γ_1, and γ_7 each once, so that given Z, there are five points P. A plane passed through P and γ_1 meets γ_7 in seven points, four of which lie on γ_1; the lines from P to the three remaining points meet γ_1, and each of these three lines determines a Z. Therefore there are eight parasitic lines of the type (d, γ_1, γ_7).

There are five other lines lying on each cubic surface, intersecting d and meeting γ_7 twice, therefore determining five points P for each Z; given P, there are five bisecants of γ_7 other than d, so that ten parasitic lines (d, γ_7^2) are noted.

If γ_1 is defined by $u = 0$, $v = 0$, its image surface has the equation $u(x)v(F) - v(x)u(F) = 0$.

7. *Discussion of Case I, A.* 3(a). In this configuration d may intersect all, two, or only one of the three straight line components of γ_8.

If d intersects the other lines γ_1, γ'_1, and γ''_1, the images of these may be denoted by 34, 56, and 35, respectively; the image of γ_5, the residual curve, may then be taken as $c_5 : 1^2 2^2 3^1 4^2 5^1 6^2$, $p = 2$. $d, \gamma_1, \gamma'_1, \gamma''_1$, meet γ_5 in one, two, two, and three points, respectively. The plane determined by d and γ_1 contains γ'_1, hence the transformation reduces to order ten; γ_1 and γ'_1 are parasitic lines. The plane determined by d and γ''_1 has one variable residual line on each cubic surface of the pencil; this line passes through the residual point of γ_5 in that plane, and has for image the line 46. In this plane there are two parasitic lines of the type $(d, \gamma''_1, \gamma_5)$. Six lines, apart from γ''_1, lie on each cubic surface of the pencil and meet d once and γ_5 twice; their images are the points 1 and 2, the lines 36, 45, and the conics 23456, 13456. From a point P on d may be drawn four bisecants of γ_5, hence there are ten parasitic lines (d, γ_5^2). There are, therefore, fifteen parasitic lines in this case.

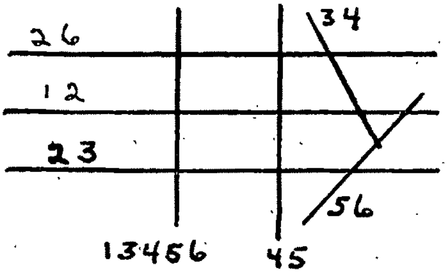
If d intersects γ_1 and γ'_1 but not γ''_1, the images of these lines may be represented by 34, 56, and 25, respectively, and the image of γ_5 by $c_5 : 1^2 2^1 3^2 4^2 5^1 6^2$, $p = 2$. d, γ_1, γ'_1, and γ''_1 meet γ_5 in two, one, two and three points, respectively. The transformation is of order ten and the number of parasitic lines is fifteen. Four lines, apart from γ'_1, on each cubic surface of the pencil meet d and γ''_1, hence given Z, there are four points P on d; given P, there are two points Z, since the plane determined by P and γ''_1 meets γ_5 in three points on γ''_1, leaving two others; there are therefore six parasitic lines of the type $(d, \gamma''_1, \gamma_5)$. Given Z, there are four lines, apart from γ_1 and γ'_1, intersecting d and meeting γ_5 twice, while given P, there are three bisecants of γ_5, apart from d, accounting for seven parasitic lines (d, γ_5^2).

Lastly, if d meets only one of the base lines γ_1, the representations of γ_1, γ'_1, and γ''_1 may be taken as 34, 26, and 15, respectively, so that the image of γ_5 is $c_5 : 1^1 2^1 3^2 4^2 5^2 6^2$, $p = 2$. These four lines meet γ_5 in three, one, two, and two points, respectively. There are two parasitic lines (d, γ_1, γ_5), since there is one variable line in the plane of d and γ_1 and through the residual point on γ_5. There are four variable lines, apart from γ_1, lying on each cubic surface and meeting both d and γ'_1; three lines may be drawn from a point P on d to meet γ'_1 and γ_5, hence there are seven parasitic lines (d, γ'_1, γ_5). Similarly there are seven parasitic lines $(d, \gamma''_1, \gamma_5)$. There is one line meeting d once and γ_5 twice, the generator of $H_2(d, \gamma_5)$ through Z and belonging to the composite cubic surface of the pencil. This case may be summarized

		parasitic lines							
d meets	order of transform	(d, γ_1, γ_5)	(d, γ'_1, γ_5)	$(d, \gamma''_1, \gamma_5)$	(d, γ_5^2)	γ_1	γ'_1	γ''_1	No.
$\gamma_1, \gamma'_1, \gamma''_1$	10	0	0	2	10	1	1	1	15
γ, γ'_1	10	0	0	6	7	1	1	0	15
γ_1	11	2	7	7	1	1	0	0	18

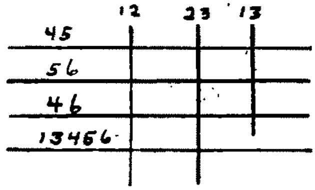
8. *Discussion of Case I, A.* 6. If γ_8 includes six straight lines and a conic, two cases arise, according as the residual line of the plane of the conic is a base line or not. In the latter case the configuration is shown by diagram (a); the section of each surface of the pencil by the plane of the conic is variable and the line forms a pencil with its vertex on one of the base lines; each of the six other lines meets the conic in one point.

A conic on F may be represented in three ways: 1. as a line through one base point; 2. as a conic through four base points; 3. as a cubic curve through all the base points and having a double point at one of them. If method 1 is chosen, and the conic is represented by $c_1 : 1$, the image of the residual base curve is represented by $c_8 : 1^2 2^3 3^3 4^3 5^3 6^3$, consisting of seven lines, six of which must meet the given base conic. The images of lines lying on the cubic surfaces of the pencil and meeting $c_1 : 1$ are the ten lines ik, $i, k \neq 1$, and the five conics containing the symbol 1. 12, 23, 26, 34, 45, 56, and 13456 may be chosen as the representations of the seven lines; the residual line in the plane of the conic is 23456 and is intersected by 12. d may be chosen as 1. intersecting four others; 2. intersecting three and forming a triangle with two of them; 3. intersecting three skew lines. The following results are obtained:

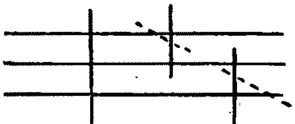


Parasitic lines	*Case* 1 $d \sim d_1 : 12$ order of transformation $= 10$	*Case* 2 $d \sim d_1 : 34$ order of transformation $= 10$	*Case* 3 $d \sim d_1 : 23$ order of transformation $= 11$
$(d, 13456, \gamma_2)$	2	3	
$(d, 45, \gamma_2)$	2	2	
$(d, 23, 26)$	2	2	
$(d, 26, \gamma_2)$	3		3
$(d, 23, \gamma_2)$	2	3	
$(d, 45, 13456)$		2	
$(d, 26, 56)$			2
$(d, 34, 13456)$			2
$(d, 34, 45)$			2
$(d, 12, 26)$			2
$(d, 12, \gamma_2)$			4
base lines intersecting d	4	3	3
total number of parasitic lines	15	15	18

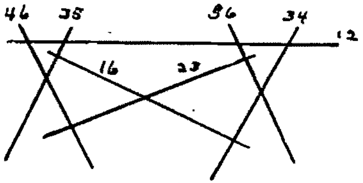
When one base line is coplanar with the conic, the residual six lines make the complete intersection with a quadric. The configuration of these lines is shown in diagram (b). d may be taken as intersecting four, three, or only two skew lines. The first case will be discussed in detail. The seven lines may be mapped on the plane by the scheme as illustrated. Since the lines 12, 13, 23 and 45, 56, 46 form two sets of skew lines, each line of either intersecting all of the other, they lie on a quadric surface and furnish a complete intersection of $|F|$. The residual base is a conic represented by $c_2 : 2$.



If $d \sim d_1 : 12$, each of the four lines meeting d is fundamental of the first kind, having the plane determined by that line and d for principal image surface, and is also parasitic. In each plane thus determined there are two additional parasitic lines; those in the plane of d and 13456 meet on 13 and the other pairs all meet on γ_2. The line represented by 23 has one additional line meeting it and d on every cubic surface of the pencil. From every point P on d it is possible to draw three lines intersecting 23 and the base curve in a second point; two of these lines intersect γ_2 and the third, 13. The latter one lies on the fixed quadric surface, and, therefore, does not participate in the correspondence between Z and P. Hence there are three parasitic lines of the type $(d, 23, \gamma_2)$. The line 13 meets γ_2 in one point, so that from P only one line may be drawn to intersect 13 and γ_2. This accounts for two parasitic lines $(d, 13, \gamma_2)$. Through the point Z on d associated with the composite cubic surface there passes a generator of the component quadric surface which is parasitic. The number of parasitic lines totals eighteen—the four base lines meeting d; eight, in pairs, in the planes determined by d and these four base lines; three of the type $(d, 23, \gamma_2)$, two of the type $(d, 13, \gamma_2)$, and the generator of the component quadric surface. The image of γ_2 is Γ_8 containing all the parasitic lines except the one on the fixed quadric surface and the two in the plane of d and 13456.



(1)



(2)

Configuration (c) results from an attempt to have only six base lines arranged in the scheme (1). This is impossible, as the residual γ_3 must be composite, having as a component a line intersecting four of the given lines (denoted by the dotted line). Adopting the scheme (2), three possible cases arise according as d intersects two pairs of intersecting lines, one pair of intersecting lines and a third line skew to these two, or three mutually skew lines. $\gamma_2 \sim c_2 : 1^1 2^1 3^0 4^1 5^1 6^0$ and with the exception of 12, the straight lines intersect γ_2 in one point. The complete table for this case follows.

Parasitic lines	*Case* 1 $d \sim d_1$: 12 order of transformation = 9	*Case* 2 $d \sim d_1$: 46 order of transformation = 10	*Case* 3 $d \sim d_1$: 16 order of transformation = 11
$(d, 23, \gamma_2)$	4		
$(d, 16, \gamma_2)$	4	3	
$(d, 23, 34)$		2	
$(d, 16, 56)$		2	
$(d, 56, \gamma_2)$		2	2
$(d, 34, \gamma_2)$		3	
$(d, 34, 46)$			2
$(d, 23, 12)$			2
$(d, 35, 56)$			2
$(d, 46, 56)$			3
$(d, 12, \gamma_2)$			4
base lines intersecting d	4	3	3
total number of parasitic lines	12	15	18

9. *Discussion of Case I, A. 7.* The nine base lines enter symmetrically; each one intersects four others forming two triangles. The representations of these lines may be taken as 12, 34, 56, 35, 46, 26, 14, 15, and 23. The equation of the pencil of cubic surfaces may be written

$$z_4x_1x_3u - z_3x_2x_4v = 0, \qquad u \equiv \sum_{i=1}^{4} a_ix_i, \qquad v \equiv \sum_{i=1}^{4} b_ix_i.$$

$$\begin{aligned}
\sigma &\equiv -\left[uv(y_1u - y_2v) + v\bar{u} - u\bar{v}\right] \\
\tau &\equiv uv(y_1\bar{u} - y_2\bar{v}) \\
\bar{u} &\equiv a_3y_1y_3u(y) + a_4y_2y_4v(y) \\
\bar{v} &\equiv b_3y_1y_3u(y) + b_4y_2y_4v(y).
\end{aligned}$$

The order of the transformation is seven. The planes $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, and $y_4 = 0$ come off as factors. If (iu), (iv), and (ij) represent $x_i = 0$, $u = 0$; $x_i = 0$, $v = 0$; and $x_i = 0$, $x_j = 0$, respectively, it may be shown that

$$\begin{aligned}
S_1 \sim S_7 &: d\,\bar{d}^2(34)^1(3v)^2(4u)^2(uv)^3\,[(14)^1(1v)^2(23)^1(2u)^2]\;4g \\
d \sim \tau_6 &: d^2\bar{d}^2(34)^1(3v)^1(4u)^1(uv)^1\,[(14)^1(1v)^1(23)^1(2u)^1]\;4g \\
\sigma_4 \sim \sigma_4 &: d\,\bar{d} \qquad\quad (3v)^1(4u)^1(uv)^2\,[\qquad (1v)^1 \qquad (2u)^1]\;4g
\end{aligned}$$

$J_{24} \equiv \tau_6{}^2\Gamma_{12}$, Γ_{12} being composed of two surfaces of order three, one of order two, and one of order four, (the images of the four lines skew to d, $-3v$, $4u$, 34, and uv, respectively).

The surfaces S_7 touch each other along every sheet through d. At every point of d one of the tangent planes to $\tau_4 \equiv y_1\bar{u} - y_2\bar{v} = 0$ coincides with that of $\sigma_4 = 0$, and has three-point contact. Moreover, the surfaces have the same tangent plane at each point of $(1v)$ and of $(2u)$ but not of (uv), $(3u)$, $(4v)$. This accounts for twelve lines, leaving four parasitic lines. The same may be obtained by the discriminant of the residual conic in the plane tangent to F_z at Z after removing d.

10. *Discussion of Case I, B.* $\gamma_8 : k\gamma_2\gamma_{8-2k}$, $k = 1, 2, 4$.

1. When $k = 1$, three cases arise:

(a) $[\gamma_2, d] = 2$. Let the representations be $d \sim d_1 : 12$, $\gamma_2 \sim c_2 : 3456$, and $\gamma_6 \sim c_6 : 1^22^23^24^25^26^2$. γ_6 lies on a quadric surface, is of genus four, and meets γ_2 twice. The transformation is of order ten and there are fifteen parasitic lines $(d, \gamma_6{}^2)$; these occur doubly on Γ, the image of γ_6; the conic is also parasitic.

(b) $[\gamma_2, d] = 1$. If $d \sim d_1 : 12$, $\gamma_2 \sim c_2 : 1345$, and $\gamma_6 \sim c_6 : 1^12^23^24^25^26^3$, it is noted that γ_6 is of genus three and meets d in three points and γ_2 in five points. The parasitic lines consist of nine meeting γ_6 twice and nine intersecting both γ_2 and γ_6.

(c) $[\gamma_2, d] = 0$. Let $d \sim d_1 : 12$, $\gamma_2 \sim c_2 : 1234$ and $\gamma_6 \sim c_6 : 1^1 2^1 3^2 4^2 5^3 6^3$, $p = 2$. γ_6 meets γ_2 in six points and d, in four. The line in the plane of the conic and lying on the cubic surface of the pencil associated with the point of intersection of d and this plane is parasitic. From any point of d it is possible to draw two bisecants of γ_6 and six lines meeting γ_2 and γ_6 each once. The lines on the F_3 of the map which meet d, γ_2 and γ_6 are represented by 1, 2, 35, 36, 45, 46, 13456, 23456. Hence there are fourteen parasitic lines (d, γ_2, γ_6); 34 meets γ_6 twice and d once, so that there are three parasitic lines $(d, \gamma_2{}^2)$.

A proper γ_6 with a quadrisecant can not be of genus two, if it lies on a quadric. Such a γ_6 consists of $\gamma_1 \sim c_1 : 56$ in the plane of the conic and intersecting d, and a $\gamma_5 \sim c_5 : 1^1 2^1 3^2 4^2 5^2 6^2$, $p = 2$. It meets γ_2 in four points and d, in three. Moreover, d and γ_5 lie on a quadric H which with the plane of γ_1 and γ_2 comprises a complete cubic surface of the pencil. The generator of H through the associated point Z of the composite surface is parasitic. There are also two parasitic lines (d, γ_1, γ_5) in the plane of d and γ_1 and fourteen parasitic lines (d, γ_2, γ_5). If Z is the intersection of γ_1 and d, the transformation is of order ten.

2. When $k = 2$ and the two conics lie in planes through d, the residual quartic must be of genus one and does not intersect d. The equation of the pencil has the form $\lambda_1 x_1 H - \lambda_2 x_2 H' = 0$, H and H' being quadratic forms. Both conics $x_1 = 0$, $H' = 0$ and $x_2 = 0$, $H = 0$ are parasitic, and the transformation is of order nine. The twelve parasitic lines are all of the form $(d, \gamma_4{}^2)$.

3. $k = 4$. If the intersection of the two quadrics $H = 0$, $H' = 0$ consists of two conics, the base curve of the pencil is composed of four conics, two of which intersect d in two points, the other two being skew to d but meeting each other in two points. In case each of the conics meets d in only one point, let $d \sim d_1 : 12$, $\gamma_2 \sim c_2 : 1345$, $\gamma'_2 \sim c'_2 : 1346$, $\gamma''_2 \sim c''_2 : 2356$, and $\gamma'''_2 \sim c'''_2 : 2456$. The parasitic lines are found to be five each of the types (d, γ_2, γ'_2) and $(d, \gamma''_2, \gamma'''_2)$ and two each of the types (d, γ_2, γ''), $(d, \gamma_2, \gamma'''_2)$, $(d, \gamma'_2, \gamma''_2)$, and $(d, \gamma'_2, \gamma'''_2)$. The image of each base curve is a surface of order five, containing nine of the parasitic lines. These two configurations of base conics are the only ones possible.

11. *Discussion of Case I, C.* $\gamma_8 : k\gamma_3\gamma_{8-3k}$. If γ_3 is skew, $k = 1$ or 2. If $k = 1$, let γ_3 and γ_5 each intersect d twice; there are eight parasitic lines $(d, \gamma_5{}^2)$ and ten of the form (d, γ_3, γ_5). If γ_3 meets d in one point, γ_5 meets it in three and the genus of γ_5 is one; there are now three parasitic lines $(d, \gamma_3{}^2)$, five of the form $(d, \gamma_5{}^2)$ and ten of the form (d, γ_3, γ_5). If γ_3 is skew to d, γ_5 has d for a quadrisecant, the genus of γ_5 is zero, and no bisecant of γ_5 intersects d; there are nine parasitic lines $(d, \gamma_3{}^2)$ and nine of the form (d, γ_3, γ_5).

When γ_3 is plane, let its image be $c_3 : 123456$, $p = 1$. The residual γ_5 has for image $c_5 : 1^1 2^1 3^2 4^2 5^2 6^2$, $p = 2$. γ_3 intersects d in one point, γ_5 in five points; d and γ_5 have three points in common. d and γ_5 lie on a quadric H; u, the plane of γ_3, and H make a composite surface of the pencil. The generator of H through the associated point Z is parasitic and is a bisecant of γ_5. All other parasitic lines are of the type (d, γ_3, γ_5).

12. *Discussion of Case I, D.* $\gamma_8 : 2\gamma_4$. Three cases are possible, genus one for each, genus one for the first and zero for the second, and genus zero for each.

When $d \sim d_1 : 12$ and both curves have genus one, they may be represented by $\gamma_4 \sim c_4 : 1^1 2^1 3^1 4^1 5^2 6^2$ and $\gamma'_4 \sim c'_4 : 1^1 2^1 3^2 4^2 5^1 6^1$. These two curves intersect in six points. Of the ten lines on a cubic surface meeting d, only one, namely, 34 meets γ_4 twice; since from any point on d only one bisecant of γ_4, apart from d, may be drawn, there are two parasitic lines $(d, \gamma_4{}^2)$. Similarly, there are two of the type $(d, \gamma'_4{}^2)$. There are eight lines on a cubic surface intersecting both γ_4 and γ'_4; from a point P on d may be drawn six secants of γ_4 and γ'_4, apart from d, hence there are fourteen parasitic lines (d, γ_4, γ'_4).

$\gamma_4 \sim c_4 : 1^1 2^0 3^1 4^2 5^2 6^2$ and $\gamma'_4 \sim c'_4 : 1^1 2^2 3^2 4^1 5^1 6^1$ represent quartic curves of genus zero and genus one, respectively. Since d intersects γ_4 in three points, there are no bisecants of γ_4 from points on d. A cubic surface contains five bisecants of γ'_4 meeting d, and from P on d may be drawn two bisecants of γ'_4, hence there are seven parasitic lines $(d, \gamma'_4{}^2)$. There are five lines intersecting both γ_4 and γ'_4 and from P on d nine such lines may be drawn but d counts for three of them, hence there are eleven parasitic lines (d, γ_4, γ'_4).

Lastly, when the genus of each curve is zero, the quartic curves must intersect in eight points, and the representations $\gamma_4 \sim c_4 : 1^2 2^0 3^1 4^2 5^1 6^2$ and $\gamma'_4 \sim c'_4 : 1^0 2^2 3^2 4^1 5^2 6^1$ may be made. In this case there are five parasitic lines of the types $(d, \gamma_4{}^2)$ and $(d, \gamma'_4{}^2)$ and eight of the type (d, γ_4, γ'_4).

13. *Cases II and III.* These cases may be considered along the lines of the preceding discussions and are, in general, special examples of Case I.

14. *Reductions in the order of the transformation, γ_8 being non-composite.* If the pencil of cubic surfaces $z_3F - z_4K = 0$ is determined by a general cubic surface $F \equiv x_1u + x_2v = 0$ (u, v being quadratic forms in x_1, x_2, x_3, x_4) and a cubic cone with vertex at $(0, 0, 0, 1)$ $K \equiv x_1u' + x_2v' = 0$ (u', v' being quadratic forms in x_1, x_2, x_3), the following table of characteristics is obtained:

$$\begin{aligned}
S_1 &\sim S_8 : d^3\,\bar{d}^2\,\gamma_8^2\,11g \\
d &\sim \tau_7 : d^3\,\bar{d}\,\gamma_8^2\,11g \\
\sigma_5 &\sim \sigma_5 : d^2\,\bar{d}\,\gamma_8\,11g \\
\gamma_8 &\sim \Gamma_{14} : d^6\,\bar{d}^4\,\gamma_8^3\,11g^2 \\
J_{28} &\equiv \tau_7^2\,\Gamma_{14}.
\end{aligned}$$

Given the pencil $z_3K - z_4K' = 0$, $K \equiv x_1u + x_2v = 0$ (u, v being quadratic forms in x_1, x_2, x_4) and $K' \equiv x_1u' + x_2v'$ (u', v' being quadratic forms in x_1, x_2, x_3). Here $K = 0$ and $K' = 0$ are general cubic cones containing d and having vertices at $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, respectively. The residual curve of intersection of K and K' is γ_8, having double points at the vertices of the cone and meeting d nowhere else. The table of characteristics is written:

$$\begin{aligned}
S_1 &\sim S_5 : d^2\,\bar{d}^2\,\gamma_8\,4g \\
d &\sim \tau_4 : d^2\,\bar{d}\,\gamma_8\,4g \\
\sigma_2 &\sim \sigma_2 : d\,\bar{d}\quad 4g \\
\gamma_8 &\sim \Gamma_8 : d^4\,\bar{d}^2\,\gamma_8\,4g^2 \\
J_{16} &\equiv \tau_4^2\,\Gamma_8.
\end{aligned}$$

Γ_8 is a ruled surface, the image of any point of γ_8 being a straight line. The section of Γ_8 by any plane through d consists of d counted four times and of four straight lines, each of which passes through a common point on d and is drawn to one of the four points of γ_8 in that plane.

Every plane through d meets σ_2 in a residual line of invariant points. Since every plane through d is invariant, the transformation in each such plane is cubic; it is a non-perspective Jonquières involution with vertex V in the point of intersection of the generator of σ_2 in that plane with d. The image of V is a conic determined by V and the four points of γ_8 in that plane and having the generator of σ_2 as a tangent line at V. This conic is also the residual section of the plane tangent to the cubic surface associated with V.

WELLS COLLEGE.

PLANAR CREMONA TRANSFORMATIONS.

By Sherburne F. Barber.

1. *Introduction.* A plane curve C_x of order γ_0 with multiplicities γ_i at $P_m{}^2$, i. e. m distinct points F_i $(i=1,\cdots,m)$, is transformed by a Cremona transformation T of order n with r_i-fold distinct F-points F_i $(i=1,\cdots,\rho,$ $\rho\leqq m)$ into a curve C'_y (in the same or a different plane) of order γ'_0 and with multiplicities γ'_j at $Q_m{}^2$, m distinct points F_j $(j=1,\cdots,m)$ of which $F_1,\cdots,F_j,\cdots,F_\rho$ $(\rho\leq m)$ are the F-points of T^{-1} where

$$(1)\qquad S:\quad \begin{aligned} \gamma'_0 &= n\gamma_0-r_1\gamma_1-\cdots-r_i\gamma_i-\cdots-r_\rho\gamma_\rho \\ \gamma'_j &= s_j\gamma_0-\alpha_{1j}\gamma_1-\cdots-\alpha_{ij}\gamma_i-\cdots-\alpha_{\rho j}\gamma_\rho \quad (j=1,\cdots,\rho) \\ \gamma'_{\rho+k} &= \gamma_{\rho+k} \qquad\qquad (k=1,\cdots,m-\rho). \end{aligned}$$

The numbers s_j are the multiplicities of T^{-1} at F_j. The directions around the points F_i correspond to the points of a rational P-curve P_i of order r_i with an α_{ij}-fold point at F_j. Two sets of points, such as $P_m{}^2$ and $Q_m{}^2$, are said to be congruent sets of points under T. The element S has the invariant quadratic and linear forms

$$(2)\qquad \gamma_0{}^2-\gamma_1{}^2-\cdots-\gamma_m{}^2$$

$$(3)\qquad 3\gamma_0-\gamma_1-\cdots-\gamma_m.$$

The numbers r_i satisfy the Diophantine equations

$$(4)\qquad \sum_{i=1}^{\rho} r_i{}^2=n^2-1, \qquad \sum_{i=1}^{\rho} r_i=3(n-1).$$

The general solution of these equations is not known nor is there any criterion for picking out from the algebraic solutions those which correspond to existent geometric situations. The only sure method of finding new types is by constructing products of known types.

The elements S constitute a group $g_{m,2}$ of finite order for $\rho\leqq 8$, of infinite order for $\rho\geqq 9$, which is generated by the permutation group Π of $\gamma_1,\cdots,\gamma_m$ and A_{123} where

$$(5)\qquad A_{123}:\quad \begin{aligned} \gamma'_j &= \gamma_j+\gamma_0-\gamma_1-\gamma_2-\gamma_3 \qquad (j=0,1,2,3) \\ \gamma'_k &= \gamma_k \qquad\qquad (k=4,\cdots,m). \end{aligned}$$

The subgroup Π of $g_{m,2}$ is generated by transpositions of the type (12), i.e.

$$(6)\qquad T_{12}:\ \begin{array}{l}\gamma'_0=\gamma_0,\quad \gamma'_1=\gamma_2,\quad \gamma'_2=\gamma_1,\\ \gamma'_j=\gamma_j\,.\end{array}\qquad (j=3,\cdots,m).$$

This is involutorial and has the space of fixed points $\gamma_1-\gamma_2$. Hence T_{12} is a central involution with space of fixed points $\gamma_1-\gamma_2$ and centre at the pole of this fixed space with respect to the invariant quadric (2). The involutions conjugate to T_{12} are determined by the spaces conjugate to $\gamma_1-\gamma_2$. Thus we find that A_{123} is determined by $\gamma_0-\gamma_1-\gamma_2-\gamma_3$. The forms conjugate to $\gamma_1-\gamma_2$ under the Cremona transformations on m F-points will be called *discriminant* or *D-conditions.* Thus we find the conjugate set:

$$(7)\qquad \begin{array}{c}\gamma_1-\gamma_2\\ \gamma_0-\gamma_1-\gamma_2-\gamma_3\\ 2\gamma_0-\gamma_1-\gamma_2-\cdots-\gamma_6\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ c_0\gamma_0-c_1\gamma_1-\cdots-c_m\gamma_m\end{array}$$

together with all that arise from these by the operations of Π.

2. *The group generated by two discriminant conditions.* The general discriminant condition ([1], p. 16) on m points $(m>9)$ is not known. We shall select two particular ones of the form

$$(8)\qquad (c_i\gamma)\equiv c_{0i}\gamma_0-c_{1i}\gamma_1-\cdots-c_{mi}\gamma_m \qquad (i=1,2)$$

where the coefficients c_{ki} are integers satisfying

$$(9)\qquad \begin{array}{l}c_{0i}{}^2-c_{1i}{}^2-\cdots-c_{mi}{}^2=-2\\ 3c_{0i}-c_{1i}-\cdots-c_{mi}=0.\end{array}$$

The element ([1], p. 17) of $g_{m,2}$ defined by (8) is

$$(10)\qquad \gamma'=(c_i\gamma)c_i+\gamma$$

where the subscript j $(j=0,1,\cdots,m)$ is to be attached to γ',γ and the unattached c_i. This element, denoted by I_i, is of period two; two such elements will in general generate a dihedral group of infinite order and we are thus led to consider the product

$$(11)\qquad W=I_1I_2:\ \gamma'=[k(c_1\gamma)+(c_2\gamma)]c_2+(c_1\gamma)c_1+\gamma$$

where $k=(c_1c_2)$. The dihedral group contains elements of two types — W^{-1}, W^0, W^1, $W^2\cdots$ and $\cdots W^{-1}I_2$, W^0I_2, $W^1I_2\cdots$—of which the first have no period and the second are of period two. For the simpler involutions we observe the discriminant conditions from which they arise

$$(12)\qquad \begin{array}{ll} W^3I_2 : & P_2(c_1\gamma) + P_1(c_2\gamma) \\ W^2I_2 : & P_1(c_1\gamma) + P_0(c_2\gamma) \\ W^1I_2 : & P_0(c_1\gamma) \\ W^0I_2 : & \qquad\qquad P_0(c_2\gamma) \\ W^{-1}I_2 : & P_0(c_1\gamma) + P_1(c_2\gamma), \end{array}$$

where $P_0 = 1$, $P_1 = k$, $P_2 = (k^2 - 1)$. We wish to ascertain the form of $P_m(c_1\gamma) + P_{m-1}(c_2\gamma)$ to which $W^{m+1}I_2$ is attached. If $P_m(c_1\gamma) + P_{m-1}(c_2\gamma)$ is a discriminant condition, on applying (9) we obtain

$$(13)\qquad P_m{}^2 + P^2{}_{m-1} - kP_mP_{m-1} = 1.$$

For the early cases, if we define $P_{-1} = 0$ we observe the recurrence relation

$$(14)\qquad P_r = kP_{r-1} - P_{r-2}.$$

If we suppose (14) true for $r = m$ we derive

$$(15)\qquad P_m{}^2 - P_{m-1}P_{m+1} = 1.$$

$$(16)\qquad P^2{}_{m+1} + P_m{}^2 - kP_mP_{m+1} = 1.$$

Hence if (16) is true, both (15) and (13) are true. But (16) holds for early values of m. Hence (13) and (15) are true by induction.

More generally to find all discriminant conditions in the pencil determined by $(c_1\gamma)$ and $(c_2\gamma)$ we seek the conditions under which

$$(17)\qquad x(c_1\gamma) + y(c_2\gamma)$$

is a discriminant condition. Analogous to (13) we seek the general solution in integers of

$$(18)\qquad x^2 + y^2 - kxy = 1,$$

of which a particular solution is $x = k$, $y = 1$.

The general solution (^{3}, p. 409) of

$$(19)\qquad Ax^2 - 2Bxy + Cy^2 = c$$

is

$$(20)\qquad Ax_{n+1} + y_{n+1}(-B \pm K^{1/2}) = [Aa + b(-B \pm K^{1/2})][p + q(-B \pm K^{1/2})]^n;$$

(a, b) is a particular solution of (19), (p, q) is the smallest solution of $p^2 - 2Bpq + ACq^2 = 1$ for which $p^2 + q^2 > 1$ and $K = B^2 - AC = (k^2 - 4)/4$. For (18), $-B \pm K^{1/2} = -k/2 \pm (k^2 - 4)^{1/2}/2 = \rho$, whence

$$\rho^2 + k\rho + 1 = 0, \qquad x_1 = a = p = k, \qquad y_1 = b = q = 1$$

and the general solution of (18) is

$$(21) \qquad x_{n+1} + \rho y_{n+1} = (x_1 + \rho y_1)(x_1 + \rho y_1)^n = (x_1 + \rho y_1)(x_n + \rho y_n).$$

Expanding and equating coefficients, we obtain

$$(22) \qquad y_{n+1} = x_n, \qquad x_{n+1} = kx_n - x_{n-1},$$

whence $x = P_m$, $y = P_{m-1}$, since the initial conditions are the same. Equation (18) can be written as $(2x - ky)^2 - (k^2 - 4)y^2 = 4$, the form of the Pell equation (3; p. 373). The fact that the solutions of the Pell equation constitute a singly infinite set shows that the sequence (22) comprises all the solutions of (18).

(23) *The form* $x(c_1\gamma) + y(c_2\gamma)$, *if* $(c_1\gamma)$ *and* $(c_2\gamma)$ *are existant discriminant conditions, is an existant discriminant condition, where* $x = P_m$, $y = P_{m-1}$. *The polynomials* P_i *are connected by the recurrence relation* (14).

For early values of k the values of the polynomials $P_i(k)$ are tabulated as follows:

(24)

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
P_{-1}	0	0	0	0
P_0	1	1	1	1
P_1	0	1	2	3
P_2	1	0	3	8
P_3	0	-1	4	21
P_4	1	-1	5	55.

For $k = 0$ and $k = 1$ we get groups of orders 4 and 6 respectively; for all other cases the group is infinite.

By direct substitution it is verified that

$$(25) \qquad P_n(k) = \sum_{r=0} (-1)^r \binom{n-r}{r} k^{n-2r}$$

satisfies (14). Between the polynomials P_i there exists the further relation

$$(26) \qquad P_r P_{s-1} - P_s P_{r-1} = P_{s-r-1} \qquad (s > r)$$

deduced by use of (14) applied r times.

The bilinear invariant of any two involutorial elements, $W^{r+1}I_2$ and $W^{s+1}I_2$, attached to the D-conditions

$$(27) \qquad P_r(c_1\gamma) + P_{r-1}(c_2\gamma) = (d_1\gamma), \quad P_s(c_1\gamma) + P_{s-1}(c_2\gamma) = (d_2\gamma)$$

is

$$(28) \qquad k' = P_{s-r-2} - P_{s-r}.$$

If $s = r + 1$ in (27) we can solve completely for $(c_1\gamma)$, $(c_2\gamma)$ since the determinant $P_{s-r-1} = P_0 = 1$, cf. (26). In other words

(29) *Any two consecutive involutions will generate the group.*

The group generated by the involutions I_1, I_2 is a dihedral group, in general of infinite order. The group may be generated by $W = I_1 \cdot I_2$ and I_2 subject to the relations $I_2^2 = 1$, $I_2 W = W^{-1} I_2$. The entire group is not in general generated by *any two* of its involutions. If $W^a I_2$ and $W^b I_2$ are any two of the involutions in the given group, these two generate a dihedral group whose cyclic subgroup is generated by $W' = W^a I_2 \cdot W^b I_2 = W^{a-b}$. Thus only the cyclic elements W^k where $k \equiv 0 \bmod (a - b)$ of the original group will occur in the group generated by $W^a I_2$. Unless the given involutions $W^a I_2$ and $W^b I_2$ are adjacent, the discriminant conditions attached to them will, when used as $(c_1\gamma)$, $(c_2\gamma)$, not determine in the above fashion all the discriminant conditions of the form (17). If they were so used, other discriminant conditions could be interpolated between them. We wish to examine the possibility of such an interpolation.

We suppose $(d_1\gamma)$, $(d_2\gamma)$ are known with $k' = P_{s-r-2} - P_{s-r}$ and we seek $(c_1\gamma)$, $(c_2\gamma)$ as in (27). The number k' may arise from the table (24) as the difference of two numbers in some column, once removed, and always appears in the row P_1; in this case $s - r = 1$ and $P_{s-r-2} - P_{s-r} = P_{-1} - P_1 = -k$. If an examination reveals it in the latter position only, $(d_1\gamma)$, $(d_2\gamma)$ are adjacent. In general such is the case since the aggregate of numbers represented by $P_{s-r} - P_{s-r-2}$ is very restricted. If, however, k' does appear as the difference of two numbers we seek to interpolate involutions between $(d_1\gamma)$, $(d_2\gamma)$. Since any two consecutive involutions generate the group, we seek that particular pair $(c_1\gamma)$, $(c_2\gamma)$ of which the former is $(d_1\gamma)$ itself. This requires that we reduce the subscripts in (27) r times and that system becomes

$$(30) \qquad (c_1\gamma) = (d_1\gamma), \qquad P_{s-r}(c_1\gamma) + P_{s-r-1}(c_2\gamma) = (d_2\gamma).$$

The coefficients c_{k2} must be integral for the solution to be geometric; this may or may not be sufficient. The final test is whether $(c_2\gamma)$ can be transformed into a form $\gamma_i - \gamma_j$. In certain particular cases however it is possible to find $(c_1\gamma)$, $(c_2\gamma)$, for example

$$(31) \qquad \begin{aligned} (d_1\gamma) &= 4\gamma_0 - 3\gamma_2 - \gamma_3 - \cdots - \gamma_{11} \\ (d_2\gamma) &= 84\gamma_0 - 8\gamma_1 - 55\gamma_2 - 21\gamma_3 - \cdots - 21\gamma_{11} \end{aligned}$$

for which $k' = -18$, $(c_1\gamma) = (d_1\gamma)$, $(c_2\gamma) = -\gamma_1 + \gamma_2$.

Since $W^m = W^{m+1} I_2 \cdot W I_2$ and the components arise from the discriminant conditions $P_m(c_1\gamma) + P_{m-1}(c_2\gamma)$ and $(c_1\gamma)$ respectively, the form of W^m is found to be

$$\gamma' = [P^2_{m-1}(c_1\gamma) + P_{m-1}P_{m-2}(c_2\gamma)]c_1 + [P_mP_{m-1}(c_1\gamma) + P^2_{m-1}(c_2\gamma)]c_2 + \gamma. \tag{32}$$

3. *Conditions for permutability of two cyclic groups generated as in* § 2. To consider the permutability of two infinite cyclic groups of the type discussed in § 2, we select four discriminant conditions $(c_1\gamma)$, $(c_2\gamma)$, $(c_3\gamma)$, $(c_4\gamma)$ which yield $I_1 \cdot I_2$ and $I_3 \cdot I_4$. The condition

$$I_1I_2 \cdot I_3I_4 = I_3I_4 \cdot I_1I_2 \tag{33}$$

may be expressed as the following bilinear form in γ and γ',

$$\begin{aligned}&(c_1\gamma)(c_4\gamma')[aAC + aB + bA + c] + (c_2\gamma)(c_3\gamma')C \\ &\quad + (c_2\gamma)(c_4\gamma')[AC + B] + (c_1\gamma)(c_3\gamma')[aC + b] \\ &= (c_3\gamma)(c_2\gamma')[aAc + AB + ab + C] + (c_4\gamma)(c_1\gamma')C \\ &\quad + (c_4\gamma)(c_2\gamma')[ac + B] + (c_3\gamma)(c_1\gamma')[Ac + b]\end{aligned} \tag{34}$$

where $k_{ij} = (c_ic_j)$ and $a = k_{12}$, $b = k_{13}$, $c = k_{14}$, $A = k_{34}$, $B = k_{24}$, $C = k_{23}$.

Since the group is to be abelian it must contain the additional elements:

$$I_4I_3I_2I_1, \qquad I_2I_1I_4I_3, \qquad I_2I_1I_3I_4, \qquad I_1I_2I_4I_3.$$

The elements of the linear group arise from each other by permutation of the indices. Regarding (1), (2), (3), (4) as the identical order of the indices, the products above arise from the transpositions (12), (34) and the permutation (13). (24). These generate the octic group

$$\begin{array}{cc} 1 & 1 \\ (13)(24) & (aA)(cC) \\ (14)(23) & (aA)(bB) \\ (12)(34) & (bB)(cC) \\ (12) & (bC)(cB) \\ (34) & (bc)(BC) \\ (13\,24) & (aA)(bCBc) \\ (14\,23) & (aA)(bcBC). \end{array} \tag{35}$$

The condition (34) is an identity in $(n+1)^2$ coefficients $\gamma_i\gamma'_j$ which involves both the coefficients c_i explicitly and their combinations k_{ij}. The resulting system of $(n+1)^2$ equations would be too complicated for immediate discussion. We therefore seek a derived system which will contain the k_{ij} alone.

The resulting values for the k_{ij} will lead to such a simplification of the identity (34) as to admit of its complete discussion.

In (34) we let γ and γ' take on the values c_1, c_2, c_3, c_4 and obtain sixteen equations in the six quantities a, b, c, A, B, C. Of these equations two pairs are identical; moreover each of these two equations appears as a linear combination of two other equations. We are thus left with only twelve equations which must be all or part of a conjugate set under the octic group (35). When we arrange the equations in conjugate sets, we find eight new ones. The twenty equations in all divide into three conjugate sets of four equations each and one conjugate set of eight equations. Sample equations from each set are listed. If a or A appears as a factor of an equation, it is cancelled; as we saw, cf. (24), the cyclic group is finite if a or A is zero. The sample equations are

$$
(36)\quad
\begin{aligned}
&aA\cdot bc + A(bB + cC) + a(b^2 + c^2) + 2(bC + cB) = 0,\\
&abcAB + ab^2B + ac^2B + bc^2A + bAB^2 + bBC - aAC\\
&\qquad + b^2c + c^3 + cB^2 - 2aB - 2bA - 4c = 0,\\
&a^2A^2C - acAC^2 + a^2AB + abA^2 - acBC - bcAC - abC^2\\
&\qquad - ABC^2 - 2a^2C - 2A^2C + acA - bcB\\
&\qquad - b^2C - C^3 - B^2C - 2ab - 2AB + 4C = 0,\\
&abcAC + aA^2C + bABC + ab^2C + b^2cA + ac^2C + 2bA^2\\
&\qquad + b^3 + bc^2 + bC^2 + aAB + cBC + 2cA - 2aC - 4b = 0.
\end{aligned}
$$

Of these four equations, the first three are members of conjugate sets of four equations and the last a member of the conjugate set of eight equations. The above are combinations of the following eight equations:

$$
(37)\quad
\begin{aligned}
&(A_1)\ \ aA\cdot bc + A(bB + cC) + a(b^2 + c^2) + 2(bC + cB) = 0,\\
&(A_2)\ \ aA\cdot BC + A(bB + cC) + a(B^2 + C^2) + 2(bC + cB) = 0,\\
&(A_3)\ \ aA\cdot bC + A(b^2 + C^2) + a(bB + cC) + 2(bc + BC) = 0,\\
&(A_4)\ \ aA\cdot Bc + A(B^2 + c^2) + a(bB + cC) + 2(bc + BC) = 0,\\
&\qquad (B_1)\ \ C(aA + bB - cC) + 2(bA + aB + 2c) = 0,\\
&\qquad (B_2)\ \ c(aA + bB - cC) + 2(AB + ab + 2C) = 0,\\
&\qquad (B_3)\ \ b(aA - bB + cC) + 2(ac + AC + 2B) = 0,\\
&\qquad (B_4)\ \ B(aA - bB + cC) + 2(cA + aC + 2b) = 0.
\end{aligned}
$$

(B_1) and (B_4) are linear and homogeneous in a, b, c and yield the values

$$
(38)\quad a = 2\rho,\ b = -\rho C,\ c = -\rho B,\quad \rho = 2(ABC + A^2 + B^2 + C^2 - 4).
$$

If these values are substituted in $(A_1)\cdots(B_2)$, (B_3) we obtain

$$
(39)\quad 2\rho(\rho^2 - 1)(ABC + B^2 + C^2) = 0,\quad C(\rho^2 - 1)(AC + 2B) = 0,
$$
$$
B(\rho^2 - 1)(AB + 2C) = 0.
$$

Suppose $\rho \neq 0$ and take $\rho^2 = 1$ to obtain

$$(40)\qquad \begin{array}{lll} I: & a = 2,\ b = b,\ c = -B, & A = A,\ B = B,\ C = -b, \\ II: & a = -2,\ b = b,\ c = B, & A = A,\ B = B,\ C = b. \end{array}$$

If $\rho^2 \neq 0, 1$ and $B \neq 0$ we obtain from (39)

$$(41)\qquad \begin{array}{lll} III: & a = a,\ b = b,\ c = b, & A = -2,\ B = B,\ C = B, \\ IV: & a = a,\ b = b,\ c = -b, & A = 2,\ B = B,\ C = -B. \end{array}$$

Finally if $B = 0$, $\rho^2 \neq 0, 1$ we obtain from (39) $b = B = c = C = 0$ and the identity (34) is satisfied. If, instead of (B_1) and (B_4), we select another pair such as (B_2), (B_3) linear in a, B, C we arrive ultimately at the same solutions.

We have to place the solutions (40) and (41) in the original condition (34). If we use the solution I and equate to zero the coefficient of $\gamma_i\gamma'_j$ we obtain:

$$(42)\qquad \begin{aligned} &[c_{i1} + c_{i2}][(B - Ab)c_{j4} - bc_{j3}] \\ &\quad - [c_{j1} + c_{j2}][(b - AB)c_{i3} - Bc_{i4}] = 0 \quad (i, j = 0, 1, \cdots, m). \end{aligned}$$

If $i = j$ since $[c_{i1} + c_{i2}]$ is not in general zero, cf. (49),

$$(43)\qquad \begin{aligned} c_{i3} &= [(2B - Ab)/(2b - AB)]c_{i4}, \\ c_{i3} + c_{i4} &= [(b + B)(2 - A)/(2b - AB)]c_{i4} \quad (i = 0, 1, \cdots, m). \end{aligned}$$

For this solution $k_{13} + k_{23} - k_{14} - k_{24} = 0$, i. e.

$$(44)\qquad (c_{01} + c_{02})(c_{03} - c_{04}) - \sum_{i=1}^{m} (c_{i1} + c_{i2})(c_{i3} - c_{i4}) = 0.$$

The permutations of (35) change this into

$$(45)\qquad (c_{03} + c_{04})(c_{01} - c_{02}) - \sum_{i=1}^{m} (c_{i3} + c_{i4})(c_{i1} - c_{i2}) = 0.$$

Equation (45) merely expresses a condition that the rôle of $(c_3\gamma)$, $(c_4\gamma)$ is the same as that of $(c_1\gamma)$, $(c_2\gamma)$. In view of (43), (45) becomes

$$(46)\qquad 2c(b + B)(2 - A)/(2b - AB) = 0.$$

If $A = 2$ in (46), from (43) $(b - B)(c_{i3} + c_{i4}) = 0$ or $b = B$. If $b = -B$ in (46), from (43) $B(A + 2)(c_{i3} + c_{i4}) = 0$, whence either $B = 0$, or $A = -2$. If $B = 0$, $b = 0$ then $c = 0$, $C = 0$ and we have an earlier case. If $c = 0$ in (46), then $B = 0$; in (42) if $i = j$, $(c_{i1} + c_{i2})(Ac_{i4} + 2c_{i3})b = 0$. If $Ac_{i4} = -2c_{i3}$, $c_{i4} = -2c_{i3}/A$ and from (9) there results $A = \pm 2$; in this case two of the discriminant conditions differ at most in sign. Hence we have the two solutions

$$(47)\quad \begin{array}{lll} I_a: & a=2,\ b=b,\ c=-b, & A=2,\ B=b,\ C=-b, \\ I_b: & a=2,\ b=b,\ c=b, & A=-2,\ B=-b,\ C=-b. \end{array}$$

Putting these values in (42) we obtain the respective conditions on the coefficients:

$$(48)\quad \begin{array}{r} (c_{i1}+c_{i2})(c_{j3}+c_{j4})-(c_{j1}+c_{j2})(c_{i3}+c_{i4})=0 \\ (c_{i1}+c_{i2})(c_{j3}-c_{j4})-(c_{j1}+c_{j2})(c_{i3}-c_{i4})=0 \\ (i, j=0, 1, \cdots, m). \end{array}$$

The assumption was made that $c_{i1}+c_{i2}\neq 0$. If $c_{i1}+c_{i2}=0$ for a particular i consider the P-curve P_i in the product $I_1 I_2$.

$$(49)\qquad \gamma'_i=[2(c_1\gamma)+(c_2\gamma)]c_{i2}+(c_1\gamma)c_{i1}+\gamma_i.$$

The coefficient of γ_i is $-2c_{i1}c_{i2}-c^2_{i2}-c^2_{i1}+1=1$; the P-curve thus has a multiplicity (-1) at F_i. To be geometric, the P-curve must reduce to the directions around F_i. For the indices j different from i we must then have

$$(50)\qquad (2c_{j1}+c_{j2}-c_{j1})c_{i2}=(c_{j1}+c_{j2})c_{i2}=0.$$

Either $c_{i2}=0$, $c_{i1}=0$ and $I_1 I_2$ belongs to $(m-1)$ instead of m points or $c_{j1}+c_{j2}=0$ and $(c_1\gamma)$, $(c_2\gamma)$ differ in sign only. These cases we do not wish to consider.

A similar treatment of solution *II* leads to the following conclusions with corresponding conditions on the coefficients:

$$(51)\quad \begin{array}{lll} II_a: & a=-2,\ b=b,\ c=b, & A=-2,\ B=b,\ C=b, \\ II_b: & a=-2,\ b=b,\ c=-b, & A=2,\ B=-b,\ C=b, \end{array}$$

$$(52)\quad \begin{array}{r} (c_{i1}-c_{i2})(c_{j3}-c_{j4})-(c_{j1}-c_{j2})(c_{i3}-c_{i4})=0, \\ (c_{i1}-c_{i2})(c_{j3}+c_{j4})-(c_{j1}-c_{j2})(c_{i3}+c_{i4})=0. \end{array}$$

Solutions *III* and *IV* of (41) duplicate the previous results.

Throughout the investigation of equations (39) the assumption has been made that $\rho\neq 0$. Consider the vanishing of ρ and its conjugates under the group (35):

$$(53)\quad \begin{array}{ll} (C_1)\ ABC+A^2+B^2+C^2-4=0, & (C_3)\ abC+a^2+b^2+C^2-4=0, \\ (C_2)\ aBc+a^2+B^2+c^2-4=0, & (C_4)\ Abc+A^2+b^2+c^2-4=0. \end{array}$$

The equations (C_2), (C_3), (C_4) are valid from another point of view; (C_4) for instance, is the factor ρ for the solution of the pair of equations (B_2), (B_3).

If the last three equations are solved for B, C, A respectively and these values substituted in (C_1) and all such possible selections are made, there result:

$$(54)\quad \begin{array}{ll} (C_5)\ \pm abc + a^2 + b^2 + c^2 - 4 = 0, & (C_7)\ \pm ABc + A^2 + B^2 + c^2 - 4 = 0, \\ (C_6)\ \pm AbC + A^2 + b^2 + C^2 - 4 = 0, & (C_8)\ \pm aBC + a^2 + B^2 + C^2 - 4 = 0. \end{array}$$

Using the plus sign we verify that

$$(55)\qquad (a - A)(b - B)(c - C) = 0$$

and using the minus sign that

$$(56)\qquad (a + A)(b + B)(c + C) = 0.$$

There are thus six cases to be considered according as

$$(57)\qquad a = A,\ b = B,\ c = C,\ a = -A,\ b = -B,\ c = -C.$$

Consider first $a = A$. $(C_1), \cdots, (C_8)$ become

$$(58)\quad \begin{array}{ll} ABC + A^2 + B^2 + C^2 - 4 = 0, & ABc + A^2 + B^2 + c^2 - 4 = 0, \\ AbC + A^2 + b^2 + C^2 - 4 = 0, & Abc + A^2 + b^2 + c^2 - 4 = 0. \end{array}$$

Subtracting these equations in pairs there arise the four possibilities

$$(59)\quad \begin{array}{l} b = B \\ c = C \end{array};\quad \begin{array}{l} b = B \\ AB + C + c = 0 \end{array};\quad \begin{array}{l} c = C \\ AC + b + B = 0 \end{array};\quad \begin{array}{l} AB + C + c = 0 \\ AC + B + b = 0 \\ Ac + B + b = 0 \\ Ab + C + c = 0. \end{array}$$

For the fourth possibility $b = B$, $c = C$ and two of the cases reduce to $a = A$, $b = B$, $c = C$. Placing these values in $(A_1) \cdots (B_1) \cdots (C_8)$ the solutions turn out to be special cases of (40) or lead to coincidences among the discriminant conditions. From (59) there remain the cases

$$(60)\quad \begin{array}{l} b = B \\ AB + C + c = 0 \end{array};\quad \begin{array}{l} c = C \\ AC + b + B = 0. \end{array}$$

We consider only the former case for which $(A_1), \cdots, (C_8)$ reduce to

$$(61)\qquad ABC + A^2 + B^2 + C^2 - 4 = 0, \qquad AB + C + c = 0,$$

and give the values

$$(62)\qquad 2B = -AC \pm [(A^2 - 4)(C^2 - 4)]^{1/2}, \quad c = -(AB + C).$$

In (34) let $\gamma = \gamma'$, $(c_i\gamma) = x_i$ and substitute to obtain

$$(63)\qquad (x_1x_4 - x_2x_3)[\pm(A^2-4)^{1/2}(C^2-4)^{1/2}] + (x_2x_4 + x_1x_3 + Ax_2x_3)(C-c) \equiv 0.$$

This identity may be satisfied in one of several ways. If it is satisfied by the vanishing of the coefficients, the situation reduces to $a = A$, $b = B$, $c = C$. Since one of the quadratics $x_1x_4 - x_2x_3$ and $x_2x_4 + x_1x_3 + Ax_2x_3$ is not a multiple of the other, we consider the possibility of a linear relation among the variables of the form

$$(64)\qquad x_4 = rx_1 + sx_2 + tx_3.$$

Again the possibilities give nothing new. Finally the variables may be twice related as

$$(65)\qquad x_3 = l_1x_1 + l_2x_2, \qquad x_4 = m_1x_1 + m_2x_2.$$

This situation was considered in § 2 (27) where it was found that the group generated by $(c_3\gamma)$, $(c_4\gamma)$ was all or part of that generated by $(c_1\gamma)$, $(c_2\gamma)$. In like fashion we may treat the other cases of (57) and we find no solutions other than (40) and (41). We sum the results of this section in

(66) *A necessary and sufficient condition for the permutability of two infinite cyclic groups generated respectively by the pairs of discriminant conditions* $(c_1\gamma)$ *and* $(c_2\gamma)$, $(c_3\gamma)$ and $(c_4\gamma)$ *all distinct, is either* (1) *that the bilinear forms be one of the aggregates*

	I_a	I_b	II_a	II_b
k_{12}	2	2	-2	-2
k_{13}	b	b	b	b
k_{14}	$-b$	b	b	$-b$
k_{34}	2	-2	-2	2
k_{24}	b	$-b$	b	$-b$
k_{23}	$-b$	$-b$	b	b

where b *is an arbitrary parameter and the coefficients* c_k $(k = 1, \cdots, 4)$ *satisfy for* I_a, I_b, II_a, II_b *respectively the following identities*

$$\begin{gathered}
(c_{i1} + c_{i2})(c_{j3} + c_{j4}) - (c_{j1} + c_{j2})(c_{i3} + c_{i4}) = 0,\\
(c_{i1} + c_{i2})(c_{j3} - c_{j4}) - (c_{j1} + c_{j2})(c_{i3} - c_{i4}) = 0,\\
(c_{i1} - c_{i2})(c_{j3} - c_{j4}) - (c_{j1} - c_{j2})(c_{i3} - c_{i4}) = 0,\\
(c_{i1} - c_{i2})(c_{j3} + c_{j4}) - (c_{j1} - c_{j2})(c_{i3} + c_{i4}) = 0,\\
(i, j = 0, 1, \cdots, m)
\end{gathered}$$

or (2) *that* $b = B = c = C = 0$.

4. *The Cremona group for* P_9^2. The determination* of the numbers n, r_i, s_j, α_{ij} for $\rho = 9$ is complete ([4]). To generate an abelian subgroup in that case two kinds of generators were used. One kind $c_{i1}^{\rho_i}$ $(i = 2, \cdots, 9)$ has the form

$$(67)\qquad \begin{aligned} &n = 36\rho_i^2 + 1, \quad r_1 = 6\rho_i(2\rho_i - 1), \quad r_i = 6\rho_i(2\rho_i + 1) \\ &r_k = s_j = 12\rho_i^2 \; (k, j = 2, \cdots, 9, k, j \neq i), \quad s_1 = r_i, \quad s_i = r_1. \end{aligned}$$

By replacing $2\rho_i$ by k_i this becomes the product of the involutions which arise from the discriminant conditions

$$(68)\qquad \begin{aligned} (c_i\gamma) &= 3k_i\gamma_0 - (k_i + 1)\gamma_1 \\ &\qquad - (k_i - 1)\gamma_i - k_i(\gamma_2 + \cdots + \gamma_{i-1} + \gamma_{i+1} + \cdots + \gamma_9), \\ (c'_i\gamma) &= -\gamma_1 + \gamma_i. \end{aligned}$$

It is readily verified that the conditions of (66) are satisfied.

There are seven distinct types of planar Cremona transformations on 9 points; we shall follow the notation in the reference cited. The two relations for a P-curve

$$(69)\qquad \sum_i \alpha_{ij} = 3s_j - 1, \qquad \sum_i \alpha^2_{ij} = s_j^2 + 1$$

lead for types *I, III, V, VII* to the following two relations

$$(70)\qquad (a)\ \sum_{i=1}^{9} \delta_i = 3\nu, \qquad (b)\ \sum_{i=1}^{9} \delta_i^2 = \nu^2 + 2\gamma.$$

For these four types we prove by direct substitution in the corresponding formulas

(71) *If* δ_i ν, γ *is a solution of* (70) *which corresponds to a transformation* T, *then* $\delta_i + \epsilon$, $\nu + 3\epsilon$, γ (ϵ *any integer positive, negative, or zero*) *is a solution of* (70) *which corresponds to the same transformation.*

In order to determine the transformations by a single set of integers $\delta_1, \cdots, \delta_9$ we determine ϵ so that ν is 0, 1, or -1 in accordance with the above theorem. Using these values for ν, in turn, equations (70) become, respectively,

$$(72)\qquad (a)\ \sum_{i=1}^{9} \delta_i = 0, 3, -3, \qquad (b)\ \sum_{i=1}^{9} \delta_i^2 = 2\gamma, 2\gamma + 1, 2\gamma + 1.$$

(73) *To write a transformation of one of the types I, III, V, VII, partition*

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* Corrections to the published formulas are as follows: for type III replace α_{jk} by $\gamma + \delta_j - \delta_k$, α_{li} by $\gamma - \epsilon_{123} + \delta_l - \delta_i$; for type VI replace α_{li} by $\gamma' + \nu - \epsilon'_{123} - \delta'_l - \delta'_i$, α_{ti} by $\gamma' + \nu - \epsilon'_{123} - \delta'_t - \delta'_i$; for type VII replace α_{jk} by $\gamma - \epsilon_{145} + \delta_j - \delta_k$.

one of the numbers $0, 3, -3$ *into nine integers* $\delta_1, \cdots, \delta_9$; *then determine* γ *from* (72); *these values, substituted in the proper formulas, yield existant transformations.*

For types *II, IV, VI* we replace ν by $\nu' - 1$ and proceed as above.

(74) *To write a transformation of one of the types II, IV, VI, partition one of the numbers* $2, -1, -4$ *into* $\delta'_1, \cdots, \delta'_9$; *then determine* γ' *from*

$$\sum_{i=1}^{9} \delta'^2_i = 2\gamma', \qquad \sum_{i=1}^{9} \delta'^2_i = 2\gamma' - 1, \qquad \sum_{i=1}^{9} \delta'^2_i = 2\gamma' \tag{75}$$

respectively. These values substituted in the proper formulas yield existent transformations.

5. *An abelian subgroup for* P_{10}^2. The following pairs of D-conditions obtained from those for nine points, satisfy the conditions of (66) and generate an abelian subgroup for ten points, depending on nine parameters:

$$\begin{array}{l}
4k_1\gamma_0 - 2k_1(\gamma_1+\gamma_2) - (k_1-1)\gamma_3 - (k_1+1)\gamma_4 - k_1(\gamma_5+\cdots+\gamma_{10}). \\
\qquad\qquad\qquad\qquad\qquad\qquad \gamma_3 - \gamma_4 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
4k_7\gamma_0 - 2k_7(\gamma_1+\gamma_2) - k_7(\gamma_3+\cdots+\gamma_8) - (k_7-1)\gamma_9 - (k_7+1)\gamma_{10} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma_9 - \gamma_{10} \\
4k_8\gamma_0 - (2k_8-1)\gamma_1 - (2k_8+1)\gamma_2 - k_8(\gamma_3+\cdots+\gamma_{10}). \\
\qquad\qquad \gamma_1 - \gamma_2 \\
(4k_9+1)\gamma_0 - 2k_9\gamma_1 - (2k_9+1)\gamma_2 - k_9\gamma_3 - (k_9+1)(\gamma_4+\gamma_5) - k_9(\gamma_6+\cdots+\gamma_{10}) \\
\qquad \gamma_0 - \gamma_2 - \gamma_4 - \gamma_5.
\end{array} \tag{76}$$

The two examples of § 4 and § 5 are sufficient to indicate that the general theorem of (66) does have application. The subgroup for P_9^2 has a finite number of conjugates whereas the one for P_{10}^2 is not so characterized. Could one exhibit a closed set of discriminant conditions for P_{10}^2 such as exist for P_9^2, one might hope to regain this property.

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SOME THEOREMS ON TRIODIC CONTINUA.*

By N. E. Rutt.

It is proposed in this paper to examine triodic continua † from a point of view somewhat different from any other now in print, and to prove for unbounded continua a statement resembling the triod theorem.‡ Two lemmas will be established first.

Lemma I. *If K is a compact indecomposable plane continuum and H is a compact plane continuum having in common with K a set T such that $H - T$ is connected and $K - T$ is nonvacuous, then there is a composant* § *of K containing no point of H.*

The proof is immediate if $T = H$ or $T = 0$, so suppose $T \neq H$ and $T \neq 0$. Moreover let the noncompact component of the plane complement of K contain $H - T$. Assume that the lemma is not true. As T is closed and compact it is the sum of a set of mutually exclusive compact continua no one of which has a complement among whose components is one which is compact and contains points of $K + H$. If to each element of this set all the bounded components of its complement are added, a collection results in one to one reciprocal correspondence with the original one, in which the elements are still mutually exclusive bounded continua having a closed sum. As no one of them divides the plane, they may serve as the only nondegenerate elements of an upper semicontinuous collection completely occupying the plane and providing a continuous transformation of the plane into itself.¶ If H, K, and T are converted into H_t, K_t, and T_t by this transformation, then it is clear that T_t is totally disconnected, that K_t is indecomposable, that all relations assumed in the hypotheses of the lemma between K and H exist also between K_t and H_t, and that some point of T_t is contained in every composant of K_t. But this implies that every point of T_t is arcwise accessible from the complement of K_t; in fact there exists a simple closed curve separating $K_t - T_t$ from $H_t - T_t$ and containing only the points T_t of $K_t + H_t$. But this is a contradiction.*

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† For definition see R. L. Moore, *Point Set Theory*, p. 254.

‡ R. L. Moore, "Concerning triods in the plane and the junction points of plane continua," *Proceedings of the National Academy of Sciences*, vol. 14, no. 1, pp. 85-88, Theorem I.

§ The term composant is used here to represent, as originally by Janiszewski, any maximal proper strongly connected subset of a continuum.

¶ R. L. Moore, "Concerning upper semicontinuous collections of continua," *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 416-428.

LEMMA II. *If K is a compact plane continuum containing the point k, and expressible as the sum of a set $[K_a]$ of continua each of which contains k and is a proper subset of K, then K is expressible as the sum of two of its proper subcontinua each of which contains k.*

If x, y, and z are any three distinct points of K and K is irreducible between x and z and between y and z then it is reducible between x and y, for let K_x and K_y be any two elements of $[K_a]$ containing x and y respectively and consider $K_x + K_y$. As K_x is a proper subcontinuum of K containing x it does not contain z, and for a similar reason K_y does not contain z. Thus as both K_x and K_y contain k and neither one contains z their sum is a proper subcontinuum of K containing $x + y$. Thus K is decomposable.

Omitting cases in which the lemma is evidently true, suppose then that K is the sum of the proper subcontinua A and B, where B does not contain k and A is irreducible between k and any point whatever of $A \cdot B$. Consider the components of $K - A$. If there are at least two of these and one of them is not contained in the limit sum of the others, let it be D. Then $A + D$ and $A + \overline{(B - D)}$ are subcontinua of K of the sort desired. On the other hand if every component of $K - A$ is contained in the limit sum of the rest, let γ be a component of $c(A)$ † and $[D_\gamma]$ be the nonvacuous collection of the components of $K - A$ contained in it. If $[D_\gamma]$ includes an element D_g such that two different components, γ_r and γ_s, of $\gamma \cdot c(D_g)$ contain points of B, then the theorem is true owing to the representation of K as the sum of the continua $A + D_g + \gamma_r \cdot B$ and $A + c(\gamma_r) \cdot B$. But in case no such two domains exist then the elements of $[D_\gamma]$ form a collection which is orderable ‡ in γ and if D_r and D_s are nonconsecutive elements of $[D_\gamma]$ both arcwise accessible from the domain upon whose boundary all of $[D_\gamma]$ have points, then, with $[D_\rho]$ and $[D_\sigma]$ the subclasses in $[D_\gamma]$ complementary to $D_r + D_s$, two elements of $[D_\gamma]$ being separated by D_r and D_s if and only if they come from different classes, a representation of K of the sort required will be

$$A + B \cdot c(\gamma) + D_r + D_s + \Sigma D_\rho \text{ and } A + B \cdot c(\gamma) + D_r + D_s + \Sigma D_\sigma.$$

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* S. Mazurkiewicz, "Sur les points accessibles des continus indecomposables," *Fundamenta Mathematicae*, vol. 14, pp. 107-115.

† In general $c(X)$ will be the complement of the point set X.

‡ N. E. Rutt, "On certain types of plane continua," *Transactions of the American Mathematical Society*, vol. 33, no. 3, pp. 806-816.

There remains only the case in which $B - B \cdot A$ is a connected set C_1. As the theorem will be true if $\bar{C}_1$ is expressible as the sum of two proper subcontinua both containing points of A suppose that it is not. If $\bar{C}_1$ is indecomposable, consider any element K_x of $[K_a]$ having a point in C_1. As K_x contains A but not K, it appears that $C_1 \cdot K_x$ is a proper subset of C_1 and $\bar{C}_1 \cdot K_x$ is a set each of whose components belongs to a composant of $\bar{C}_1$ and contains points of $A \cdot B$. By varying x it is seen that every composant of $\bar{C}_1$ contains a point of A, a contradiction of Lemma I.

Finally suppose $\bar{C}_1 = P_1 + Q_1$ where P_1 and Q_1 are proper subcontinua of $\bar{C}_1$ and $Q_1 \cdot A = 0$. Let $A_1 = A + P_1$. As in the case of A it may be concluded that $A_1 - A_1 \cdot Q_1$ is connected and that A_1 is irreducible between k and every point of $A_1 \cdot Q_1$. As in the case of B it may be shown that $C_2 = K - A_1$ is connected and that Q_1 is the sum of its proper subcontinua Q_2 and R_2 with $Q_2 \cdot A_1 = 0$, and so on. In general a well ordered sequence $Q_1, Q_2, Q_3, \cdots, Q_\omega, \cdots, Q_\eta, \cdots$ results. If the sequence has a last element let it be Q_λ, and if it has no last element let the product of its elements be Q_λ. If nondegenerate, Q_λ is necessarily indecomposable and the theorem follows. If Q_λ is a single point it must be the infinite product of the elements of the sequence. But in this case K is not reducible between k and Q_λ, as any continuum containing k and Q_λ must contain ΣA_η and $\Sigma A_\eta \supset K - Q_\lambda$.

Corollary. *If K is a compact continuum in the plane containing the subcontinuum k and expressible as the sum of a collection of proper subcontinua $[K_a]$ each of which contains k, then it is the sum of two proper subcontinua each of which contains k.*

The corollary follows easily from the preceding lemma by means of an upper semicontinuous transformation of space.

Theorem I. *If K is a plane bounded continuum not separating the plane and expressible for any point k of K as the sum of two of its proper subcontinua each of which contains k, then K is triodic.*

Let k be any point of K, and let K_a and K_b be subcontinua of K of the type assumed. The set $K_c = K_a \cdot K_b$ is connected as otherwise $K_a + K_b$ separates the plane. Consider now any point x of $K_a - K_c$, and let K_x be an irreducible subcontinuum of K_a containing x and K_c. If for each x such a continuum exists which is proper in K_a then it follows from the corollary above that the theorem is true, as K_a is expressible as the sum of the two continua K_p and K_q both containing K_c and neither one containing $K_a - K_c$, so that the continua K_p, K_q, and $K_b + K_p \cdot K_q$ exhibit K as triodic, the set $K_p \cdot K_q$ being clearly a continuum with a point in common with K_b.

On the other hand if there exists an x in $K_a - K_c$ such that the only subcontinuum of K_a containing x and K_c contains K_a; then consider x. By supposition K is the sum of two of its proper subcontinua X_r and X_s, each of which contains x. As K_a is irreducible between x and K_c, if $X_r \cdot X_b \neq 0$, then $X_r \supset K_a - K_c$, and consequently both X_r and X_s contain $K_a - K_c$, a set clearly connected, and have points in K_b including of course the continuum $D = K_b \cdot \overline{(K_a - K_c)}$. Thus $Y_r = K_b \cdot X_r$ and $Y_s = K_b \cdot X_s$ are continua neither one of which contains the other and both of which contain D, so that the continua Y_r, Y_s, and $\overline{(K_a - K_c)} + Y_r \cdot Y_s$ exhibit K as a triod.

Lemma III. *Let K be a bounded continuum, the domain δ be a component of $c(K)$ with boundary Δ, and F_1 and F_2 be two bounded continua such that*

(a) *$K \cdot F_1$, $K \cdot F_2$, $\delta \cdot F_1$, and $\delta \cdot F_2$ are all nonvacuous,*
(b) *$F_1 - K \cdot F_1$ and $F_2 - K \cdot F_2$ are connected but $(K + F_1) \cdot (K + F_2)$ is not, and*
(c) *$K \cdot F_1 + K \cdot F_2$ belongs to a proper subcontinuum D of Δ.*

Then $K + F_1 + F_2$ is a continuum F containing a triod.

Clearly the set $(D + F_1) + (D + F_2)$ separates δ between two points f_1 and f_2 which are not separated by either $K + F_1$ or $K + F_2$. Let A be an arc in δ connecting f_1 and f_2 but not intersecting F_2, and let C be a simple closed curve in δ containing $f_1 + f_2$ but no other points of $A + F_1$. Complementary to $A + C$ are three domains two of which must contain points of $(D + F_1) \cdot (D + F_2)$ belonging to different components of $c(A + C) \cdot F$ and having points of C as limits, that is having subsets in $F_2 \cdot c(K + F_1)$. But then these components contain subsets U and V which are subcontinua of F_2 with points in $D + F_1$ and in $c(K + F_1)$. Thus $\Delta + F_1$, $D + U + F_1$, $D + V + F_1$ contains a triod.

Lemma IV. *When K is a bounded continuum, the domain δ is a component of $c(K)$, and F is a closed set such that $H = \delta \cdot F$ has only a finite number of components then:*

(a) *If $\bar{H} \cdot K$ has three or more components, $K + F$ contains a triod,*
(b) *If $\bar{H} \cdot K$ has exactly two components and their sum is a subset of D, a proper subcontinuum of K, then $K + H$ contains a triod.*

In (a) if $\bar{H} \cdot K$ contains components F_1, F_2, F_3 let η_1, η_2, η_3 be mutually exclusive neighborhoods of these with simple closed curves as boundaries not intersecting $\bar{H} \cdot K$. Let K_1, K_2, K_3 be the components of $K + H$ in η_1, η_2, η_3 respectively containing F_1, F_2, F_3. Now $\bar{K}_1, \bar{K}_2, \bar{K}_3$ are mutually exclusive continua no one of which is contained in K, so $K + K_1$, $K + K_2$, $K + K_3$ clearly contains a triod.

- In (b) if the two components of $\bar{H} \cdot K$ are F_1 and F_2, form K_1 and K_2 as above. Then K, $D + K_1$, $D + K_2$ clearly contains a triod.

THEOREM II. *If K is a bounded continuum expressible for any point k as the sum of three of its subcontinua each one containing k, and no one contained in the sum of the other two, then K contains a triod.*

If for some point k there exist subcontinua K_1, K_2, K_3 as assumed in the theorem such that $K_1 \cdot K_2$, $K_3 \cdot K_1$, $K_2 \cdot K_3$ are all connected, that K contains a triod is readily seen. So suppose that $K_1 \cdot K_2$ is disconnected and let δ' be a component of $c(K_1 + K_2)$ containing points of K_3. Let Δ be the outer boundary of δ' and δ be the component of $c(\Delta)$ containing δ'. Clearly $\delta \cdot K \neq 0$. It is easy to find a triod in K when more than two components of $c(\Delta)$ contain points of K. When exactly two contain points of K, each must include just one component of $K \cdot c(\Delta)$ and a situation arises identical with one dismissed below in which one domain contains two distinct components. So it may as well be supposed that only δ of the components of $c(\Delta)$ contains any points of K. The many cases remaining will be carefully listed below, but a full discussion will be given only in a few of typical sorts. Whenever for given point k, the sets K_1, K_2, K_3 are continua of the type supposed in the theorem, then each of the three contains points, hereafter to be called *essential*, which are not contained by either of the other two.

The cases in which $\overline{K - \Delta} \not\supset \Delta$ *will be treated first.*

(A) The set $K - \Delta$ consists of more than two components. This case is obvious.

(B) The set $K - \Delta$ consists of a pair of components R and S whose limit sets G_r and G_s are mutually separated. Clearly G_r and G_s are connected (Lemma IV). If Δ contains a continuum D such that $D \cdot G_r \neq 0$, $D \cdot G_s \neq 0$, and $\Delta - (D + G_r + G_s) \neq 0$ use Lemma IV. Suppose that no such continuum as D exists. If in $\bar{R}$ there are two continua R_p and R_q each having points in both G_r and $c(G_r)$, then when $(R_p + G_r) \cdot (R_q + G_r)$ is disconnected Lemma III applies and when it is not $\Delta + R_p \cdot R_q$, $G_r + R_p$, $G_r + R_q$ clearly contains a triod unless either $R_p \supset R_q$ or $R_q \supset R_p$. So of any two subcontinua of $\bar{R}$ or of $\bar{S}$ having points in both Δ and $c(\Delta)$ one contains the other. Let $R \supset k$ and consider K_1, K_2, K_3. Among the components of $K_1 \cdot \bar{S}$, $K_2 \cdot \bar{S}$, and $K_3 \cdot \bar{S}$ is one containing $\bar{S}$. If it belongs to K_1, then $K_1 \supset \Delta + S$ and all essential points of K_2 and K_3 belong to R. Let

$P_1 = R \cdot K_1, P_2 = K_2 \cdot c(K_1)$; and $P_3 = K_3 \cdot c(K_1)$. Clearly P_1, P_2, and P_3 are connected and if $(\bar{P}_1 + P_2 + G_r) \cdot (\bar{P}_1 + P_3 + G_r)$ is disconnected then Lemma III applies, while if not then $P_2 \cdot P_3 + K_1$, $P_2 + P_1 + G_r$, $P_3 + P_1 + G_r$ contains a triod. In this case the theorem is true.

(C) The set $K - \Delta$ consists of a pair of components R and S whose limit sets G_r and G_s are not necessarily mutually separated. If either G_r or G_s is disconnected it consists of precisely two components. If either one is disconnected an argument resembling the proof of Lemma IV may be used, and if neither is then the methods of (B) apply.

(D) The set $K - \Delta$ is a single component H whose limit set in Δ is disconnected into two sets G_r and G_s, both evidently connected. Let k be a point of H, and let $\Delta \cdot K_i$ be the set of components, evidently not more than two in number, K'_i and K''_i. By methods employed in other connections all possibilities except two are easily dismissed. The two will be discussed.

(i) Suppose $\Delta = K'_1 + K'_2$ where $K'_1 \cdot G_r \neq 0$, $K'_1 \cdot G_s = 0$, $K'_2 \cdot G_r = 0$, and $K'_2 \cdot G_s \neq 0$. In order that $\Delta = K'_1 + K'_2$, the set $K'_1 \cdot K'_2$ must be disconnected. Accordingly $K'_2 \cdot (K'_1 + H)$ has at least three components and Lemma IV applies.

(ii) Suppose $\Delta = K'_1$. There are various subcases.

(a) All three of K_1, K_2, and K_3 have essential points in H. Consider the sets $P_1 = \delta \cdot K_1$, $P_2 = \delta \cdot K_2$, and $P_3 = \delta \cdot K_3$.

(1) If any one of these has a component with limits in both G_r and G_s, as Δ contains mutually exclusive subcontinua D_r and D_s with points in one and only one of G_r and G_s respectively, then, P being the component,

$$P + G_r + G_s + D_r, \quad P + G_r + G_s + D_s, \quad P + G_r + G_s + K_2 \cdot c(\Delta + P)$$

contains a triod.

(2) If none of these has such a component, let P'_1, P'_2, P'_3 represent the particular components of P_1, P_2, P_3 containing essential points. Two of these must have limit points in the same one of G_r and G_s. Suppose that P'_1 and P'_2 have limits in G_r. It is easily seen that K contains a triod unless $(G_r + P'_1) \cdot (G_r + P'_2)$ is connected and when it is connected then $\Delta + P'_1 \cdot P'_2$, $G_r + P'_1$, $G_r + P'_2$ contains a triod.

(b) Finally let K_1 have no essential points in H.

(1) If $\bar{H}$ contains a proper subcontinuum with points in both G_r and G_s, a triod can be found as in (1) above.

(2) If two components from the same or from different ones of P_1, P_2, P_3 have limits in the same one of G_r and G_s, triods may be found as in (2) above.

(3) Neither P_2 nor P_3 may have a component with limits in G_r and another with limits in G_s, since if one did then it would contain Δ and K_1 could have no essential points at all.

(4) When P_2 is connected and has limit points only in G_r and P_3 is connected and has limit points only in G_s, the existence of a triod in K is clear unless of any two subcontinua of P_2 or P_3 with points in Δ and in $c(\Delta)$, one necessarily contains the other. Under these circumstances let k be a point of $\Delta - (G_r + G_s)$ and consider with respect to this new k a new set of continua K_1, K_2, and K_3. One of these three must contain a point of $P_2 \cdot P_3$ and thus, as (b) (1) has already been treated, must contain all of $P_2 - P_2 \cdot P_3$ or $P_3 - P_2 \cdot P_3$, while the same one or some other must contain all the remainder of H. In either case K is the sum of just two of the three.

(E) The set $K - \Delta$ is a single component of H, and $\bar{H} \cdot \Delta$ is a proper connected subset of Δ. Unless of any two subcontinua of $\bar{H}$ with points in Δ and in $c(\Delta)$ one contains the other, the existence of a triod in K is evident. When of any two one does contain the other, let $H \supset k$ and $K_1 \supset H$, as it is clear that one of the three must contain H. Thus both K_2 and K_3 have essential points in Δ and it is possible to apply Lemma III.

Now the cases in which $\overline{K - \Delta} \supset \Delta$ *will be treated.*

(A) The set $K - \Delta$ consists of more than two components. This case is obvious.

(B) The set $K - \Delta$ consists of two components R and S with limiting sets G_r and G_s in Δ. If $\bar{G}_r$ were to contain two continua neither one of which contained the other, both of them containing points of Δ and of $c(\Delta)$ belonging to a proper subcontinuum of Δ, then a triod could readily be found in K.

(i) If $G_r \not\supset \Delta$ and $G_s \not\supset \Delta$ let $R \supset k$ and consider sets K_1, K_2, K_3 for this selection of K. Owing to comments just made above it is clear that the only situation remaining uncompleted requires one of these sets to contain R and another to contain S, so that K is the sum of just two of them, a contradiction.

(ii) If $G_r \supset \Delta$ and $G_s \not\supset \Delta$ let $S \supset k$ and among the related sets K_1, K_2, K_3 let $K_1 \supset S$. Suppose $P_2 = c(\bar{S}) \cdot K_2$ and $P_3 = c(\bar{S}) \cdot K_3$. If either of these were to be disconnected it would be evident that K contains a triod. If both are connected then whether or not $(P_2 + G_s) \cdot (P_3 + G_s)$ is connected it appears that K contains a triod upon using Lemma III.

(iii) Finally let $G_r \supset \Delta$ and $G_s \supset \Delta$. Let $\Delta \supset k$, and consider K_1, K_2, K_3.

It is clear that $c(\Delta)$ contains essential points of all three and that two of the three, say K_1 and K_2, have essential points in one, say R, of R and S. Let $P_1 = K_1 \cdot R$ and $P_2 = K_2 \cdot R$, both of them clearly connected sets. If $(P_1 + \Delta) \cdot (P_2 + \Delta)$ is a connected set D, then $S + \Delta + D$, $\Delta + D + P_1$, $\Delta + D + P_2$ contains a triod, and if it is not connected then Lemma III may be used.

(C). The set $K - \Delta$ consists of a single component H. Suppose $\Delta \supset k$ and consider K_1, K_2, and K_3. It is readily seen that H contains essential points of all three of these. Let $P_1 = c(\Delta) \cdot K_1$, $P_2 = c(\Delta + K_1) \cdot K_2$, and $P_3 = c(\Delta + K_1 + K_2) \cdot K_3$. It is clear from Lemma IV that $P_i \cdot P_{i+1}$ $(i = 1, 2, 3)$ has at most two components, G'_i and G''_i, and from this fact and other considerations that P_i has at most two components, that when it has two, say P'_i and P''_i, the limit set of each is a continuum and when they are distinct they must be G'_i and G''_i respectively, and that when it has only one the limit set is either a continuum or a pair of mutually exclusive continua, namely G'_i and G''_i.

(i) Let P'_1 and P''_1 exist.

(a) When both P'_2 and P''_2 exist let F'_2 and F''_2 be subcontinua of $\bar{P}_1$ whose sum does not include $P_1 \cdot c(G'_2 + G''_2)$, each containing points of Δ, and one in addition containing points of $\bar{P}'_2$ and the other of $\bar{P}''_2$. It is clear that both of these exist, but they may of course be subsets of Δ. Then $\Delta + F'_2 + F''_2 + \bar{P}'_2 + G''_2$, $\Delta + F'_2 + F''_2 + G'_2 + \bar{P}''_2$, $\Delta + K_1$ contains a triod.

(b) When both G'_2 and G''_2 exist but P''_2 does not, obtain mutually exclusive subcontinua H'_2 and H''_2 of $\bar{P}'_2$ containing points of P'_2 and of G'_2 and G''_2 respectively, and then in the sentence just above replace $\bar{P}'_2$ by $G'_2 + H'_2$, and $\bar{P}''_2$ by $G''_2 + H''_2$.

(c) When neither G''_2 nor P''_2 exist examining the relations of P_3 with the rest of K reduces all cases occurring to ones essentially equivalent to the two just completed except when neither G''_3 nor P''_3 exist. In this case let F'_2 and F''_3 be subcontinua of $\bar{P}'_1$ and $\overline{P'_1 + P_2}$ respectively having a sum which contains neither $P_1 \cdot c(G'_2 + G'_3)$ nor $P_2 \cdot c(G'_3)$, each one containing a point of Δ, and the first containing a point of G'_2 while the second contains one of G'_3. Clearly these continua exist as the position of k and the distribution of the essential points of K_1, K_2, K_3 show easily. They may of course reduce considerably in special cases, even down to subsets of Δ. Then

$$\Delta + F'_2 + G'_2 + F'_3 + G'_3 + P_1, \quad \Delta + F'_2 + \bar{P}'_2 + F'_3 + G'_3,$$
$$\Delta + F'_2 + G'_2 + F'_3 + \bar{P}'_3$$

contains a triod.

(ii) Let G'_1 and G''_1 exist but not P''_1.

(a) When both P'_2 and P''_2 or both G'_2 and G''_2 exist, the sets F'_2 and F''_2 may be constructed as in (a) and (b) above, and no new cases needing special mention arise.

(b) When neither P''_2 nor G''_2 exist, depending upon whether both P'_3 and P''_3 or both G'_3 and G''_3, or whether neither P''_3 nor G''_3 exist, a pair of continua F'_3 and F''_3 (like F'_2 and F''_2 except that it must be specified that their sum does not contain $(P_1 + P_2) \cdot c(G'_3 + G''_3)$), or a pair of continua F'_2 and F'_3 (as in (c) just above) may be found, and then a triod is contained either in

$\Delta + P_1 + P_2$, in $\Delta + F'_3 + F''_3 + P'_3 + G''_3$, in $\Delta + F'_3 + F''_3 + G'_3 + P''_3$, in this same set with $\bar{P}'_3$ replaced by $H'_3 + G'_3$ and $\bar{P}''_3$ by $H''_3 + G''_3$, or in the set of the last sentence of (i) (c).

(iii) Finally let neither G''_1 nor P''_1 exist.

(a) When either both of P'_2 and P''_2 or both of G'_2 and G''_2 exist upon regarding momentarily $\Delta + P_1$ as Δ and reviewing (i) (a) and (i) (b) it is seen that the arguments used there are applicable here also.

(b) When neither P''_2 nor G''_2 exist but either both of P'_3 and P''_3 or both of G'_3 and G''_3 exist then the methods of (ii) (b) may be used without important change.

(c) Finally when none of P''_2, G''_2, P''_3, G''_3 exist, the procedure of (i) (c) may be used again.

The theorem is seen to be proved at last.

Remarks. A plane bounded dendron or acyclic continuous curve having at least three end points is a convenient simple example of a set satisfying the hypotheses of Theorem I. A plane bounded cyclicly connected continuous curve which is not a mere simple closed curve is a convenient example of a set satisfying the hypotheses of Theorem II. It is evident that each of these sets contains at least one simple triod and therefore is triodic.

Neither Lemma I nor Lemma II is true as stated in space of more than two dimensions. This may be seen by means of easy examples. The condition in Theorem I is obviously necessary that K be a triod whether K divides the plane or not. That it is not, however, sufficient if K divides the plane is made evident at once by any simple closed curve. Upon examining the simple continuous arc it is seen that the condition of Theorem I must apply to all points of K except at most one, whereas that one exception is actually allowable the argument itself shows readily enough. The condition of Theorem II is not necessary. A simple triod consisting of three arcs has three points, namely the free ends of the arcs, which do not admit being used as the point k.

The triod theorem referred to at the beginning of the paper states that no plane collection of mutually exclusive bounded triodic continua can be uncountable. Theorem III below makes a somewhat similar statement concerning a plane collection of unbounded continua. Professor Kuratowski has described a complicated continuum* not containing a bounded subcontinuum which is triodic and satisfying the supposition made in this theorem about the continua involved. Other much simpler examples † serving equally well exist, however, and concerning collections of continua of either of the types referred to, the theorem states that the elements can not be both uncountable in number and mutually exclusive. It will be seen that Theorem III is not true if the collection includes elements separating the plane, and this any uncountable system of parallel straight lines in the plane shows at once.

LEMMA V. *If K_a and K_b are unbounded plane continua whose sum does not divide the plane, then the components of $K_a \cdot K_b$ are all unbounded.*

If one of the components were bounded and the plane were inverted using as center of inversion a point not belonging to $K_a + K_b$ then the set into which $K_a \cdot K_b$ would invert would be disconnected. Such being the case $K_a + K_b$ must have divided the plane before inversion. When $K_a + K_b$ occupies the whole plane other easy choices may be made for the center of inversion when inversion is useful.

LEMMA VI. *If K is an unbounded plane continuum not dividing the plane and is the sum of two of its unbounded subcontinua K_a and K_b neither one of which contains the other, then K contains two distinct points a and b and a closed set H not containing either a or b such that any simple closed curve C enclosing a and b and excluding H has upon it two distinct points m and n each of which belongs to an unbounded component of $K \cdot ci(C)$.‡*

As K is a continuum, $K_a \cdot K_b \neq 0$, and $K_a \cdot K_b$ is a closed set. Let it be H. Let a and b be points respectively of $K_a - H$ and $K_b - H$, and let C be any simple closed curve enclosing $a + b$ and not intersecting H. Then $e(C) \supset H$, and $ce(C) \cdot K$ consists of components no one of which contains both a and b. Now both K_a and K_b have points both inside and outside C so in $ci(C)$ each of these has an unbounded component with a point on C

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* B. Knaster and C. Kuratowski, "Sur les continus non-bornés," *Fundamenta Mathematicae*, vol. 5, see figure on page 43.

† J. H. Roberts, "Concerning atriodic continua, *Monatshefften für Mathematik und Physik*, 37, Band 2, example on page 223.

‡ When C is a simple closed curve in the plane $i(C)$ is its interior, $e(C)$ is its exterior, and $ci(C)$ and $ce(C)$ are the complements of these sets.

which is not contained in the other. Let these points be m and n, the corresponding components just identified being M and N whether distinct or not.

THEOREM III. *If* $[W_a]$ *is a collection of unbounded plane continua none of them separating the plane and each one having two unbounded subcontinua neither one of which contains the other, then the elements of the collection are not both mutually exclusive and uncountable in number.**

For the typical element W of $[W_a]$ let U and V be a typical pair of subcontinua of the sort assumed. Suppose that the theorem is not true. Then for uncountably many of the distinct elements of $[W_a]$ either (1) $U \cdot V \neq 0$, or (2) $U \cdot V = 0$.

If (1) occurs let $U + V = K$ and consider the uncountable collection of mutually exclusive continua $[K_\beta]$. Let $[H_\beta]$, $[a_\beta]$, and $[b_\beta]$ be the collections of sets and points derived in the proof of Lemma VI. It is readily seen that circles C_a and C_b each contained in the exterior of the other and uncountable subcollections $[H_\gamma]$, $[a_\gamma]$, $[b_\gamma]$ out of corresponding elements of a subcollection $[K_\gamma]$ of $[K_\beta]$ exist such that $i(C_a)$ contains all of $[a_\gamma]$ and $i(C_b)$ contains all of $[b_\gamma]$, and $e(C_a) \cdot e(C_b)$ contains all of $[H_\gamma]$. Let K_p and K_q be elements of $[K_\gamma]$ neither of which contains a or b. As $K_p + K_q$ does not separate the plane it does not separate a from b and so there is a simple closed curve enclosing $a + b$ and excluding $K_p + K_q$. Out of it and $C_a + C_b$ may be constructed a simple closed curve C such that $i(C) \supset (i(C_a) + i(C_b))$, that C consists of a subarc A of C_a, a subarc B of C_b, and two arcs P and Q complementary to these having only their end points in $C_a + C_b$, and that $i(C) \cdot c(i(C_a) + i(C_b))$ contains no points of $K_p + K_q$. Consider now M_p and N_p, M_q and N_q as determined in Lemma VI. Clearly $M_p \cdot C$ and $M_q \cdot C$ are on A, and $N_p \cdot C$ and $N_q \cdot C$ are on B, and because of this it may readily be seen that no component of any element of $[H_\gamma]$ can contain a point of $P + Q$ so that $e(C)$ contains all of $[H_\gamma]$.

It is now possible to use Lemma VI to advantage. Points m_g and n_g exist for every element K_g of $[K_\gamma]$, and no such pair of points can be separated on C by any other pair. Thus C contains an uncountable infinity of mutually exclusive arcs. This contradiction shows (1) impossible. In (2) the points of the collections $[a_\beta]$ and $[b_\beta]$ must be chosen from corresponding elements of the collections $[U_\beta]$ and $[V_\beta]$. The case can then be reduced to an absurdity along much the same but less complicated lines as above. The contradiction is thus general, and the theorem proved.

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* As this theorem has been anticipated in an example by J. H. Roberts, "Concerning atriodic continua," *loc. cit.*, its proof will be reduced here to a summary.

CYCLIC ELEMENTS OF HIGHER ORDERS.

By G. T. Whyburn.

In this paper it is purposed to develop a decomposition of closed and bounded (compact) subsets M of n-dimensional euclidean space R^n into sets E_r for each $r = 0, 1, 2, \cdots$ which is analogous to the decomposition of locally connected continua into true cyclic elements.* The sets E_r are defined with respect to the r-dimensional complete cycles in M, and in case $r = 0$ they are maximal subcontinua of M which are not separated by any one point. Thus in case M is locally connected, the sets E_0 are exactly the true cyclic elements of M; and since the properties exhibited by the sets E_r for arbitrary r with respect to r-cycles are parallel to those exhibited by the sets E_0 with respect to 0-cycles, these sets will be called the *r-th order cyclic elements* of M. Most of the results established below for the sets E_r reduce to or follow from known properties of the true cyclic elements in case M is locally connected and $r = 0$.

The usual terminology and notation of point set theory will be employed, the diameter of a set X being denoted by $\delta(X)$, etc. The letter M will designate a compact subset of R^n as indicated above. The results themselves do not seem to require anything more of M than that it be compact and metric, but the proofs lean heavily on linkings and on the duality theorem, so that such an imbedding requirement is necessary for the present at least. The notions of a complete cycle and its associated concepts are used in the sense of Vietoris and Alexandroff.† Complete cycles in M are denoted by γ^r and ordinary geometric cycles in $R^n - M$ are denoted by Γ^r, in each case the superscript indicating the dimensionality of the cycle. Homologies are indicated by the symbol $\sim$ and bounding relations by $\rightarrow$. All bounding relations and cycles are understood to be taken modulo 2, and orientation of cycles plays no part in our treatment. A 0-cycle Γ^0 is understood to be any *even number* of 0-cells (i. e., points).

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* See my paper "Concerning the structure of a continuous curve," *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194; also Kuratowski and Whyburn, "Sur les éléments cycliques et leurs applications," *Fundamenta Mathematicae*, vol. 16 (1930), pp. 305-331, and note bibliography at the end of this article. That decompositions such as we obtain in the present paper might be possible has been predicted by R. L. Wilder (see his first paper cited below).

† See Vietoris, "Uber den höheren Zusammenhang kompakter Räume . . . ," *Mathematische Annalen*, vol. 97, pp. 545-572; Alexandroff, "Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimensionen," *Annals of Mathematics*, ser. 2, vol. 30 (1928), pp. 101-187; see also Lefschetz, "Topology," *American Mathematical Society Colloquium Publications*, vol. 12 (1930).

1. The sets T_r and E_r. A closed set of points which carries no essential * complete r-dimensional cycle (i. e., no γ^r) will be called a *T_r-set,* or simply a T_r. Thus any closed subset of a T_r-set is itself a T_r-set. A non-degenerate subset X of M will be called an E_r-set, or merely an E_r, provided X is not separated (disconnected) by any T_r in X and X is saturated in M relative to this property. The existence of an E_r in M containing any given non-degenerate set X having the property not to be separated by any T_r has been established by the author as a particular case of a general existence theorem on maximal sets.† However, for convenience of reference we list this result together with some properties of the sets E_r as

(1. 1) *For any non-degenerate subset X of M which is not separated by any T_r, there exists a unique E_r in M containing X. Furthermore, each E_r is a continuum; and the common part of any two distinct sets E_r is a T_r-set.*

(1. 2) *Suppose the compact set B carries the homology $\gamma^r = \gamma_1{}^r + \gamma_2{}^r \sim 0$ and that X is a closed subset of B giving the separation $B - X = B_1 + B_2$, where $\gamma_i{}^r \subset B_i + X$, $(i = 1, 2)$. Then if every r-cycle in X is ~ 0 in X,* i. e., *if* ‡ $p^r(X) = 0$, *we have $\gamma_i{}^r \sim 0$ in $B_i + X$, $(i = 1, 2)$.*

This result is a generalization of a lemma proved by the author in the article just cited, and it is established by the same argument as was given to prove the lemma.§

(1. 3) *If $\gamma^r \subset E_r$ and $\gamma^r \sim 0$ in M, then any irreducible membrane B carrying the homology $\gamma^r \sim 0$ in M is contained in E_r so that $\gamma^r \sim 0$ in E_r.*

This has also been proven by the author in the above mentioned article. It will be noted that it follows readily from (1. 2).

(1. 4) *Any carrier $C \subset M$ of an essential γ^r contains a subcontinuum K which is not disconnected by any T_{r-1}-set. Thus $K \subset$ some E_{r-1}.*

We may suppose that γ^r is not ~ 0 in C. Then $p^r(C) > 0$, and since by the duality theorem,* $p^r(C) = p^{n-r-1}(R^n - C)$, there exists a Γ^{n-r-1} in $R^n - C$ which is linked with C. Now C contains a subset K which is irreducibly linked with Γ^{n-r-1}. By a theorem of Alexandroff,† any set F which disconnects K must have $p^{n-(n-r-1)-2}(F) = p^{r-1}(F) > 0$; and thus no such set F is a T_{r-1}-set.

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* A γ^r is essential if it has at least one carrier in which it is not ~ 0.

† See *Bulletin of the American Mathematical Society,* vol. 40 (1934).

‡ Here $p^r(X)$ denotes the r-th connectivity (Betti or Brouwer) number of the set X. See, for example, Alexandroff, *loc. cit.*; also Wilder, *Bulletin of the American Mathematical Society,* vol. 38 (1932), pp. 649-692.

§ See my paper, *loc. cit.*

(1.41) *If M contains an essential γ^r, then M contains non-degenerate sets E_{r-1}.*

(1.42) *Any closed subset K of M which is irreducibly linked by some Γ^{n-r-1} in $R^n - K$ is contained wholly in some single E_{r-1}.*

(1.5) *Every T_{r-1}-set is a T_r-set, $(r = 1, 2, 3, \cdots)$.*

Suppose, on the contrary, that some T_{r-1} contains an essential r-cycle. Then by (1.41), T_{r-1} has at least one set E_{r-1} of itself. But then if F is any set separating E_{r-1}, F carries some essential $(r-1)$-cycle. Thus T_{r-1} carries an essential $(r-1)$-cycle, contrary to definition.

(1.6) *For each r and each E_r in M there exists a sequence of sets $E_0, E_1, \cdots, E_{r-1}$ such that $E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_{r-1} \supset E_r$.*

This results at once from (1.5). For since no E_r is disconnected by any T_r and since every T_{r-1} is a T_r, therefore E_r is not disconnected by any T_{r-1}; and hence it lies wholly in some single E_{r-1}. Similarly E_{r-1} lies in some E_{r-2}, and so on.

(1.7) *For each r, dim $E_r \geq r + 1$.*‡

Proof. This is immediate for $r = 0$, since each E_0 is a non-degenerate continuum and hence $\dim E_0 \geq 1$. Suppose it to have been proven for $r = k - 1$. Then for each E_{k-1}, we have

(i) $$\dim E_{k-1} \geq k.$$

Now let us consider any set E_k. By definition of E_k, any set F which separates E_k must carry some essential k-cycle. Thus by (1.41) any such set F contains a set $E^F{}_{k-1}$ of itself. By (i) we have

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* See Alexandroff, *loc. cit.*, p. 156; Frankl, *Wiener Akademie Anzeiger*, Abt., 2A, vol. 136 (1927), pp. 689-699; Lefschetz, *Annals of Mathematics*, vol. 29 (1928), pp. 232-254; Alexander, *Transactions of the American Mathematical Society*, vol. 23 (1922), pp. 333-349.

† See Alexandroff, *loc. cit.*, p. 153, Satz IV.

‡ The notation *dim X* means the Menger-Urysohn dimension of the set X.

(ii) $$\dim F \geqq \dim E^F{}_{k-1} \geqq k.$$

Thus any set separating E_k is of dimension $\geqq k$, and consequently $\dim E_k \geqq k+1$.

Thus the truth of our theorem for $r=k-1$ implies its truth for $r=k$. Accordingly, by complete induction, the theorem is established.

2. Connectivity Numbers. The formula $p^r(M) = \Sigma\, p^r(E_{r-1})$.

(2.1) *If F is closed and $F = \sum_{i=1}^{\infty} T^i{}_{r-1}$, then F is a T_r.*

For suppose on the contrary that F contains an essential r-cycle. Then by (1.41) there exists at least one set $E^F{}_{r-1}$ in F. Thus we have

$$E^F{}_{r-1} = \sum_{i=1}^{\infty} E^F{}_{r-1} \cdot T^i{}_{r-1}.$$

But each term in this sum is a closed set which must be non-dense on $E^F{}_{r-1}$, since any open subset of $E^F{}_{r-1}$ contains sets disconnecting $E^F{}_{r-1}$ which cannot be contained in any T_{r-1}; and clearly this is impossible.

(2.11) $\sum_{i=1}^{k} T^i{}_{r-1}$ *is a T_r, for every k.*

We proceed to establish the following formula for the r-th connectivity number of the sum of any finite number of sets E_{r-1}:

(2.2) $$p^r\Big(\sum_{i=1}^{k} E^i{}_{r-1}\Big) = \sum_{i=1}^{k} p^r(E^i{}_{r-1}).$$

This is trivial in case $k=1$. Suppose it proven for $k=m-1$. Then $p^r(\sum_{i=1}^{m-1} E^i{}_{r-1}) = \sum_{i=1}^{m-1} p^r(E^i{}_{r-1})$. Now

$$E^m{}_{r-1} \cdot \sum_{i=1}^{m-1} E^i{}_{r-1} = \sum_{i=1}^{m-1} E^m{}_{r-1} \cdot E^i{}_{r-1} = \sum_{i=1}^{m-1} T^i{}_{r-1},$$

since by (1.1) the common part of any two sets E_{r-1} is a T_{r-1}. Thus by (2.1),

$$p^r\Big(E^m{}_{r-1} \cdot \sum_{i=1}^{m-1} E^i{}_{r-1}\Big) = p^r\Big(\sum_{i=1}^{m-1} T^i{}_{r-1}\Big) = 0.$$

Whence, by an addition theorem of Mayer,* we have

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* See W. Mayer, *Monatshefte für Mathematik und Physik*, vol. 36 (1929), p. 40, formula (96). This formula has been extended to complete cycles by E. R. van Kampen and P. M. Swingle.

(i) $$p^r(\sum_{i=1}^{m} E^i_{r-1}) \geq \sum_{i=1}^{m} p^r(E^i_{r-1}).$$

Now, denoting $\sum_{i=1}^{m-1} E^i_{r-1}$ by K, let us take any γ^{r-1} in $K \cdot E^m_{r-1}$ which is ~ 0 in K. Let B be an irreducible membrane in K carrying the homology $\gamma^{r-1} \sim 0$. By (1. 3), we have $B \subset E^m_{r-1}$. Whence $\gamma^{r-1} \sim 0$ in $K \cdot E^m_{r-1}$. Thus any γ^{r-1} in $K \cdot E^m_{r-1}$ which is ~ 0 in K is ~ 0 in $K \cdot E^m_{r-1}$; consequently, by the addition theorem quoted above we have

(ii) $$p^r(\sum_{i=1}^{m} E^i_{r-1}) = p^r(K + E^m_{r-1}) \leq \sum_{i=1}^{m} p^r(E^i_{r-1}).$$

This together with (i) gives the desired formula for $k = m$. Thus by induction the formula is established for all values of k.

(2. 3) *If $\gamma_i^r \subset E_{r'}^i$ $(i = 1, 2, 3, \cdots, k;\ r' \leq r)$, where the sets $E_{r'}^i$ are all distinct, and $\gamma_i^r \not\sim 0$ in $E_{r'}^i$, then $\sum_{i=1}^{k} \gamma_i^r \not\sim 0$ in M.*

Let us suppose this is not so, and let us choose k as the least integer for which it fails. It follows readily from (1. 2) and (1. 3), that $k > 1$. By supposition there exists a set of cycles $[\gamma_i^r]$, where $\gamma_i^r \subset E_{r'}^i$, $\gamma_i^r \not\sim 0$ in $E_{r'}^i$, $(i = 1, 2, \cdots, k;\ r' \leq r)$, and $\gamma^r = \sum_{i=1}^{k} \gamma_i^r \sim 0$ in M. Let B be an irreducible membrane carrying the homology $\gamma^r \sim 0$ in M, and let N denote the set $B + \sum_{i=1}^{k} E_{r'}^i$. Since $k > 1$, there must exist some $T_{r'}$-set $T_{r'}$ in N giving a separation $N - T_{r'} = N_1 + N_2$. Since for each i, $E_{r'}^i - T_{r'} \cdot E_{r'}^i$ is connected, each of the sets $E_{r'}^i$ is contained wholly in either $N_1 + T_{r'}$ or $N_2 + T_{r'}$. Thus γ^r breaks up into two cycles δ_1^r and δ_2^r, where $\delta_i^r \subset N_i + T_{r'}$, $(i = 1, 2)$, where δ_1^r is made up of all those cycles γ_i^r such that $E_{r'}^i \subset N_1 + T_{r'}$ and δ_2^r consists of all the remaining ones. Since every $T_{r'}$ is a T_r, by (1. 2) we have $\delta_i^r \sim 0$ in $N_i + T_{r'}$, $(i = 1, 2)$. Since k was chosen a minimum, either δ_1^r or δ_2^r, say δ_1^r, must be identical with γ^r. But then by (1. 2) we have $\gamma^r \sim 0$ in $B \cdot (N_1 + T_{r'})$, contrary to the fact that B is an irreducible carrier of the homology $\gamma^r \sim 0$. Thus the supposition that our theorem is not true leads to a contradiction.

(2. 4) *If there are only a finite number of sets E_{r-1} in M such that $p^r(E_{r-1}) > 0$ and if K denotes the sum of all such sets E_{r-1}, then $p^r(M) \leq p^r(K)$.*

Suppose, on the contrary, that $p^r(M) > p^r(K)$. Then by the duality theorem quoted above, $p^{n-r-1}(R^n - M) > p^{n-r-1}(R^n - K)$. Thus if Γ_1^{n-r-1},

$\Gamma_2^{n-r-1}, \cdots, \Gamma_t^{n-r-1})$ is a base for the $(n-r-1)$-cycles in $R^n - M$, then since $t > p^{n-r-1}(R^n - K)$, the cycles in this base cannot be independent in $R^n - K$. Accordingly there exists some combination of them, say Γ^{n-r-1}, which is ~ 0 in $R^n - K$. This means that Γ^{n-r-1} links M but does not link K. This is impossible, for if any Γ^{n-r-1} links M, it irreducibly links some subset F of M; and by (1.42) we have $F \subset$ some $E_{r-1} \subset K$, so that Γ^{n-r-1} links K. This contradiction proves our result.

(2.5) $p^r(M) = \Sigma\, p^r(E_{r-1})$, *the summation being extended over all sets* E_{r-1} *in* M.

Proof. It follows by (2.3) that we have

$$\text{(i)} \qquad p^r(M) \geqq \Sigma\, p^r(E_{r-1}).$$

For, taking $r' = r - 1$ in that theorem, it follows that if we choose in each E_{r-1} a basis (γ_i^r), $[i = 1, 2, \cdots p^r(E_{r-1})]$, for the r-cycles in that set, then there will exist no relation of the form $\gamma^r \sim 0$ between any of the whole aggregate of cycles $[(\gamma_i^r)]$ obtained for all sets E_{r-1} in M. Thus the cycles in this aggregate are independent in M and (i) follows.

Now to prove the reverse inequality, we note that we can suppose $\Sigma\, p^r(E_{r-1})$ to be finite; for if it is infinite, then by (i) it follows that $p^r(M)$ is infinite and this gives the desired relation. Thus by (i) there can be only a finite number, say k, of sets E_{r-1} with $p^r(E_{r-1}) > 0$. Let K denote the sum of all such sets E^i_{r-1}. Then by (2.2) we have

$$\text{(ii)} \qquad p^r(K) = \sum_{i=1}^{k} p^r(E^i_{r-1}).$$

Now by (2.4) together with (ii) we obtain

$$\text{(iii)} \qquad p^r(M) \leqq p^r(K) = \sum_{i=1}^{k} p^r(E^i_{r-1}).$$

Whence

$$\text{(iv)} \qquad p^r(M) \leqq \sum p^r(E_{r-1}),$$

since for any E_{r-1} not in K, $p^r(E_{r-1}) = 0$.

The inequalities (i) and (iv) give our desired formula.

(2.51) *In order that* $p^r(M) = 0$ *it is necessary and sufficient that* $p^r(E_{r-1}) = 0$, *for each* E_{r-1} *in* M.*

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* For the case where M is a locally connected continuum and $r = 1$, this reduces to a result recently established by Borsuk; see *Fundamenta Mathematicae*, vol. 20 (1933), p. 230. In connection with this same case, see R. L. Wilder, *Annals of Mathematics*, vol. 34 (1933), pp. 441-449.

(2.52) *In order that M should separate R^n it is necessary and sufficient that some E_{n-2} in M should separate R^n.*

This follows at once from (2.51) together with the duality theorem quoted above. For, taking $r = n - 1$ and applying (2.51), we have that $p^{n-1}(M) > 0$ if and only if $p^{n-1}(E_{n-2}) > 0$ for some E_{n-2}. Thus by the duality theorem, $p^0(R^n - M) > 0$ if and only if $p^0(R^n - E_{n-2}) > 0$ for some E_{n-2}, and this is equivalent to (2.52).

(2.6) *For each $r' \leq r$, $p^r(M) \geq \Sigma\, p^r(E_{r'})$, the summation being extended over all sets $E_{r'}$ in M.*

This follows from (2.3) by the same argument as given to prove (i) under (2.5).

(2.61) *If $p^r(M) = 0$, then $p^r(E_{r'}) = 0$ for each $r' \leq r$ and each set $E_{r'}$ in M.*

(2.7) *For each $r' < r$, $p^r(M) = \Sigma\, p^r(E_{r'})$, the summation being extended over all sets $E_{r'}$ in M.*

This is an immediate consequence of (2.5) and (1.6).

3. E_r-sums. A set X is said to be *γ^r-connected* provided that every γ^r in X is ~ 0 in X, i. e., provided $p^r(X) = 0$; a closed, connected, and γ^r-connected set X will be called a *γ^r-continuum.* A self-compact set X is said to be *locally γ^r-connected* provided that for each $\epsilon > 0$ a $\delta_\epsilon > 0$ exists such that every γ^r in X which is carried by some set of diameter $< \delta_\epsilon$ is ~ 0 in some subset of X of diameter $< \epsilon$.*

A closed subset K of M having the property that K contains every E_r in M such that $E_r \cdot K$ carries some essential γ^r will be called a *K_r-set,* or simply a K_r; in other words, if $K_r \cdot E_r \neq T_r$ always implies $E_r \subset K$, then K is a K_r. A γ^r-connected K_r will be called a *C_r-set* or simply a C_r. Finally, a closed subset A of M will be called an *A_r-set* or merely an A_r provided that for each γ^r in A, every irreducible membrane in M carying the homology $\gamma^r \sim 0$ in M is contained wholly in A. These sets K_r, C_r, and A_r are analogous to the so called *A-sets* in locally connected continua,† and indeed they all reduce to the A-sets in case M is a locally connected continuum and $r = 0$. On account of this analogy they may be thought of as E_r-sums.

By (1.1) and (1.3) we have at once

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* Compare with Alexandroff, *loc. cit.*, p. 181, and Lefschetz, *loc. cit.*

† See Kuratowski and Whyburn, *loc. cit.*

(3.01) *Every* E_r *is a* $K_r \cdot A_r$ (*i. e., both a* K_r *and an* A_r).

(3.1) *For any closed set* F *in* M *and any* K_r, *every* γ^r *in* $F \cdot K_r$ *which is* ~ 0 *in* F *and also in* K_r *is* ~ 0 *in* $F \cdot K_r$.

Suppose, on the contrary, that some γ^r in $F \cdot K_r$ is ~ 0 in F and in K_r but not in $F \cdot K_r$. Then by a result of Alexandroff,* there exists a Γ^{n-r-2} in $R^n - (F + K_r)$ such that $\Gamma^{n-r-2} \sim 0$ in $R^n - F$ and also in $R^n - K_r$ but $\Gamma^{n-r-2} \not\sim 0$ in $R^n - (F + K_r)$. In other words, Γ^{n-r-2} links $F + K_r$ but does not link either F or K^r. Now $F + K_r$ contains a subset B which irreducibly links Γ^{n-r-2}. By (1.42), B is contained wholly in some E_r. Now since Γ^{n-r-2} links B but links neither F nor K_r, B is not contained in either F or K_r. But then $E_r \cdot (K_r - F \cdot K_r) \neq 0 \neq E_r \cdot (F - F \cdot K_r)$. Thus $F \cdot K_r \cdot E_r$ disconnects E_r, and hence this set contains an essential γ^r. Thus $K_r \cdot E_r$ is not a T_r, and this gives $E_r \subset K_r$, contrary to the fact that B is not contained in K_r. Thus the supposition that (3.1) is not true leads to a contradiction.

(3.11) *If* F *is a* γ^r*-continuum in* M, *so also is every set* $F \cdot C_r$.

(3.12) *If* M *is* γ^r*-connected, every* E_r *is a* C_r.

(3.13) *If* M *is locally* γ^r*-connected, so also is every set* E_r.

(3.2) *Every* C_r *is an* A_r.

For let B be any irreducible membrane in M carrying the homology $\gamma^r \sim 0$, where $\gamma^r \subset C_r$. Then since $\gamma^r \sim 0$ in C_r and also in B, by (3.1), $\gamma^r \sim 0$ in $B \cdot C_r$. And since B is irreducible, this gives $B \subset C_r$.

(3.3) *The product of any number of sets* $\begin{Bmatrix} A_r \\ K_r \\ C_r \end{Bmatrix}$ *is a set* $\begin{Bmatrix} A_r \\ K_r \\ C_r \end{Bmatrix}$.

In other words, we have

$$\Pi A_r = A_r \tag{3.31}$$

$$\Pi K_r = K_r \tag{3.32}$$

$$\Pi C_r = C_r. \tag{3.33}$$

The first of these is an immediate consequence of the definition of an A_r. To prove (3.32), let E_r be any E_r-set with $E_r \cdot \Pi K_r \neq T_r$. Then we have $E_r \cdot K_r \neq T_r$ for every K_r in our collection. Whence, $E_r \subset K_r$ for every K_r, and this gives $E_r \subset \Pi K_r$.

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* See Alexandroff, *loc. cit.*, pp. 178-179.

To prove (3. 33), we note in the first place that since each C_r is a K_r, then by (3. 32), ΠC_r is a K_r; and by (3. 2) and (3. 31) it is also an A_r. Thus if we take any γ^r in ΠC_r and any irreducible membrane B in some one C_r of our collection so that $\gamma^r \sim 0$ in B, we have $B \subset \Pi C_r$. Whence $\gamma^r \sim 0$ in ΠC_r. Thus ΠC_r is γ^r-connected and hence is a C_r.

(3. 4) *If M is* $\left\{\begin{array}{l}\gamma^r\textit{-connected}\\ \textit{locally } \gamma^r\textit{-connected}\end{array}\right\}$, *every A_r is* $\left\{\begin{array}{l}\gamma^r\textit{-connected}\\ \textit{locally } \gamma^r\textit{-connected}\end{array}\right\}$.

This is an immediate consequence of the definition of A_r.

(3. 41) *If M is γ^r-connected, every $K_r \cdot A_r$ is a C_r.*

(3. 42) *If M is locally γ^r-connected, so also is every C_r.*

This is a consequence of (3. 2) and (3. 4).

(3. 5) *If X is any* $\left\{\begin{array}{l}\text{(a) } A_r\\ \text{(b) } K_r\\ \text{(c) } C_r\end{array}\right\}$ *and T_r is any T_r-set in X giving a separation $X - T_r = X_1 + X_2$, then $X_i + T_r$, $(i = 1, 2)$, is a* $\left\{\begin{array}{l}\text{(a) } A_r\\ \text{(b) } K_r\\ \text{(c) } C_r\end{array}\right\}$.

To prove (a), let γ^r be any r-cycle in $X_i + T_r$ and let B be any irreducible carrier of the homology $\gamma^r \sim 0$ in M. Then $B \subset X$. Thus if B is not contained in $X_i + T_r$, we would have the separation $B - B \cdot T_r = B \cdot X_1 + B \cdot X_2$. This is impossible by (1. 2), since γ^r is carried by $B \cdot X_i + T_r$ and B is irreducible. Thus $B \subset X_i + T_r$, which shows that $X_i + T_r$, $(i = 1, 2)$, is an A_r.

Part (b) is immediate; for, if an E_r is such that $E_r \cdot (X_i + T_r)$ is not a T_r-set, then $E_r \cdot X_i \neq 0$; and since $E_r - E_r \cdot T_r$ is connected, we have $E_r \subset X_i + T_r$.

To prove (c), we note that by parts (a) and (b), $X_i + T_r$ is both an A_r and a K_r, since every C_r is an $A_r \cdot K_r$. Thus since X is γ^r-connected and $X_i + T_r$ is an $A_r \cdot K_r$ relative to X, we have by (3. 41) that $X_i + T_r$ is a C_r.

(3. 6) *If $C = \sum_{i=1}^{\infty} K_r^i$ is a γ^r-continuum in M, then C is a C_r.*

Proof. We have only to show that C is a K_r. To this end, take any E_r such that $C \cdot E_r$ contains an essential γ^r. Since C is γ^r-connected, there exists an irreducible membrane B in C carrying the homology $\gamma^r \sim 0$. By (1. 3), we have $B \subset E_r$. Since $B = \sum_{i=1}^{\infty} B \cdot K_r^i$, at least one of the sets $[B \cdot K_r^i]$, say $B \cdot K_r^k$, contains an open subset R of B such that $B - R$ carries γ^r. Since

R contains subsets which separate B into two sets one of which carries γ^r, and since by (1.2) every such set contains an essential r-cycle, it follows that $B \cdot K_r^k$ and hence $E_r \cdot K_r^k$ contains an essential r-cycle. But then $E_r \subset K_r^k \subset C$, and therefore C is a K_r.

(3.61) *Any γ^r-connected $C_r^1 + C_r^2$ or $K_r^1 + K_r^2$* is a C_r.

Definition. A closed subset A of M will be said to have property P provided each point of A either belongs to some E_r which is contained in A, to no E_r whatever, or to an E_r with $E_r \cdot A = T_r$. A closed set having property P will be called a P_r.

We have immediately

(3.71) *Every K_r has property P.*

(3.72) *If no irreducible carrier of an homology $\gamma^r \sim 0$ in any E_r is contained wholly in the sum of the remaining E_r-sets, where γ^r is an essential cycle, then the property of being a C_r is equivalent to the property of being γ^r-connected and having property P.*

Proof. The first property implies the second by (3.71). To show the reverse implication, let C be any γ^r-continuum having property P and let us take any E_r such that $E_r \cdot C$ contains an essential r-cycle γ^r. Since $\gamma^r \sim 0$ in C, we may choose an irreducible carrier B of this homology, and by (1.3) we have $B \subset E_r$. By hypothesis there exists a point x of B which is contained in no E_r-set other than E_r. Thus since $E_r \cdot C \neq T_r$ and $x \epsilon E_r$, we have $E_r \subset C$ by property P. Thus C is a K_r and hence a C^r.

(3.73) *If no E_r intersects uncountably many other E_r-sets, the hypothesis of (3.72) is satisfied; and hence the two properties there mentioned are equivalent.*

We have only to show that the hypothesis of (3.72) is satisfied. If this is not so, then there exists an irreducible carrier B of an homology $\gamma^r \sim 0$ in some E_r, where γ^r is an essential cycle, and a countable sequence (E_r^i) of other E_r-sets such that $B \subset \sum_{i=1}^{\infty} E_r^i$. But then some one of these, say E_r^k, contains an open subset R of B such that $B - R$ carries γ^r; and this is impossible, because by the argument given under (3.6), every such subset of B contains an essential r-cycle whereas $E_r^k \cdot E_r = T_r$.

(3.74) *In case $r = 0$, the hypothesis of* (3.73) *and hence also of* (3.72) *is satisfied. Thus in this case the two properties in* (3.72) *are equivalent.*

This is a known [*] property of the sets E_0.

From the definition of property P we obtain at once

(3.75) *Any closed set which is obtained by adding together E_r-sets is a P_r. Also any closed set which is the sum of E_r-sets plus points belonging to no E_r sets is a P_r.*

(3.8) *Notes.* Any closed set K such that for each E_r, $E_r \cdot K = T_r$ is a K_r. In particular, if K intersects no E_r, then it is a K_r. Such sets K are analogous to the closed sets of cut points and end points of a locally connected continuum. We have similar results concerning them. For example, if each of any number of them is γ^r-connected, so also is their product [by (3.33)]. In case M has no true E_r-sets, i. e., if M is *γ^r-acyclic*, then (a) every closed subset of M is a K_r and (b) every γ^r-connected closed subset of M is a C_r. In general, every such set K as mentioned above is γ^r-acyclic. If M is locally γ^r-connected, every γ^r-connected closed subset K such that for each E_r, $K \cdot E_r = T_r$ is a C_r, and hence by (3.42) is locally γ^r-connected.

The following table gives at a glance the relations existing between the various types of sets considered in this section. A symbol + or — in a given row and column indicates the truth or falsity of the statement: *Every set of the type designating the row is a set of the type heading the column.*

	E_r	K_r	C_r	A_r	P_r
E_r	+	+	—	+	+
K_r	—	+	—	—	+
C_r	—	+	+	+	+
A_r	—	—	—	+	—
P_r	—	—	—	—	+

4. Extensible and reducible properties. A property X will be said to be *E_r-extensible* provided that when each E_r in M has property X, then M itself has property X. A property X is said to be *E_r-reducible* provided that when M has property X, so also does each E_r in M. Some properties of this type have already been established, e. g., (2.51) and (2.52), and more will be proven in this section. The E_r-extensible and reducible properties are strictly analogous to the *cyclicly extensible* and *reducible* properties [†] of locally connected continua, to which they reduce in case M is a locally connected continuum and $r = 0$.

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[*] See my paper in *American Journal of Mathematics*, vol. 55 (1933), p. 456.

[†] See my paper "Concerning the structure of a continuous curve," *loc. cit.*, and Kuratowski and Whyburn, "Sur les éléments cycliques . . . ," *loc. cit.*

(4.1) *Let N be a γ^r-subcontinuum of M such that for all save a null sequence*[*] $[E^i_{r-1}]$ *of sets E_{r-1} in M we have $N \cdot E_{r-1} = T_r$.*[†] *Then if $N \cdot E^i_{r-1}$ is locally γ^r-connected for each i, N itself is locally γ^r-connected.*

Proof. Suppose, on the contrary, that N is not locally γ^r-connected. Then it follows at once that there exists an $\epsilon > 0$ and an infinite null sequence of closed subsets $X_1, X_2, \cdots$ of N with $\epsilon > \delta(X_i) \rightarrow 0$ such that each X_i carries an r-cycle which is not homologus to 0 in any subset of N of diameter $< 4\epsilon$. Then if $V_\epsilon(X_i)$ denotes the set of all points at a distance $< \epsilon$ from X_i, it follows by the duality theorem that X_i links some Γ_i^{n-r-1} in $R^n - \overline{V_\epsilon(X_i)} \cdot N$; and thus X_i contains a closed subset Y_i which is irreducibly linked with Γ_i^{n-r-1}. Now for each i, by (1.42), there exists an $E^{n_i}_{r-1}$ in M such that $Y_i \subset E^{n_i}_{r-1}$. Since Y_i carries some γ_i^r which is not homologous to 0 in any subset of N of diameter $< \epsilon$, no E_{r-1} can contain infinitely many of the sets Y_i by hypothesis. Thus by the condition on the diameters of the sets E^i_{r-1}, there exists an integer k such that $\delta(E^{n_k}_{r-1}) < \epsilon$. But $\gamma_k^r \sim 0$ in N. Whence $\gamma_k^r \sim 0$ in $N \cdot E^{n_k}_{r-1}$, which is impossible since $\delta(N \cdot E^{n_k}_{r-1}) < \epsilon$.

(4.11) *Suppose M is γ^r-connected and that the sets E_{r-1} in M form a null family. Then if each E_{r-1} is locally γ^r-connected, so also is M.*

(4.12) *If the sets E_{r-1} in M form a null family, then any subcontinuum N of M such that $N \cdot E_{r-1}$ is either vacuous or locally γ^r-connected for each E_{r-1} is itself locally γ^r-connected.*

Definition. M will be said to be *hereditarily locally γ^r-connected* provided every γ^r-subcontinuum of M is locally γ^r-connected.

(4.2) *If the sets E_{r-1} in M form a null family, the property of being hereditarily locally γ^r-connected in M is E_{r-1}-extensible and reducible.*

The reducibility is obvious and the extensibility follows immediately from (4.12).

(4.3) *Suppose M is locally γ^r-connected and let B be any γ^r-subcontinuum of M which is not locally γ^r-connected. Then for some $\epsilon > 0$ there exists a null sequence of closed subsets $[D_i]$ of B such that for each i, D_i is contained wholly in some $E_r{}^i$ and D_i carries some $\gamma_i{}^r$ such that every carrier of the homology $\gamma_i{}^r \sim 0$ in B is of diameter $> \epsilon$.*

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[*] A sequence or family $[X_i]$ of sets is called a *null sequence* or *null family* provided that at most a finite number of the sets exceed any given $\epsilon > 0$ in diameter.

[†] i. e., this set is a T_r-set.

Proof. By virtue of the non-local-γ^r-connectivity of B it follows that there exists an $\epsilon > 0$ and a null sequence $[Y_i]$ of closed subsets of B, where each Y_i carries some γ^r, say C_i^r, such that every carrier of the homology $C_i^r \sim 0$ in B is of diameter $> 4\epsilon$. Furthermore since M is locally γ^r-connected, we may suppose that, for each i, the homology $C_i^r \sim 0$ in M is carried by some subset of M of diameter $< \epsilon$. Then if $F_i = \overline{V_\epsilon(Y_i)} \cdot M$, we have $C_i^r \sim 0$ in F_i and $C_i^r \sim 0$ in B but $C_i^r \not\sim 0$ in $B \cdot F_i$. By a result of Alexandroff's,[*] there exists a Γ^{n-r-2} in $R^n - (B + F_i)$ which links $B + F_i$ but links neither B nor F_i. Then $B + F_i$ contains a closed subset X_i which irreducibly links Γ^{n-r-2}. By another result of Alexandroff's [†] it follows that some E_r^i contains X_i for each i. Since Γ^{n-r-2} links neither B nor F_i, there exist complexes K_b and K_f in $R^n - B$ and $R^n - F_i$ respectively, each bounded by Γ^{n-r-2}. Then $\Gamma^{n-r-1} = K_b + K_f$ is a cycle in $R^n - B \cdot F_i$.

Now Γ^{n-r-1} must link $X_i \cdot B \cdot F_i$. For if $\Gamma^{n-r-1} \sim 0$ in $R^n - X_i \cdot B \cdot F_i$, there exists a complex Q in $R^n - X_i \cdot B \cdot F_i$ bounded by Γ^{n-r-1}. We may suppose Q subdivided so finely that no simplex of Q has points in common with each of the sets $X_i \cdot B$ and $X_i \cdot F_i$, because Q does not intersect the product of these sets. Let K be the complex consisting of all simplexes of Q which have points in common with $X_i \cdot B$ and let $K \rightarrow \Delta^{n-r-1}$. Let P denote the complex $\Delta^{n-r-1} + K_f$ (mod 2), i. e., the complex consisting of all simplexes of Δ^{n-r-1} which are not in K_f + all simplexes of K_f which are not in Δ^{n-r-1}. Then since $\Delta^{n-r-1} \rightarrow 0$, we have $P \rightarrow \Gamma^{n-r-2}$. Another way to define P is as follows: Let S be the complex consisting of all simplexes of Δ^{n-r-1} which are not in K_f and let T be the complex consisting of the remaining simplexes in Δ^{n-r-1}. Let R be the complex consisting of all simplexes of K_f which are not in Δ^{n-r-1} (i. e., the ones not in T). Then if $S \rightarrow Z^{n-r-2}$, we have $T \rightarrow Z^{n-r-2}$ and $R \rightarrow \Gamma^{n-r-2} + Z^{n-r-2}$. Thus if we set $P = S + R$, we have $P \rightarrow \Gamma^{n-r-2}$.

Now by virtue of the definition of K we have

$$P \cdot X_i \cdot B = X_i \cdot B \cdot S + X_i \cdot B \cdot R = 0.$$

Also $\quad P \cdot X_i \cdot F_i = 0, \quad$ because $\quad R \cdot X_i \cdot F_i \subset K_f \quad$ and $\quad K_f \cdot F_i = 0$

and $S \cdot X_i \cdot F_i = 0$ by virtue of the definition of K, i. e., each simplex of S is on a simplex of K which intersects $B \cdot X_i$ and which therefore cannot intersect $X_i \cdot F_i$. But then

$$P \cdot X_i = P \cdot X_i \cdot B + P \cdot X_i \cdot F_i = 0, \text{ and } \Gamma^{n-r-2} \sim 0 \text{ in } R^n - X_i,$$

contrary to the fact that Γ^{n-r-2} links X_i.

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[*] See Alexandroff, *loc. cit.*, p. 178.

[†] *Loc. cit.*, p. 153.

Therefore Γ^{n-r-1} links $X_i \cdot B \cdot F_i$. Consequently it links some γ_i^r in $X_i \cdot F_i \cdot B$; and since Γ^{n-r-1} is in $R^n - B \cdot F_i$, $\gamma_i^r \not\sim 0$ in $B \cdot F_i$. Now if there were a carrier H_i of the homology $\gamma_i^r \overset{\bullet}{\sim} 0$ in B of diameter $< \epsilon$, we would have $H_i \subset F_i$ which would give $\gamma_i^r \sim 0$ in $B \cdot F_i$. Thus there is no such carrier of diameter $< \epsilon$. Hence if we set $D_i = X_i \cdot B \cdot F_i$, all the conditions for our theorem are satisfied.

(4. 4) *In addition to the hypothesis of* (4. 3), *let us suppose that* B *intersects only a null sequence of sets* E_r *in* M *in more than a* T_r*-set. Then there exists some* E_r *in* M *such that* $E_r \cdot B$ *is not locally* γ^r*-connected.*

This follows at once from (4. 3). As a consequence of (4. 4), we have immediately

(4. 5) *If* M *is locally* γ^r*-connected and the sets* E_r *form a null family* then the property of being hereditarily locally* γ^r*-connected is* E_r*-extensible and reducible in* M.

By way of recapitulation we list here the extensible and reducible properties which have been established in the preceding sections.

The following properties are E_r *extensible and reducible:*

(1) to have a vanishing r'-th Betti number for any $r' > r$, [(2. 51) and (2. 7)]
(2) to be $\gamma^{r'}$-connected for any $r' > r$ [same as (1)]
(3) to separate R^{r+2}, when $M \subset R^{r+2}$, [(2. 52)]
(4) to be a T_{r+1}, [(1. 41)].

In case M is γ^{r+1}-connected and the sets E_r form a null family we have also

(5) to be locally γ^{r+1}-connected [(3. 13), (4. 11)]
(6) to be hereditarily locally γ^{r+1}-connected [(4. 2)].

In case M is locally γ^r-connected and the sets E_r form a null family we have

(7) to be hereditarily locally γ^r-connected [(4. 5)].

The following properties are E_r*-reducible:*

(i) to be $\gamma^{r'}$-connected, for any $r' \geq r$, [(2. 6)].
(ii) to be locally $\gamma^{r'}$-connected, for any $r' \geq r$.

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* For $r = 0$ this condition follows from the local connectivity hence is superfluous.

APPLICATION OF BERNOULLI POLYNOMIALS OF NEGATIVE ORDER TO DIFFERENCING (SECOND PAPER).*

By B. F. Kimball.

1. This paper along with the preceding one by the author † gives a solution of the problem of the expansion of the n-th difference of a function in terms of its derivatives. In the author's first paper the case of real, equal difference intervals was studied and certain problems of interest were treated by the technique there developed. If $\overset{n}{\underset{\omega_1\cdots\omega_n}{\Delta}} f(x)$ denotes the n-th difference of $f(x)$ with complex difference intervals

$$\omega_1\omega_2\cdots\omega_n, \quad \text{and} \quad \tau_n = \tfrac{1}{2}(\omega_1+\omega_2+\cdots\omega_n)$$

we have ‡

$$\overset{n}{\underset{\omega_1\cdots\omega_n}{\Delta}} f(x) = \sum_{m=0}^{\infty}(1/2m!)B_{2m}^{-n}(-\tau_n)f^{[n+2m]}(x+\tau_n). \tag{1.1}$$

The results outlined in § 5 of the present paper taken along with the theorem of § 4 give a means for studying the rapidity of convergence of the above series when the difference intervals are unequal and complex; and in some cases a means for finding the asymptotic value of $\overset{n}{\underset{\omega_1\cdots\omega_n}{\Delta}} f(x)$ as $n \to \infty$.

Nörlund § has developed the theory of the Bernoulli polynomials of negative order. The author in dealing with specific series of the above type has been interested to carry the study of these polynomials further.

2. *Resumé of some of the properties of* $B_r^{-n}(x, \omega_1, \omega_2\cdots\omega_n)$. In order to simplify the notation in this paper the Bernoulli polynomials with difference intervals $w_1w_2\cdots w_s$ or $\omega_1\omega_2\cdots\omega_s$ will in many cases be written simply as $B_r^{-s}(x)$. We recall that

$$\frac{d\,B_r^{-n}(x)}{dx} = r\,B_{r-1}^{-n}(x), \qquad B_0^{-n}(x) \equiv 1. \tag{2.1}$$

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* Presented to the Society, April 14, 1933.

† "Application of Bernoulli polynomials of negative order to differencing," *American Journal of Mathematics*, vol. 55 (1933), p. 399.

‡ Kimball, *loc. cit.*, formula (1.3).

§ Nörlund, *Differenzenrechnung*, Julius Springer (1924), Chapter VI. Further references to this book will be denoted by the letter N.

Let the difference intervals, real or complex, be denoted by $\omega_1\omega_2\cdots\omega_n$ and set $\tau_n = \frac{1}{2}(\omega_1 + \omega_2 + \cdots \omega_n)$. Then (N, p. 140)

$$B^{-n}_{2m+1}(-\tau_n) = 0 \tag{2.2}$$

for all positive integral m and n and for $m = 0$ or $n = 0$. Also (N, p. 140)

$$\begin{aligned} (2.3)\quad & B^{-n}_{2m}(-\tau_n) \\ & = \frac{1}{2^{2m}} \sum \frac{2m!}{(2s_1+1)!\,(2s_2+1)!\cdots(2s_n+1)!}\,\omega_1^{2s_1}\cdot\omega_2^{2s_2}\cdots\omega_n^{2s_n} \end{aligned}$$

where s_k is a positive integer or zero such that $s_1 + s_2 + \cdots s_n = m$ and the summation extends over all possible values of the numbers s_k. Thus we see that $B^{-n}_{2m}(-\tau_n)$ *is a homogeneous, symmetric polynomial of degree* $2m$ *in* $\omega_1\omega_2\cdots\omega_n$.

If all the intervals ω_k are known to be real, we denote them by w_k and set $t_k = \frac{1}{2}(w_1 + w_2 + \cdots w_n)$. The above formula shows that *if all the difference intervals are real, the value of* $B^{-n}_{2m}(-t_n)$ *is unchanged in replacing one or more negative difference intervals, by their absolute values.* Again, we note that *for real difference intervals,* $B^{-n}_{2m}(-t_n)$ *is positive.* If the difference intervals are all equal to unity, $B^{-n}_{2m}(-\frac{1}{2}n)$ is more easily calculated by a formula developed by the author * than by Nörlund's formula (2.3). If the difference intervals are all equal, say to ω, we have

$$B^{-n}_{2m}(-\tau_n) = \omega^{2m} B^{-n}_{2m}(-\tfrac{1}{2}n) \tag{2.4}$$

where $B^{-n}_{2m}(-\frac{1}{2}n)$ denotes polynomial with difference intervals equal to unity. When the difference intervals are unequal, formula (2.3) serves to determine the structure of the polynomial.

Now from (2.1) and (2.2), using Taylor's expansion, we have

$$B^{-n}_{2m}(x) = \sum_{\mu=0}^{m} \binom{2m}{2\mu} B^{-n}_{2(m-\mu)}(-\tau_n)\cdot(x+\tau_n)^{2\mu} \tag{2.5}$$

and

$$B^{-n}_{2m+1}(x) = \sum_{\mu=0}^{m} \binom{2m+1}{2\mu+1} B^{-n}_{2(m-\mu)}(-\tau_n)\cdot(x+\tau_n)^{2\mu+1}. \tag{2.6}$$

It is thus noted that for x real or complex,

$$B^{-n}_{2m}(-\tau_n + x) = B^{-n}_{2m}(-\tau_n - x) \tag{2.7}$$

and

$$B^{-n}_{2m+1}(-\tau_n + x) = -B^{-n}_{2m+1}(-\tau_n - x). \tag{2.8}$$

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* Kimball, *loc. cit.*, formula (2.12).

Thus *if the intervals* ω_k *are all real,* $B^{-n}_{2m}(x)$ *is positive for all real values of* x *with its minimum value at* $x = - t_n$; *and* $B^{-n}_{2m+1}(x)$ *is positive for* x *real and greater than* $- t_n$ *and negative for* x *real and less than* $- t_n$.

3. *Reduction formula for* $B_r^{-n}(x)$. Nörlund has obtained a reduction formula for the Bernoulli polynomial with equal difference intervals (cf. N, p. 145). The analogous formula for unequal difference intervals can be obtained in a somewhat similar manner as follows. By the differentiation of what Nörlund calls the generating function of the Bernoulli polynomials of negative order [cf. N, p. 142, *Erzeugende Funktion*—formula (75)] one obtains the relation

$$(3.1)\qquad \omega_1(\partial/\partial\omega_1)\, B_r^{-n}(x, \omega_1 \cdots \omega_n) = B_r^{-(n-1)}(x + \omega_1, \omega_2, \cdots \omega_n) - B_r^{-n}(x, \omega_1 \cdots \omega_n).$$

Since the following difference equation holds

$$(3.2)\qquad B_r^{-(n-1)}(x + \omega_1) - B_r^{-(n-1)}(x) = \omega_1 r\, B^{-n}_{r-1}(x)$$

we may write (3.1) in the form (cf. N (79), p. 144)

$$(3.3)\qquad \omega_1(\partial/\partial\omega_1)\, B_r^{-n}(x, \omega_1 \cdots \omega_n) = \omega_1 r\, B^{-n}_{r-1}(x, \omega_1 \cdots \omega_n) + B_r^{-(n-1)}(x, \omega_2 \cdots \omega_n) - B_r^{-n}(x, \omega_1 \cdots \omega_n).$$

Summing this equation for all the intervals ω_i one obtains

$$(3.4)\qquad \sum_{i=1}^{n} \omega_i(\partial/\partial\omega_i) B_r^{-n}(x) = - nB_r^{-n}(x) + r\sum_{i=1}^{n} \omega_i B^{-n}_{r-1}(x) + \sum_{i=1}^{n} B_r^{-(n-1)}(x, [\omega_i])$$

where we have denoted by $B_r^{-(n-1)}(x, [\omega_i])$ the polynomial of $n - 1$-th order with difference intervals $\omega_1\omega_2 \cdots \omega_n$, ω_i being omitted. Now

$$(3.5)\qquad x(d/dx)B_r^{-n}(x) = xrB^{-n}_{r-1}(x).$$

But referring to (2.5) and (2.6) we note that $B_r^{-n}(x)$ is a homogeneous polynomial of degree r in $x, \omega_1\omega_2 \cdots \omega_n$. Hence using the theorem of Euler

$$x(d/dx)B_r^{-n}(x) + \sum_{i=1}^{n} \omega_i(\partial/\partial\omega_i)B_r^{-n}(x) = rB_r^{-n}(x).$$

Thus adding equations (3.4) and (3.5) and solving for $B_r^{-n}(x)$ we have:

$$(3.6)\qquad B_r^{-n}(x) = [1/(n + r)]\, \{ \sum_{i=1}^{n} B_r^{-(n-1)}(x, [\omega_i]) + r(x + \sum_{i=1}^{n} \omega_i)B^{-n}_{r-1}(x)\}.$$

One notes that if all the difference intervals are equal to unity, this relation reduces to that given by Nörlund (cf. N (81), p. 145).

4. *Upper bound of* $B_{2m}^{-n}(-t_n)$. We take the difference intervals w_k as positive. By virtue of (3. 6) we have:

$$(4.\,1)\qquad B_{2m}^{-n}(-t_n) = [1/(n+2m)]\sum_{i=1}^{n} B_{2m}^{-(n-1)}(-t_n, [\omega_i]).$$

Denote by $\sum\limits_{j}^{i}$ a summation extended over $j = 1, 2, 3, \cdots, n$ omitting $j = i$.

Thus we have:

$$(4.\,2)\quad B_{2m}^{-(n-1)}(-t_n, [\omega_i])$$
$$= \frac{1}{n-1+2m}\left\{\sum_{j}^{i} B_{2m}^{-(n-2)}(-t_n, [w_i w_j]) + 2m(t_n - w_i)B_{2m-1}^{-(n-1)}(-t_n, [w_i])\right\}$$

since $-t_n + \sum\limits_{j}^{i} w_j = -t_n + 2t_n - w_i$, where $B_{2m}^{-(n-2)}(x, [w_i w_j])$ is the polynomial with w_i and w_j omitted from the difference intervals $w_1, w_2 \cdots w_n$. Thus

$$(4.\,3)\quad \sum_{i=1}^{n} B_{2m}^{-(n-1)}(-t_n, [w_i])$$
$$= \frac{1}{2m+n-1}\left\{\sum_{i=1}^{n}\sum_{j}^{i} B_{2m}^{-(n-2)}(-t_n, [w_i w_j]) + 2m\sum_{i=1}^{n}(t_n - w_i)B_{2m-1}^{-(n-1)}(-t_n, [w_i])\right\}.$$

Recalling the properties of the polynomial of odd degree (v. (2. 8) et seq.) we note that:

$$(4.\,4)\qquad B_{2m-1}^{-(n-1)}(-t_n, [w_i]) = B_{2m-1}^{-(n-1)}(-t_{n-1} - \tfrac{1}{2}w_i, [w_i]) < 0.$$

We now consider two cases: Case 1 where $t_n \geqq w_i$ for $i = 1, 2, \cdots n$, and Case 2 where $t_n < w_i$ for some $i = 1, 2, \cdots n$.

Case 1. $t_n \geqq w_i$ for $i = 1, 2, \cdots n$. Thus $t_n - w_i$ is positive or zero. Let M be the minimum absolute value of $B_{2m-1}^{-(n-1)}(-t_n, [w_i])$. Now $B_{2m-1}^{-(n-1)}(-t_n, [w_i])$ is negative. Hence

$$(4.\,5)\quad \sum_{i=1}^{n}(t_n - w_i)B_{2m-1}^{-(n-1)}(-t_n, [w_i]) \leqq -M\sum_{i=1}^{n}(t_n - w_i) = -(n-2)t_n M$$

and we have

$$(4.\,6)\quad B_{2m}^{-n}(-t_n) \leqq \frac{1}{(2m+n)(2m+n-1)}\sum_{i=1}^{n}\sum_{j}^{i} B_{2m}^{-(n-2)}(-t_n, [w_i w_j])$$

where if $n=2$ the equality sign holds but if $n>2$, the inequality sign applies. One notes that if $n=2$, $t_2=\frac{1}{2}(w_1+w_2)$ and hence the condition of Case 1 does not hold unless $w_1=w_2$ in which case the left-hand term of (4. 5) is equal to zero and hence the equality sign must be used.

Case 2. $t_n<w_i$ *for some* $i=1,2,\cdots n$. One has to be satisfied with the relation (4. 1). Now from the equality [N, p. 14, formula (31)]

$$\overset{k}{\underset{w_1\cdots w_k}{\Delta}} f(x_0)=f^{[k]}[x_0+\theta(w_1+w_2+\cdots w_k)],\quad 0<\theta<1,\quad w_i \text{ real},$$

it is seen that

$$B^{-(n-1)}_{2m}(-t_n,[w_i])=[-t_n+h]^{2m},\quad 0<h<2t_n-w_i.$$

Hence

$$B^{-(n-1)}_{2m}(-t_n,[w_i])<t_n^{2m},\quad (i=1,2,\cdots n)$$

and we have

$$B^{-n}_{2m}(-t_n)<[n/(n+2m)]t_n^{2m}.$$

Under Case 1 above, we now distinguish between the cases, Case (1. 1) when $t_n\geqq w_i+w_j$, $i,j=1,2,\cdots,n$, $i\neq j$ and Case (1. 2) when $t_n<w_i+w_j$ for some i and j, $i\neq j$. We find that under the condition of Case (1. 1),

$$B^{-(n-2)}_{2m}(-t_n,[w_iw_j])\leqq\frac{1}{2m+n-2}\sum_k^{ij}B^{-(n-3)}_{2m}(-t_n,[w_iw_jw_k])$$

where the summation is for $k=1,2,3,\cdots,n$ with the omission of $k=i$ and $k=j$. Under Case (1. 2) we cannot go further than to state that

$$B^{-n}_{2m}(-t_n)<\frac{n(n-1)}{(2m+n)(2m+n-1)}t_n^{2m}.$$

For the general case the reasoning is the same. Thus we have the following theorem:

THEOREM. *Let* $w_1w_2\cdots w_n$ *denote the* n *positive difference intervals of* $B^{-n}_{2m}(x,w_1w_2\cdots w_n)$. *Set* $t_n=\frac{1}{2}(w_1+w_2\cdots w_n)$. *Let* $\sum_r w_i$ *represent the sum of* r *intervals* w_i *with different subscripts taken from the above group. If* r *is an integer (or zero) such that*

$$t_n-\sum_r w_i\geqq 0$$

for all possible sums $\sum_r w_i$ *taken from the above group, then*

$$B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n) \leqq \frac{n(n-1)\cdots(n-r)}{(n+2m)(n-1+2m)\cdots(n-r+2m)}\,t_n^{2m}$$

where the equal sign does not apply when $n > 2$.

One notes that if the difference intervals are all equal, r can be taken as $[\frac{1}{2}n]$ and no larger. This gives the result announced in the previous paper by the author.*

5. *Discussion of $B_{2m}^{-n}(-\tau_n)$ when ω_i is complex.*

(A) If all the difference intervals ω_i are pure imaginaries, we set $\omega_i = iw_i$ where w_i is real. Since $B_{2m}^{-n}(-\tau_n)$ is a homogeneous polynomial of degree $2m$ in ω_i we find that

$$(5.1)\qquad B_{2m}^{-n}(-\tau_n, \omega_1\omega_2\cdots\omega_n) = (-1)^m B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n).$$

Thus the coefficients of $(x+\tau_n)^k$ in the expansion of $B_r^{-n}(x, \omega_1\omega_2\cdots\omega_n)$ are real quantities alternating in sign. [v. formulae (2.5) and (2.6)].

(B) In general, when ω_i is complex, referring to formula (2.3)—which holds equally well for complex difference intervals—we note that the coefficient of any term in $\omega_1^{2s_1}\omega_2^{2s_2}\cdots\omega_n^{2s_n}$ is a positive real number. Thus by virtue of the fact that the absolute value of a sum is less than or equal to the sum of the absolute values of the terms in the sum, we have the relation

$$\left|\sum \frac{2m!}{(2s_1+1)!\,(2s_2+1)!\cdots(2s_n+1)!}\,\omega_1^{2s_1}\omega_2^{2s_2}\cdots\omega_n^{2s_n}\right|$$
$$\leqq \sum \frac{2m!}{(2s_1+1)!\,(2s_2+1)!\cdots(2s_n+1)!}\,|\omega_1|^{2s_1}|\omega_2|^{2s_2}\cdots|\omega_n|^{2s_n}.$$

Hence it follows that

$$(5.2)\qquad |B_{2m}^{-n}(-\tau_n, \omega_1\omega_2\cdots\omega_n)| \leqq B_{2m}^{-n}(-t_n, w_1w_2\cdots w_n)$$

where $w_i = |\omega_i|$ and $t_n = \frac{1}{2}(w_1 + w_2 + \cdots w_n)$.

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* Cf. Kimball, *loc. cit.*, Theorem 2.

NON-HOLONOMIC GEOMETRIES.

By John L. Vanderslice.

Of great significance in the unification and development of nineteenth-century geometry was the famous observation of Felix Klein that a geometry could be defined as the invariant theory of a transformation group acting upon a manifold. Today it is common knowledge that this point of view has only limited application and that a strict adherence to it has long been abandoned. Nevertheless its fruits are far from exhausted. Consider the case of the so-called non-holonomic geometries of which the Riemannian geometry is the most familiar type. These form probably the most notorious exceptions to Klein's rule. Yet it is a fact first emphasized by Cartan * and Schouten † and foreshadowed in the work of Ricci, König, Weyl, and others that these geometries have much in common with those of the classical type, that a prolific source of new non-holonomic geometries and a powerful aid in the study of the old ones lies in the adoption of a standpoint which is a direct generalization of Klein's. Roughly speaking this standpoint consists in regarding a non-holonomic geometry as the theory of isomorphic displacements of spaces with classical geometries along arbitrary curves of an underlying manifold. Up to the present time only the Riemannian (generalized euclidean) and the generalized affine, projective, non-euclidean and conformal geometries have been treated extensively in this way. In the literature there are also several brief discussions of the general case in which the displaced spaces carry Klein geometries with arbitrary fundamental groups.‡

The present paper develops the general theory of non-holonomic geometries as generalizations of Klein geometries starting from a set of fundamental assumptions presented in the form of postulates. There will be four groups of these defining respectively, the initial structure of the underlying manifold, the associated Klein spaces, the relationship between the underlying space and

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* E. Cartan, "Les récentes généralisations de la notion d'espace," *Bulletin des Sciences mathématiques*, vol. 48 (1924), pp. 294-320.

† J. A. Schouten, "Erlanger Programm und Uebertragungslehre," *Rendiconti del Circolo Matematico di Palermo*, vol. 50 (1925), pp. 142-169.

‡ E. Cartan, "Sur les variétés à connexion affine et la théorie de la relativité généralisée." *Annales de l'École Normale*, vol. 40 (1923), pp. 383-390; J. A. Schouten, *loc. cit.*. p. 163 ff.; H. Weyl, "On the foundations of general infinitesimal geometry," *Bulletin of the American Mathematical Society*, vol. 35 (1929), pp. 716-725; H. P. Robertson and H. Weyl, "On a problem in the theory of groups etc.," *ibid.*, pp. 686-690.

each associated space, and the relationship between the different associated spaces. In giving these postulates we are not so much interested in logical rigor as in stating explicitly the initial assumptions involved in the construction of a non-holonomic geometry so as to furnish a common basis and a means of parallel development to the important special types. Whenever possible and sometimes at the expense of brevity we state the postulates in purely geometrical form. For the most part the terminology is that of Veblen and Whitehead in the *Cambridge Tract,* "Foundations of differential geometry" and in several instances knowledge of the contents of this book will be assumed.

We do not take the position that the non-holonomic geometries defined by our postulates represent the only significant generalized geometries, nor do we wish to minimize the importance of other points of view toward these same geometries. Our treatment is not in conflict with the conception of a geometry as the theory of a geometric object; * the analytical development soon gives rise to a "geometric object" $f_j{}^i$ (p. 161) upon which the subsequent discussion is based. Rather does our theory furnish one method among many of discovering geometrical objects which are of significance.

We wish at this point to express our appreciation to Professor Veblen for his valuable inspiration and advice in the preparation of this paper.

1. The underlying space; Postulates *A*. As our Postulates *A* characterizing the underlying manifold of a non-holonomic geometry we shall use without modification the recent differential geometric axioms of Veblen and Whitehead.† Essentially, these axioms limit the underlying space to being a topological manifold having an arbitrary but fixed class u (degree of smoothness) as defined through the intermediation of a very general class of "allowable" coördinate systems related to each other by transformations of class u (number of continuous derivatives). Any n-dimensional space satisfying these axioms will be called an A_n. Subsequent developments will show that it is necessary always to choose $u \geqq 3$. (Cf. $D(3)$; (p. 161), and 7.1).

2. A class of Klein spaces; Postulates *B*. Thus far we have used the term Klein space rather vaguely as representing a space carrying a transformation group. By means of Postulates B we shall now give an explicit definition of a class of Klein spaces ‡ which we believe includes all those that may be profitably generalized to the non-holonomic case. In selecting this restricted class we have been guided by the general considerations that the structure of any space of the class be definable in its entirety by means of preferred coördinate systems, that in any discussion each space permit simple treatment as a whole without recourse to infinitesimal methods. It is also to be characteristic of this class that any two spaces bearing the same group are isomorphic, indeed the postulates select from all spaces admitting the same group one of simple topological character. Thus for the euclidean group all space forms are excluded except the open n-cell. In the subsequent generalization these spaces are to serve as tangent spaces to an A_n to aid in the infinitesimal investigation of the latter and for such a purpose we consider the above properties essential.

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* Cf. O. Veblen and J. H. C. Whitehead, "The Foundations of Differential Geometry" (*Cambridge Tract*), p. 46 ff.

† *Op. cit.*, Chapter IV. Also, "A set of axioms for differential geometry," *Proceedings of the National Academy of Sciences*, vol. 17 (1931), pp. 551-561.

‡ It has points of similarity with the class of "homogeneous spaces with a Lie

In form the postulates are a rather direct extension of the axioms G of Veblen and Whitehead [*] to apply to spaces which are not necessarily homeomorphic to the arithmetic space.[†] Any space satisfying the postulates B is termed a B_n.

Definition. G is a family of continuous transformations between regions of the *arithmetic* space of n-dimensions.

$B(1)$ Each preferred coördinate system is a one-to-one transformation of a subset of the space B_n into the arithmetic space of n-dimensions.

$B(2)$ Each point is represented in at least one preferred coördinate system.

$B(3)$ Any transformation between two preferred coördinate systems (defined on the common domain of these systems) belongs to G.

$B(4)$ The domains of any three preferred coördinate systems have at least one point in common.[‡]

$B(5)$ With every preferred coördinate system and transformation of G there exists a unique preferred coördinate system related to the first on their common domain by the transformation of G.

$B(6)$ If a transformation of G leaves pointwise invariant a region of its domain of definition it is the identity.

The geometries defined by the axioms G of Veblen and Whitehead, of

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group" recently studied by Cartan. See, for example, his *Mémorial, La théorie des groupes finis et continus et l'analysis situs* (Paris, 1930).

[*] *Op. cit.*, p. 24.

[†] E.g. projective, conformal and non-euclidean spaces.

[‡] And hence also a region in common if we define a region of B_n as the image in a preferred coördinate system of a region of the arithmetic space. It should be remarked that with this definition of region postulates B endow a B_n locally with the topological properties of the arithmetic space.

which the affine and euclidean are the most notable, form a sub-class of those defined by postulates B above. Indeed one of the leading virtues of the postulates we have given is the fact that they are also satisfied by the classical projective, conformal, and non-euclidean geometries.

THEOREM 2. 1. *The family G is the realization of a group.*

In the first place it is immediately evident from the postulates that the identity is in G together with the inverse of any transformation of G. Secondly, suppose T_1 and T_2 are transformations of G and that T_1 determines by $B(5)$ a preferred system X_1 from a preferred system X and T_2 a preferred system X_2 from X_1. Then T_2T_1 defines on the common domain D of X, X_1 and X_2 ($B(4)$) the transformation from X to X_2. But by $B(3)$ the transformation between X and X_2 is a transformation, say T_3, of G defined on the common domain D_1 of X and X_2. Hence $T_3 = T_2T_1$ on D. There can be no other transformation T'_3 of G with this property for if so $T_3{}^{-1}T'_3 \neq 1$ would contradict $B(6)$. This proves the theorem.

Two spaces $B_n{}^{(1)}$, $B_n{}^{(2)}$ with the same group G will be called isomorphic if there exist a one-to-one point correspondence between them such that the domain of each preferred coördinate system of $B_n{}^{(1)}$ is mapped on the domain of a preferred system of $B_n{}^{(2)}$, and conversely, points with the same coördinates correspond. Of course two spaces isomorphic in the above sense have identical properties.

This definition of isomorphism at first appears too strong; it might seem necessary to allow for the equivalence of spaces with simply isomorphic fundamental groups. This is not the case however; the actual analytic form of G is essential because of the precise nature of preferred coördinate systems. For example, preferred systems for euclidean geometry are rectangular Cartesian systems and hence the group for this geometry must have the precise form of the euclidean group expressed in Cartesian coördinates.

THEOREM 2. 2. *Two spaces $B_n{}^{(1)}$, $B_n{}^{(2)}$ with the same group G admit a family of isomorphisms in one-to-one correspondence with the transformations of G.*

Let D_1 be the domain of a preferred coördinate system X_1 in $B_n{}^{(1)}$, D_2 the domain of X_2 in $B_n{}^{(2)}$. Applying any transformation T of G to X_1 and X_2 one obtains by $B(4)$ two new preferred systems X_{1T}, X_{2T} of domains D_{1T} and D_{2T} respectively. It follows readily from the axioms that to every transformation T of G will correspond in the above manner one and only one preferred coördinate system X_{1T} of $B_n{}^{(1)}$ and X_{2T} of $B_n{}^{(2)}$, and that all preferred coördinate systems in either space are obtained in this way. Consequently a one-to-one correspondence is set up between the preferred systems of the two spaces. Now consider the representation of $B_n^{(1)}$ on $B_n^{(2)}$ in which each domain D_{1T} is mapped on the corresponding domain D_{2T} making points with the same coördinates coincide. This representation is an isomorphism in the sense defined above. For proof it is only necessary to show that the representation is one-to-one, or what is equivalent, to show that the transformation T_1 of G defined by $B(3)$ on the common domain of any pair of preferred systems in $B_n^{(1)}$ is identical with the transformation T_2 of G defined in the common domain of the corresponding pair in $B_n^{(2)}$. Consider the domain D^*_1 of $B_n^{(1)}$ common to the given pair and the original D_1; also the similarly defined D^*_2 of $B_n^{(2)}$. That these common domains exist follows from $B(5)$ and of course in the above representation they correspond. Then from the manner in which the correspondence was established between the coördinate systems of $B_n^{(1)}$ and $B_n^{(2)}$ it is evident that T_1 and T_2 are identical in so far as they act upon the corresponding domains D^*_1 and D^*_2. Hence T_1 and T_2 are completely identical; if not $T_1T_2^{-1}$ would be a transformation of G other than the identity leaving an n-dimensional arithmetic domain pointwise invariant contrary to the hypothesis on G. Thus for every choice of preferred system X_2 as correspondent of a fixed X_1 a unique and distinct isomorphism is determined between the given spaces, and indeed all isomorphisms are obtained in this way. The resulting one-to-one correspondence set up between the preferred coördinate systems of $B_n^{(2)}$ and the isomorphisms of $B_n^{(1)}$ and $B_n^{(2)}$ establishes the theorem.

As an immediate corollary of theorem 4. 2. it follows that a space B_n admits a group of automorphisms isomorphic to its fundamental group G. Thus having started with the notion of preferred coördinate systems we arrive at the existence of a group of motions, the fundamental concept of the classical geometries.

For the future development it will be necessary to consider the analytic expression of the isomorphisms of two B_n's. This requires a fixed coördinate system in both $B_n^{(1)}$ and $B_n^{(2)}$ and since a coördinate system is not in general defined over the entire space an isomorphism is not analytically expressible in its entirety but only locally in the domains of the chosen coördinate systems. Let X_1 and X_2 of domains D_1 and D_2 be fixed preferred systems in $B_n^{(1)}$ and $B_n^{(2)}$. Let X_{2T} be any other preferred system of $B_n^{(2)}$ related to X_2 by the transformation T of G. The isomorphic representation of $B_n^{(1)}$ on $B_n^{(2)}$ determined by the correspondence of X_1 and X_{2T} is expressible analytically in the systems X_1 and X_2 by the transformation $X_2 = T^{-1}X_1$, which is defined for any T of G in some n-cell of D_1. Thus we have the following theorem

THEOREM 2. 3. *Expressed in two fixed coördinate systems* X_1 *and* X_2 *of* $B_n^{(1)}$ *and* $B_n^{(2)}$ *the family of isomorphic representations of* $B_n^{(1)}$ *on* $B_n^{(2)}$ *is given by* $X_2 = TX_1$ *where* T *runs through all transformations of the fundamental group* G.

Before proceeding to the generalization of spaces of type B we impose two additional postulates in order to make possible a differential geometric treatment. Their introduction has been delayed because the results of this section are independent of them.

$B(7)$ Each transformation of G is of class* $v \geqq 2$.

$B(8)$ The set G forms† an A_r of class $w \geqq 2$.

In the presence of $B(7)$ and $B(8)$, Theorem (2. 1) becomes

THEOREM 2. 1a. *The set* G *forms an r-parameter Lie transformation group of class* $\geqq 2$.

The above restriction that the class be $\geqq 2$ makes possible the construction of one parameter subgroups of G from its infinitesimal transformations, a construction of fundamental importance in the sequel.

It will often be convenient to use in a given B_n the class of coördinate systems obtained by submitting each preferred system to the set of all affine (linear) transformations. The introduction of these systems which will be called quasi-preferred allows, for example, the use of oblique coördinate systems in euclidean geometry; in projective geometry on the other hand no extension takes place. In general the transformations between quasi-preferred coördinate systems do not form a group.

3. Generalized Klein spaces. The next step is the generalization of the class of Klein spaces defined above to the non-holonomic type. The procedure is as follows. With every point P of an A_n (§ 1) there is associated a B_n (§ 2) each $B_n(P)$ having the same fundamental group G. A first set of postulates, C, defines a class of point correspondences between each $B_n(P)$ and the neighborhood of P in A_n establishing a relation of tangency while a second set, D, defines point correspondences between spaces associate to different points of A_n. The former will be called local correspondences, the latter displacements.

Local correspondence, postulates C.

$C(1)$ A local correspondence for a $B_n(P)$, $P \subset A_n$ is a one-to-one bicontinuous point transformation of a region of $B_n(P)$ into a region of A_n.

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* When $v = \omega$ (analytic functions) $B(6)$ is redundant.

† In the sense of postulates A.

$C(2)$ If C_1, C_2 are two local correspondences for $B_n(P)$, the product $C_1C_2^{-1}$ (a point transformation in A_n) is defined in a region containing P and as expressed in an allowable coördinate system x of domain containing P is of order greater than one in the increments * of the coördinates x about P.

$C(3)$ If C_1, C_2 are transformations satisfying $C(1)$, $C(2)$ and if C_1 is a local correspondence, then so also is C_2.

$C(4)$ For each $B_n(P)$, $P \subset A_n$ there exists at least one local correspondence.

With these postulates we have attempted to define the notion of tangency abstractly without the use of an ambient space.

THEOREM 3.1. *For a given allowable coördinate system x of A_n and point P of coördinates q there exists at least one quasi-preferred coördinate system Y of $B_n(P)$ in which the local correspondences at P are expressed as*

$$(3.1) \qquad Y^i = x^i - q^i + \epsilon^i(x-q)$$

where the ϵ^i run through all functions of class > 0 *and order* > 1.

In the first place, it follows from $C(2)$ that P has a unique correspondent, say P', in $B_n(P)$. Now let Y be any quasi-preferred system of $B_n(P)$ in which the coördinates of P' vanish and let

$$(3.2) \qquad Y^i = F^i(\delta x) \qquad F^i(0) = 0$$

be a local correspondence. From $C(2)$

$$(\partial F^i/\partial(\delta x^j))_P = a_j^i$$

where the a_j^i form a non-singular matrix. If Y^i be that quasi-preferred system related to Y^i by

$$Y^i = A_j^i Y^j \qquad A_j^i a_k^j = \delta_k^i$$

(3.2) becomes in the new coördinates

$$Y^i = x^i - q^i + \epsilon_1{}^i(x-q)$$

where $\epsilon_1{}^i$ is of class > 0 and order greater than one. That all correspondences are obtained by letting the ϵ^i vary as in the statement of the theorem then follows immediately from $C(3)$, $C(4)$.

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* Thus $C_1C_2^{-1}$ is expressible in an allowable coördinate system x containing P as $\delta x^{i'} = \phi^i(\delta x)$ where the ϕ^i are of class > 0 by $C(1)$ and $\lim_{\delta x \to 0} \phi^i(\delta x)/\delta x^i = 0$ $(i, j = 1, \cdots, n)$ by $C(2)$.

This theorem makes clear how the local correspondences lend tangential character to $B_n(P)$.

Any quasi-preferred coördinate system in $B_n(P)$ in which the local correspondences are given by (3.1) will be called a tangential coördinate system associated with the system x.

Displacements; * *Postulates D.*

Definition. A one-cell of A_n is called allowable if it is simple and of class † $\geqq 2$.

$D(1)$ Given any two points P and Q of A_n and an allowable one-cell C joining them, there is a unique isomorphic representation $D_c(P, Q)$ of $B_n(Q)$ on $B_n(P)$.

$D(2)$ If P, Q, and R are any three points of an allowable one-cell C, then $D_c(P, Q)$ followed by $D_c(Q, R)$ is identical with $D_c(P, R)$.

Two immediate consequences of these first two postulates are that $D_p(P, P)$ is the identity and that $D_c(P, Q)$ is the inverse of $D_c(Q, P)$.

Definition. An allowable coördinate system x of domain D_x in A_n together with a quasi-preferred coördinate system Z_p in each $B_n(P)$, $P \subset D_x$ is called an admissible reference scheme for a domain $D \subset D_x$ if

(i) the domain of Z_p, $P \subset D$, contains the contact point, and

(ii) $D_c(P_1, P_2)$, (C, P_1, P_2 contained in D), expressed as a transformation between Z_{p_1} and Z_{p_2} is of class $\geqq 2$ as function of P_2 on C at each point of the domain of Z_{p_1} at which the transformation is defined.

Thus in the coördinate systems Z_p, $D_c(P_1, P_2)$ is a transformation $Z^i_{p_2} = F^i(Z_{p_1}, P_2)$ mapping the point of coördinates $Z^i_{p_1}$ in $B_n(P_1)$ upon the point of coördinates $Z^i_{p_2}$ in $B_n(P_2)$ and (ii) asserts that F^i are of class $\geqq 2$ in P_2 for all values of P_2 and $Z^i_{p_1}$ for which F^i are finite. Incidentally it follows from postulates B and the fact that $D_c(P_1 P_2)$ is an isomorphism, that for any point P_2 of C, F^i are defined in some subdomain of the domain of Z_{p_1} and this is sufficient to define the complete isomorphism between $B_n(P_1)$ and $B_n(P_2)$ (cf. Theorem 2.3).

Definition. $D_c(P, P + dP) = \lim\limits_{\Delta P \to 0} [D_c(P, P + \Delta P)/\Delta P]dP$, where

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[*] Veblen and Whitehead, *op. cit.*, p. 91, have given a very general set of axioms for a displacement theory, but to permit of differential geometric treatment a more specialized set is necessary here.

[†] In other words, if it has no double points and is representable in an allowable coördinate system x by $xi = xi(t)$ of class $\geqq 2$ in the parameter t.

$D_c(P, P + \Delta P)$ is expressed in an admissible reference scheme, is called an infinitesimal displacement from P along C.

In terms of these definitions the final two displacement postulates are:

$D(3)$ For any point P of A_n there exists an admissible scheme of *tangential* coördinate systems the domain of which contains P.

$D(4)$ If two allowable one-cells C, C' are tangent* at a point P, the infinitesimal transformations $D_C(P, P + dP)$, $D_{C'}(P, P + dP)$ are identical as expressed in an admissible reference scheme.

An A_n together with its associated B_n's satisfying postulates A, B, C, D will be called an X_n and it is with the geometry of an X_n that we are here concerned. Such a geometry is conveniently termed non-holonomic since in general the isomorphisms between different associated spaces are only unique as defined along a curve.

4. Differential equations of displacement. As a consequence of postulates D a set of differential equations can be derived which when integrated along a one-cell furnish the corresponding displacement.

Let (x, Y) be an admissible reference scheme of tangential coördinate systems for a domain D of R. Consider the family of curves

$$x^i = x_0{}^i + p^i t \tag{4.1}$$

of which there is one curve passing through x_0 in each direction p^i. This family will fill out a certain neighborhood of x_0 in D. The displacements $D_C(x_0, x)$ from x_0 along the curves of the family can be written

$$Y^i = F^i(Y_0, t, p) \tag{4.2}$$

where Y_0 are tangential coördinates in $B_n(x_0)$ and where the F^i are continuous of class $\geqq 2$ in t since the Y's form an admissible reference scheme. The infinitesimal displacements defined at x_0 along the curves (4.1) and obtained by differentiating (4.2) at x_0 have the form

$$dY^i = f_j{}^i(Y, p) p^j \, dt \tag{4.3}$$

where $f_j{}^i$ are homogeneous of degree zero in p^i since the displacements are independent of the choice of parameter t. Then by virtue of $D(4)$ the infinitesimal displacement at x_0 along any allowable one-cell in the direction p^i is given by (4.3). x_0 being any point in D, there is a well-defined set of functions $f_j^i(Y, x, dx)$ determining an infinitesimal displacement in each direction at each point x of D, viz.

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* More generally tangency may be replaced by contact of order $p > 0$. Certain non-holonomic geometries have been considered in which $p > 1$. See e. g., H. V. Craig, "On parallel displacement in non-Finsler space," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 125-142; A. Kawaguchi, "Die Differentialgeometrie in der Verallgemeinsten Mannigfaltigkeit," *Rendiconti del Circolo matematico di Palermo*, vol. 61 (1932), pp. 245-271, "The foundation of the theory of displacements," *Proc. imp. Acad.* (Japan), vol. 9 (1933), pp. 351-354.

$$dY^i - f_j^i(Y, x, dx)\, dx^j = 0. \tag{4.4}$$

The finite displacements between the associated spaces along an allowable one-cell are reconstructed by the integration of (4.4) along the one-cell. Since the finite displacements are of class $\geqq 2$ along one-cells of class $\geqq 2$, the f_j^i must be of class $\geqq 1$ in their arguments so long as x is in D and $\Sigma(dx^i)^2 \neq 0$.

It is often desirable to write the differential equations (4.4) in terms of preferred coördinate systems. In order to do this it will first be necessary to show the existence of admissible reference schemes of preferred systems. We return to the family of curves (4.1) and note that since the functions F^i of (4.2) may be obtained by the integration of (4.4) along the family, the F^i must be of class $\geqq 2$ in *all* arguments. Since besides p^i and t can occur only in the combination $p^i t$ the transformation (4.2) may be written

$$Y^i = F^i(Y_0, x) \tag{4.5}$$

where F^i are of class $\geqq 2$ in x within a sufficiently limited domain $D_1 \supset x_0$. Now since the displacements are isomorphisms a family of preferred coördinate systems $\{X\}$ exists, one in each $B_n(P)$ such that the displacements along (4.1) are represented in these coördinates by

$$X^i = X_0^i. \tag{4.6}$$

Furthermore, from the continuity of the displacements in tangential coördinates it follows that in a sufficiently restricted domain $D_2 \supset x_0$ each of these systems contains the contact point provided the system X_0^i in $B_n(x_0)$ does. The systems Y of (4.5) will be related to the systems X of 4.6 by, say

$$X^i = G^i(Y, x). \tag{4.7}$$

Since (4.6) must be a consequence of (4.5) in $D_3 = D_2 D_1$ by virtue of (4.7) it follows that (4.7) are continuous of class $\geqq 2$ in Y and x for $x \subset D_3$. Therefore the preferred coördinate systems X together with the underlying system x form an admissible preferred reference scheme for the domain D_3. This important result we state as

THEOREM 4.1. *For each point P of the domain of an admissible tan-*

gential reference scheme (x, Y) *there exists an admissible preferred reference scheme* (x, X) *for a domain* $D \supset P$.

Suppose that G has r essential parameters and that $\xi_\alpha{}^i(X)$ $(\alpha = 1, \cdots r)$ be a set of independent infinitesimal generators of G. When the differential equations of displacement are written in terms of an admissible preferred reference scheme, they must then be of the form

$$dX^i + \lambda_j{}^\alpha(x, dx)\,\xi_\alpha{}^i(X)\,dx^j = 0 \tag{4.8}$$

where $\lambda_j{}^\alpha$ are homogeneous of degree zero in dx and continuous of class $\geqq 2$ in the arguments. (4.8) have the advantage of exhibiting explicitly the precise functional form of the most general displacement equations for a given fundamental group G.

5. Curvature; Holonomic spaces. In a given $B_n(P)$, $P \subset R$ there is a unique automorphism associated with every closed curve O of class $\geqq 2$ containing P and contained in R, namely the automorphism obtained by mapping $B_n(P)$ upon itself by displacement around O. The set of all these automorphisms of $B_n(P)$ forms an invariant subgroup H of G. The group H is independent of the particular point P originally chosen and may be called following Cartan the holonomy group of X_n for the region R.

At any point P there will be a family of infinitesimal transformations of H defined by displacements about infinitesimal circuits at P. At a point at which this family reduces to the identity, the X_n is said to have no curvature. The analytical expression of these infinitesimal transformations is readily obtained by the use of the familiar device of displacing $B_n(P)$ about an infinitesimal parallelogram of edges dx, δx. Expressed in an admissible preferred reference scheme (x, X) for the neighborhood of P, the resulting infinitesimal displacement is *

$$\Delta X^i = R^i{}_{jk}(x, X, dx, \delta x)\,dx^i \delta x^k \tag{5.1}$$

where

$$R^i{}_{jk} = \frac{\partial \lambda_k{}^\alpha(x, \delta x)}{\partial x^j}\,\xi_\alpha{}^i(X) - \frac{\partial \lambda_j{}^\alpha(x, dx)}{\partial x^k}\,\xi_\alpha{}^i(X)$$
$$+ \lambda_j{}^\alpha(x, dx)\,\lambda_k{}^\beta(x, \delta x)\left(\frac{\partial \xi_\alpha{}^i(X)}{\partial X^l}\,\xi_\beta{}^l(X) - \frac{\partial \xi_\beta{}^i(X)}{\partial X^l}\,\xi_\alpha{}^l(X)\right).$$

By the use of the relations

$$\xi_\beta{}^l \partial \xi_\alpha{}^i / \partial X^l - \xi_\alpha{}^l \partial \xi_\beta{}^i / \partial X^l = c^\gamma{}_{\alpha\beta}\,\xi_\gamma{}^i$$

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* Cf. Weyl, *loc. cit.*, p. 717.

where $c^{\gamma}{}_{\alpha\beta}$ are the structure constants of G

$$(5.2)\quad R^{i}{}_{jk} = \left[\frac{\partial \lambda_{k}{}^{a}(x, \delta x)}{\partial x^{j}} - \frac{\partial \lambda_{j}{}^{a}(x, dx)}{\partial x^{k}} + c^{a}{}_{\beta\gamma}\lambda_{j}{}^{\beta}(x, dx)\lambda_{k}{}^{\gamma}(x, \delta x)\right]\xi_{a}{}^{i}(X)$$

showing explicitly that ΔX^{i} is an infinitesimal transformation of G.

From (5.1) the necessary and sufficient conditions for the curvature to vanish at x are

$$(5.3)\qquad (a)\qquad R^{i}{}_{jk}(X, x, dk, \delta x) = 0$$

which imply

$$(5.3)\qquad (b)\qquad \frac{\partial \lambda_{i}{}^{a}(x, dx)}{\partial (dx^{j})} = 0.$$

Since (5.3) are the conditions of integrability of (4.8) we have

THEOREM 5.1. *A necessary and sufficient condition that the displacements $D_C(P_1, P_2)$ defined in a simply-connected region R be independent of the path $C \subset R$ is that the curvature vanish at every point of R.*

The geometry is said to be holonomic in the given region.

When the curvature vanishes at every point of a region R, the equations of displacement (4.8) are completely integrable and any displacement $D_C(P_1, P_2)$ is independent of the one cell C. When this is the case, the geometry is said to be holonomic in R.

6. Preferred coördinate systems in A_n. In the holonomic case it is possible in an invariantive way to introduce coördinate systems locally in the underlying space A_n which are related to each other by transformations of G. This fact expresses that a holonomic space is locally of the same character as the associated spaces. We shall discuss this question, first geometrically, and then analytically.

We confine attention to the domain of an admissible preferred reference scheme (x, X) of R. Let x_1, q_1 be a pair of points in the chosen domain and let $X_0(x)$ be the coördinates * of the contact point in $B_n(x)$. Since the space is assumed holonomic there is a unique isomorphic representation of $B_n(q_1)$ upon $B_n(x_1)$ in which $X_0(q_1)$ is mapped upon a point $X(x_1, q_1)$ of $B_n(x_1)$. As q_1 varies within a sufficiently restricted domain D, while x_1 remains fixed, a correspondence

$$(6.1)\qquad X^{i} = X^{i}(x_1, q)$$

is established between D and a region of $B_n(x_1)$. This correspondence may be interpreted as an introduction of preferred coördinates in D. Allowing x_1 to vary, a family of coördinate systems $X(x, q)$ is obtained the members of which are related by transformations * of G and serve also as preferred systems for D. And having once introduced a preferred system $X(x, q)$, all the preferred systems of $B_n(x)$ having a domain in common with the image of D, serve equally well as preferred systems for D.

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* The functions X_0 will be continuous of order $\geqq 2$ by virtue of (4.7).

We now proceed to the derivation of a set of partial differential equations having for solution these preferred systems as functions of q.

Since (5. 3) are satisfied, (4. 8) may be written as a set of total differential equations

$$\partial \bar{X}^i/\partial x^j = f_j{}^i(x, \bar{X}) \tag{6.2}$$

which are completely integrable. The complete solution depending on n arbitrary constants may be written

$$\bar{X}^i = \phi^i(x, q, X) \tag{6.3}$$

where X are the n arbitrary initial values assumed by $\bar{X}$ at $x = q$. Thus $X^i = \phi^i(q, q, X)$. If now X is set equal to the contact point $X_0(q)$ of $B_n(q)$ equations (6. 3) furnish the correspondence (6. 1) which served to introduce preferred coördinates in A_n.

It is a well-known property of the solutions of equations of the form (6. 2) that

$$X^i = \phi^i(q, x, \bar{X})$$

where x and q have been interchanged in (6. 3). The functions $\phi^i(q, x, X)$ are a set of first integrals of (6. 2) and therefore satisfy

$$\partial\psi/\partial x^j + (\partial\psi/\partial \bar{X}^l) f_j{}^l(x, \bar{X}) = 0$$

identically in x, $\bar{X}$ and the parameters q. Therefore

$$\frac{\partial\phi^i(x, q, X)}{\partial q^j} = -\frac{\partial\phi^i(x, q, X)}{\partial X^l} f_j{}^l(q, X). \tag{6.4}$$

Hence † for $X^i = X_0{}^i(q)$

$$\frac{\partial\bar{X}^i}{\partial q^j} = \frac{\partial\bar{X}^i}{\partial X_0{}^l}\left(\frac{\partial X_0{}^l}{\partial q^j} - f_j{}^l(q, X_0)\right). \tag{6.5}$$

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* If $X(x_1, q)$, $X(x_2, q)$ are two systems of the family, the transformation between them is just that produced by displacement from x_1 to x_2 according to (4. 8).

† Note that there are two meanings to $\partial/\partial q^i$, according as it acts upon q only as it occurs explicitly in the function or upon q totally. In the first case we write $\partial\phi(\ )/\partial q$, in the second, $\partial\bar{X}/\partial q$ and similarly for other functions.

Higher derivatives of $\bar{X}$ with respect to q are expressible in analogous form by differentiation of (6. 5) totally with respect to q and use of the derivatives of (6. 4) with respect to X. Thus

$$(6.\,6)\quad \frac{\partial^2 \bar{X}^i}{\partial q^j \partial q^k} = \frac{\partial^2 \bar{X}^i}{\partial X_0{}^l \partial X_0{}^m}\left(\frac{\partial X_0{}^l}{\partial q^j} - f_j{}^l(q, X_0)\right)\left(\frac{\partial X_0{}^m}{\partial q^k} - f_k{}^m(q, X_0)\right)$$
$$+ \frac{\partial \bar{X}^i}{\partial X_0{}^l}\left(\frac{\partial^2 X_0{}^l}{\partial q^j \partial q^k} - \frac{\partial f_j{}^l(q, X_0)}{\partial X_0{}^m}\,\frac{\partial X_0{}^m}{\partial q^k}\right.$$
$$\left. - \frac{\partial f_k{}^l(q, X_0)}{\partial X_0{}^m}\,\frac{\partial X_0{}^m}{\partial q^j} - \frac{\partial f_j{}^l(q, X_0)}{\partial q^k} + f_j{}^m(q, X_0)\,\frac{\partial f_k{}^l(q, X_0)}{\partial X_0{}^m}\right).$$

The symmetry of these derivatives in j, k is a consequence of the integrability conditions of (6. 2).

In general

(6. 7)

$$\frac{\partial^p \bar{X}^i}{\partial q^{i_1} \cdots \partial q^{i_p}} = \frac{\partial^p \bar{X}^i}{\partial X_0{}^{j_1} \cdots \partial X_0{}^{j_p}}\left(\frac{\partial X_0{}^{j_1}}{\partial q^{i_1}} - f_{i_1}{}^{j_1}(q, X_0)\right) \cdots \left(\frac{\partial X^{j_p}}{\partial q^{i_p}} - f_{i_p}{}^{j_p}(q, X_0)\right) + \bigstar$$

where ★ represents terms linear in the derivatives of X with respect to X_0 of lower order than p.

It will be necessary here to make the assumption

$$(6.\,8)\qquad \det|\partial X_0{}^i/\partial q^j - f_j{}^i(q, X_0)| \neq 0 \quad \text{(generically).}$$

Geometrically this condition requires that the contact points of the B_m's in a neighborhood of q form an n-dimensional set when mapped upon $B_n(q)$ by (6. 2). With this assumption equations (6. 7) for any p may be solved for $\partial^p \bar{X}^i/\partial X_0{}^{j_1} \cdots \partial X_0{}^{j_p}$ linearly in terms of $\partial^s \bar{X}^i/\partial q^{i_1} \cdots \partial q^{i_s}$ $(s = 1, \cdots, p)$.

We now make use of the fact that the transformations of the fundamental group G are given by the complete solution of a system of mixed total differential equations, the so-called Mayer-Lie system associated with the group.* They have the form

$$(6.\,9)\quad \begin{cases} \dfrac{\partial^a \bar{X}^i}{\partial X^{i_1} \cdots \partial X^{i_a}} = F^i{}_{i_1 \ldots i_a}\left(\bar{X}, \dfrac{\partial \bar{X}}{\partial X}, \cdots, \dfrac{\partial^{a-1} \bar{X}}{\partial X, \cdots, \partial X}\right) \\ \qquad 0 \qquad = R^\mu\left(\bar{X}, \dfrac{\partial \bar{X}}{\partial X}, \cdots, \dfrac{\partial^{a-1} \bar{X}}{\partial X, \cdots, \partial X}\right) \quad (\mu = 1, \cdots, m) \end{cases}$$

$$n\left(1 + \tbinom{n}{1} + \tbinom{n+1}{2} + \cdots + \tbinom{n+a-2}{a-1}\right) - \mu = r = \text{order of } G.$$

These equations can be combined with (6. 7) for $p = 1, \cdots, \alpha$ and with $X^i = X_0{}^i(q)$ to give rise to a mixed system

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* Bianchi, *Teoria dei gruppi continui finiti* (1918), p. 40.

$$(6.10)\quad \begin{cases} \dfrac{\partial^\alpha \bar{X}^i}{\partial q^{i_1}\cdots\partial q^{i_\alpha}} = \Phi^i_{i_1\ldots i_\alpha}\left(q, \bar{X}, \dfrac{\partial \bar{X}}{\partial q}, \cdots, \dfrac{\partial^{\alpha-1}\bar{X}}{\partial q\cdots\partial q}\right) \\ 0 \qquad = \Omega^\mu\left(q, \bar{X}, \dfrac{\partial \bar{X}}{\partial q}, \cdots, \dfrac{\partial^{\alpha-1}\bar{X}}{\partial q\cdots\partial q}\right). \end{cases}$$

These finally are the equations which we set out to derive; the complete set of solutions depending obviously on r parameters gives the family of preferred systems for domains of A_n. The proof is almost immediate. From the manner of forming (6.10) it is evident that for the initial conditions on the parametric derivatives obtained from (6.7) by making

$$(6.11)\quad q = x,\ \bar{X}^i = X_0{}^i(x),\ \left(\frac{\partial \bar{X}^i}{\partial X^j}\right)_{q=x} = \delta_j{}^i, \cdots \left(\frac{\partial^s \bar{X}^i}{\partial X^{j_1}\cdots\partial X^{j_s}}\right)_{q=x} = 0$$
$$(s = 2, \cdots \alpha - 1)$$

the solution is given by (6.3) with $X = X_0(q)$, i.e.

$$\bar{X}^i = \phi^i(x, q, X_0(q)),$$

and since there can be only one solution with given initial conditions it is unique. But now every solution $\bar{X}^i = \bar{X}^i(X)$ of (6.9), that is, every transformation of G furnishes at least formally a solution of (6.10)

$$(6.12)\qquad \bar{X}^i = \bar{X}^i(\phi(x, q, X_0(q))).$$

These solutions depend upon r-parameters and hence form the complete solution of (6.2). Also they obviously give the desired family of preferred coördinate systems. This completes the proof. It is to be noted that the complete integrability of (6.10) is a consequence of the integrability of (6.2).

Formally the equations (6.10) also exist when the integrability conditions (5.3) are not satisfied, the functions Φ and Ω being in general homogeneous of degree zero in dq. And they will be of geometrical significance since by their use preferred coördinates can at least be introduced along curves of the A_n. Important use of this possibility will be made in §§ 11, 12.

Any passive system of differential equations having as complete solution the transformations of G could be used in place of the Mayer-Lie system (6.9) in forming the fundamental differential equations. Although the alternative system to (6.10) thus obtained may no longer be total, it may have the merit of greater simplicity.

It should be remarked that the equations (6.10) could be used as a basis for the generalization of a classical Klein geometry instead of the notion of displacement. The general method would be to write down the completely integrable differential equations for the introduction of preferred coördinates in a B_n referred to general coördinates and then retaining the form of the equations to remove the restriction that they be completely integrable.*

7. Transformation of coördinates. In order to preserve the continuity of the fundamental displacements, the allowable coördinate transformations in X_n must be restricted to those between admissible reference schemes. Furthermore, for all our purposes, quasi-preferred coördinate systems are the most general that we need use in the associated spaces. We therefore restrict the allowable coördinate transformations to those between admissible quasi-preferred reference schemes (x, Y), viz.

$$(7.1) \qquad \begin{array}{ll} (a) & \bar{x}^i = \bar{x}^i(x) \\ (b) & \bar{Y}^i = \bar{Y}^i(Y, x) \end{array}$$

where $\bar{x}^i(x)$ are continuous of class u and $\bar{Y}^i$ of class ≥ 2. In particular, if Y and $\bar{Y}$ are tangential coördinate systems (7.1(b)) have the general form

$$(7.1) \qquad (c) \qquad \bar{Y}^i = Y^j \, \partial\bar{x}^i/\partial x^j + \epsilon^i(Y, x)$$

where the ϵ^i are of order > 1 in Y.

Under a transformation of the form (7.1) the displacement equations (4.4) take the form

$$d\bar{Y}^i - \bar{f}_j{}^i(\bar{Y}, \bar{x}, d\bar{x})\, d\bar{x}^j = 0$$

where the parameters $\bar{f}_j{}^i$ in the new coördinate scheme $(\bar{x}, \bar{Y})$ are related to those in the original scheme (x, Y) by

$$(7.2) \qquad \bar{f}_j{}^i(\bar{Y}, \bar{x}, d\bar{x}) = \left[f_s{}^r(Y, x, dx) \frac{\partial \bar{Y}^i}{\partial Y^r} + \frac{\partial \bar{Y}^i}{\partial x^s} \right] \frac{\partial x^s}{\partial \bar{x}^j} .$$

The quantities $f_j{}^i$ are therefore the components of a "geometric object" and, as we have remarked in the introduction, a non-holonomic geometry with (4.4) as equations of displacement may be conceived as the theory of this "geometric object."

8. Local isomorphism and invariant theory.

Two non-holonomic spaces X_n, $\bar{X}_n$ are said to be locally isomorphic if there exists

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* As an example of this method as applied to projective geometry, cf. Veblen, "Generalized projective geometry," *Journal of the London Mathematical Society*, vol. 4 (1929), p. 142.

(i) *a one-to-one correspondence of class u,* $\bar{x} = f(x)$*, between a region* R *of* A_n *and a region* $\bar{R}$ *of* $\bar{A}_n$.

(ii) *an isomorphic correspondence between* $B_n(x)$ *and* $B_n(\bar{x})$ *for every pair of corresponding points* x, *and* $\bar{x}$ *of* R *and* $\bar{R}$ *respectively, by which the local correspondences and displacements in* X_n *are carried into those in* $\bar{X}_n$.

The preservation of the local correspondences is best expressed by the use of tangential coördinate systems since in them the local correspondences receive their simplest expression. Thus referring X_n, $\bar{X}_n$ to admissible tangential reference schemes, if

$$\bar{x}^i = \bar{x}^i(x) \tag{8.1}$$

is a correspondence between regions of A_n and $\bar{A}_n$ the isomorphisms between the associated spaces as expressed in the tangential reference schemes must be (cf. 7.1 (c))

$$\bar{Y}^i = Y^j \partial x^i / \partial \bar{x}^j + \epsilon^i(Y, x) \tag{8.2}$$

where ϵ^i are of order > 1 in Y and such that $\bar{Y}$ is one of the tangential coördinate systems associated with $\bar{x}$ as a system of coördinates in A_n.

If furthermore the displacements are to be preserved, (8.1), (8.2) must be so selected that they transform the displacement parameters $f_j{}^i$ (Y, x, dx) of X_n into the parameters $\bar{f}_j{}^i(\bar{Y}, \bar{x}, d\bar{x})$ of $\bar{X}_n$, that is to say (cf. 7.2), so that

$$\bar{f}_j{}^i(\bar{Y}, \bar{x}, d\bar{x}) = \Big[f_s{}^r(Y, x, dx) \frac{\partial \bar{x}^i}{\partial x^r} + \frac{\partial^2 \bar{x}^i}{\partial x^s \partial x^r} Y^r$$
$$+ f_s{}^r(Y, x, dx) \frac{\partial \epsilon^i(Y, x)}{\partial Y^r} + \frac{\partial \epsilon^i(Y, x)}{\partial x^s} \Big] \frac{\partial x^s}{\partial \bar{x}^j} .$$

This very general discussion of the equivalence problem indicates that in constructing a local invariant theory of our X_n it is sufficient to use only tangential coördinate systems and hence only the simultaneous coördinate transformations (8.1, 8.2). An attempt to construct an invariant theory for the general case of an X_n with any fundamental group G is of little practical value. For any particular geometry it is better to use the special devices there available. Thus one important artifice often applicable is that of linearizing the fundamental group by the use of supernumerary coördinates. In these coördinates the displacement equations become linear of the general form

$$dZ^\alpha + \Lambda^\alpha{}_{\beta j}(x; dx) Z^\beta dx^j = 0 \qquad (\alpha, \beta = 1, \cdots, m \geqq n).$$

The quantity with the components $\Lambda^\alpha{}_{\beta j}$ is called a linear connection, and concerning the general theory of such a connection much is known.*

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* The first general treatment of linear connections was given by R. König in

9. The tangent space of differentials. Let dq^i be the differentials of a system of underlying coördinates at the point $x = q$. It is characteristic of differentials that there is a family of relations

$$dq^i = x^i - q^i + \epsilon^i(x - q) \tag{9.1}$$

where the ϵ^i are of order > 1 in $x - q$, between them and the increments of the x's about q, and that under allowable change of coördinates in A_n

$$\bar{x}^i = \bar{x}^i(x) \tag{9.2}$$

they transform according to

$$d\bar{q}^i = (\partial \bar{x}^i / \partial x^j)_q \, dq^j. \tag{9.3}$$

As (9.2) run through all allowable coördinate transformations, (9.3) run through all transformations of the *centered* affine group. These remarks show that the various systems of differentials at q may be interpreted as the preferred coördinate systems of a centered affine space * which by virtue of (9.1) is tangent to A_n at q, with the center $(0, 0 \cdots 0)$ as contact point.

The question presents itself as to the possibility of representing the space of differentials at a point upon the associated B_n at that point. (9.1) indicates that dq^i could be represented by the point Q of coördinates dq^i in some tangential system Y of $B_n(q)$, and that then under (9.2, 9.3) there would be a unique tangential system $\bar{Y}$ associated with $\bar{x}$ in which the point Q has coördinates $d\bar{q}^i$. But since there is no distinction between the tangential coördinate systems in $B_n(q)$ which are associated with a given underlying system x, it is impossible to choose one of them as representing the differentials without introducing extraneous elements into the non-holonomic geometry. However, there is an invariantive one-many correspondence which always exists, namely, one in which a set of differentials dq^i corresponds to all points in $B_n(q)$ which have coördinates dq^i in some tangential system associated with x.

10. Preservation of local correspondence under infinitesimal displacement. Let $B_n(q)$ be mapped upon $B_n(q + dq)$ by displacement from q in the direction dq. The local correspondences existing between $B_n(q)$ and any

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"Beiträge zu einer allgemeinen Mannigfaltigkeitslehre," *Jahr. der D. Math. Ver.*, vol. 28 (1920), pp. 213-228. The most recent publication on the subject is D. J. Struik, "Theory of linear connections," *Erg. d. Math.*, vol. 2, ser. 3, in which there is an extensive bibliography.

* That is, a B_n in the sense of postulates B in which G is the group of linear homogeneous transformations in n variables.

ϵ-neighborhood[*] N_ϵ of q in A are carried into a set of correspondences $\{C\}$ between $B_n(q+dq)$ and N_ϵ. When the set $\{C\}$ is identical to within terms of the second order in ϵ, dq with the *local* correspondences defined between $B_n(q+dq)$ and N_ϵ we say that the local correspondences are preserved under displacement.

The necessary and sufficient condition for this to be the case is

Assumption D $\qquad f_j{}^i(0, x, dx) = -\delta_j{}^i$

where the $f_j{}^i$ are expressed in an admissible tangential reference scheme. The proof is as follows. Applying an admissible tangential reference scheme, a point Y of $B_m(q)$ is mapped under displacement upon the point

$$Y^{i'} = Y^i + f_j{}^i(Y, q, dq)\,dq^j + \delta^i(Y, q, dq) \tag{10.1}$$

of $B_n(q+dq)$ where the δ^i are of order > 1 in dq. The point Y^i of $B_m(q)$ and $Y^{i'}$ of $B_m(q+dq)$ are related to A_n by the respective families of local correspondences

$$Y^i = x^i - q^i + \epsilon^i(x-q) \tag{10.2}$$

$$Y^{i'} = x^i - q^i - dq^i + \epsilon^i(x-q-dq). \tag{10.3}$$

From (10.1), (10.2) the correspondences $\{C\}$ become

$$Y^{i'} = x^i - q^i + f_j{}^i(0, q, dq)\,dq^j + \epsilon^i(x-q, dq) \tag{10.4}$$

where the ϵ^i are of order > 1 in the combined arguments. By a comparison of (10.3) and (10.4) our assertion is proved.

In the general development of a non-holonomic geometry it is not desirable to impose Assumption D *ab initio*. But the investigation once having been made in its full generality, it will be found that the use of this assumption produces a marked simplification in the analytic treatment. It might be mentioned in this connection that in viewing a non-holonomic geometry as immersed in a B_m $m > n$, the assumption D is satisfied.

11. Generalized affine geometry. We now proceed to the application of the general theory just developed to certain special geometries. The generalization of the classical affine geometry is one of the simplest to carry out because of the fundamental rôle played by the affine group in the earlier work. This particular non-holonomic geometry has been very extensively treated and for the most part as a theory of vector displacement. We shall give a brief formulation on the basis of the preceding pages thus taking point displacements as fundamental.

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[*] By an ϵ-neighborhood of q we mean the set of all points x of A_n for which $|x^i - q^i| < \epsilon$.

The group G is in the affine case

$$X^{i'} = a_j{}^i X^j + a^i$$

where the $n^2 + n$ parameters a are subject only to the mild restriction, $\det | a_j{}^i | \neq 0$. A B_n is now a classical affine space with the domains of the preferred coördinate systems coinciding with the space itself. Also there is no distinction between preferred and quasi-preferred systems and in a given B_n there is but one tangential coördinate system associated with any given allowable coördinate system in the underlying A_n. Hence, in conformity with § 9 a set of differentials dq has a unique representation as a point in $B_n(q)$ so that $B_n(q)$ and the space of differentials at q may be identified.

A complete set of infinitesimal generators of the group is given by the array

$$\xi^i{}_{(a)} : \left(\begin{array}{cccccccccccccc} X^1 & X^2 & \cdots & X^n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & X^1 & X^2 & \cdots & X^n & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & X^1 & X^2 & \cdots & X^n & 0 & 0 & \cdots & 1 \end{array}\right)$$

$i =$ row; $\alpha =$ column.

The equations of displacement can therefore be written

$$(11.1) \qquad dX^i + \Gamma^i{}_{jk}(x, dx)\, X^j\, dx^k + \Gamma_k{}^i(x, dx)\, dx^k = 0$$

where the Γ's are of degree zero in dx and of a certain class $\geqq 1$. The geometry will be holonomic provided these equations are completely integrable. The conditions are

$$(11.2) \quad \begin{array}{ll} \text{(a)} & R^i{}_{jkl} = \partial\Gamma^i{}_{jk}/\partial x^l - \partial\Gamma^i{}_{jl}/\partial x^k + \Gamma^i{}_{ml}\Gamma^m{}_{jk} - \Gamma^i{}_{mk}\Gamma^m{}_{jl} = 0 \\ \text{(b)} & S^i{}_{jk} = \partial\Gamma^i{}_j/\partial x^k - \partial\Gamma^i{}_k/\partial x^l + \Gamma^m{}_k\Gamma^i{}_{mj} - \Gamma^m{}_j\Gamma^i{}_{mk} = 0 \\ \text{(c)} & \Gamma^i{}_{jk'l} = \partial\Gamma^i{}_{jk}/\partial(dx^l) = 0 \\ \text{(d)} & \Gamma^i{}_{j'l} = \partial\Gamma^i{}_j/\partial(dx^l) = 0. \end{array}$$

Confining the transformation theory to tangential coördinate systems in conformity with § 8 the allowable transformations are

$$(11.3) \quad \begin{array}{ll} \text{(a)} & \bar{x}^i = \bar{x}^i(x) \\ \text{(b)} & \bar{X}^i = X^j\, \partial\bar{x}^i/\partial x^j \end{array}$$

while the law of transformation of the parameters of displacement become

$$(11.4)\qquad \begin{aligned} &(a)\quad \bar{\Gamma}^i{}_{jk} = \left(\Gamma^r{}_{st}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k}\right)\frac{\partial \bar{x}^i}{\partial x^r} \\ &(b)\quad \bar{\Gamma}^i{}_j = \Gamma^r{}_s \frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^r}. \end{aligned}$$

By virtue of (11. 3) the formalism of tensor analysis is available for the development of the invariant theory. For example, the quantities set equal to zero in (11. 2) become tensors of the character shown by the indices.

Let us now construct the differential equations (6. 10) for the introduction of affine coördinate systems in the underlying A_n. We use tangential coördinate systems in which the law of displacement is (11. 1). Of course in each B_n the contact point is then $(0, 0, \cdots, 0)$. Referring to (6. 5), (6. 6) there results

$$(11.5)\qquad \begin{aligned} &(a)\quad \frac{\partial \bar{X}^i}{\partial q^j} = \frac{\partial \bar{X}^i}{\partial X_0{}^l}\Gamma_j{}^l \\ &(b)\quad \frac{\partial^2 \bar{X}^i}{\partial q^j \partial q^k} = \frac{\partial^2 \bar{X}^i}{\partial X_0{}^l \partial X_0{}^m}\Gamma_j{}^l\Gamma_k{}^m + \frac{\partial \bar{X}^i}{\partial X_0{}^l}\left(\frac{\partial \Gamma_j{}^l}{\partial q^k} + \Gamma^m{}_j\Gamma^l{}_{mk}\right). \end{aligned}$$

By virtue of (6. 8), (11. 5(a)) can be solved in the form

$$(11.6)\qquad \partial \bar{X}^i/\partial X_0{}^l = \gamma_l{}^j\,\partial \bar{X}^i/\partial q^j \qquad \gamma_l{}^j\Gamma_k{}^l = \delta_k{}^i.$$

Also, the Mayer-Lie system (6. 9) for the affine group is

$$(11.7)\qquad \partial^2 \bar{X}^i/\partial X^j \partial X^k = 0.$$

From (11. 5(b)), (11. 6) and (11. 7) the system (6. 10) becomes

$$(11.8)\qquad \frac{\partial^2 \bar{X}^i}{\partial q^j \partial q^k} - \overset{0}{\Gamma}{}^l{}_{jk}\frac{\partial \bar{X}^i}{\partial q^l} = 0$$

where

$$(11.9)\qquad \overset{0}{\Gamma}{}^l{}_{jk} = \gamma^l{}_m(\partial \Gamma^m{}_j/\partial q^k + \Gamma^r{}_j\Gamma^m{}_{rk}).$$

From the form of equations (11. 8) and the fact that they are invariant under the transformations (11. 3) it follows that the functions $\overset{0}{\Gamma}{}^l{}_{jk}$ are the components of an affine connection, that is to say, they transform under (11. 3) according to (11. 4(a)). $\overset{0}{\Gamma}{}^i{}_{jk}$ will be called the derived affine connection to distinguish it from $\Gamma^i{}_{jk}$.

Although the affine displacement determines the derived connection the converse is not true. From (11. 9)

$$(11.10)\qquad \Gamma^i{}_{jk} = \gamma^l{}_j(\Gamma^i{}_m\overset{0}{\Gamma}{}^m{}_{lk} - \partial \Gamma^i{}_l/\partial q^k)$$

and thus for given $\overset{0}{\Gamma}{}^i{}_{jk}$ the choice of $\Gamma^i{}_j$ remains arbitrary.

An important identity arises from expressing the conditions of integrability of (11. 9) viewed as linear differential equations for $\Gamma^i{}_j$. It is

$$\Gamma^i{}_s \overset{0}{R}{}^s{}_{jkl} = \Gamma^s{}_j R^i{}_{skl} \tag{11. 11}$$

where $\overset{0}{R}{}^i{}_{jkl}$ is the curvature tensor formed from $\overset{0}{\Gamma}{}^i{}_{jk}$. We note also that

$$\Gamma^i{}_l S^l{}_{jk} = 2\overset{0}{\Gamma}{}^i{}_{[jk]}. \tag{11. 12}$$

A consideration of (11. 2), (11. 10), (11. 11) and (11. 12) leads to

THEOREM 11. 1. *The integrability of* (11. 1) *implies the integrability** *of* (11. 8). *Conversely the integrability of* (11. 8) *implies the integrability of* (11. 1) *provided* (11. 2d) *are satisfied.*

Of special importance are the solutions of (11. 8) which satisfy the initial conditions $(X^i)_{q=x} = 0$, $(\partial X^i/\partial q^j)_{q=x} = \Gamma_j{}^i(x)$. (6. 11), (11. 5(a)), (11. 6) and the above theorem show that as functions of x these solutions satisfy the displacement equations (11. 1) and transform as a contravariant vector (11. 3b).

Non-integrability of the fundamental equations (11. 8) by no means destroys their utility. Over certain subspaces of A_n they may be integrable and serve to introduce affine coördinates in these regions. In particular this is always the case along curves of A_n and leads to the notion of generalized straight lines, those curves of A_n along which the affine coördinates satisfy the usual linear equations. In terms of a suitable parameter t this condition is

$$d^2X^i/dt^2 = 0. \tag{11. 13}$$

Using (11. 8)

$$\begin{aligned} \frac{d^2X^i}{dt^2} &= \frac{\partial^2 X^i}{\partial q^j \partial q^k}\frac{dq^j}{dt}\frac{dq^k}{dt} + \frac{\partial X^i}{\partial q^j}\frac{d^2q^j}{dt^2} \\ &= \frac{\partial X^i}{\partial q^j}\left(\frac{d^2q^j}{dt^2} + \overset{0}{\Gamma}{}^j{}_{kl}\frac{dq^k}{dt}\frac{dq^l}{dt}\right) = 0. \end{aligned} \tag{11. 13(a)}$$

Since det $|\partial X^i/\partial q^j| \neq 0$ we have as differential equations of the generalized straight lines

$$\frac{d^2x^i}{dt^2} + \overset{0}{\Gamma}{}^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{11. 14}$$

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* This also follows directly from the general theory of § 6.

The following alternative geometrical definition of the curves defined by (11. 14) is noteworthy. Let the spaces $B_n(x)$ associated with the points x of one of these curves be isomorphically represented on one of their number, say $B_n(x_0)$, by means of the fundamental displacement law. Then the contact points of the $B_n(x)$ are represented upon $B_n(x_0)$ as a straight line.

This interpretation points the way to the definition of more general families of curves similar to the one defined by (11. 14). Instead of requiring the contact points to be mapped into a straight line we could require the same of some arbitrary field of points, say $X^i(x)$. The differential equations for the resulting curves may be found as follows. Form the equations (6. 10) as before but using the field $X^i(x)$ in place of the contact point $(0, 0, \cdots 0)$; then proceed as in (11. 13(a)). We need only require of $X^i(x)$ that (6. 8) is satisfied with $X^i(x)$ replacing $X_0{}^i$. The final result corresponding to (11. 14) is

$$\frac{d^2x^i}{dt^2} + \overset{x}{\Gamma}{}^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{11. 15}$$

where

$$\overset{x}{\Gamma}{}^i{}_{jk} = H^i{}_l \biggl(\frac{\partial^2 X^l(x)}{\partial x^j \partial x^k} + \frac{\partial X^m(x)}{\partial x^k} \Gamma^l{}_{mj} + \frac{\partial X^m(x)}{\partial x^j} \Gamma^l{}_{mk} \tag{11. 16}$$
$$+ \frac{\partial \Gamma^l{}_{mj}}{\partial x^k} X^m(x) + \Gamma^l{}_{rk}\Gamma^r{}_{mj} X^m(x) + \frac{\partial \Gamma^l{}_j}{\partial x^k} + \Gamma^l{}_{mk}\Gamma^m{}_j \biggr),$$

and

$$H^i{}_l(\partial X^l(x)/\partial x^j + \Gamma^l{}_{mj} X^m(x) + \Gamma^l{}_j) = \delta^i{}_j.$$

Of course, when $X^i(x) = 0$, $\overset{x}{\Gamma}{}^i{}_{jk} = \overset{0}{\Gamma}{}^i{}_{jk}$.

If $X^i(x)$ does not satisfy (6. 8) the quantity $\overset{x}{\Gamma}{}^i{}_{jk}$ is not uniquely determined.

The argument used to prove $\overset{0}{\Gamma}{}^i{}_{jk}$ to be an affine connection suffices to prove the same concerning $\overset{x}{\Gamma}{}^i{}_{jk}$.

The displacement (11. 1) is composed of an infinitesimal affine transformation

$$dX^i = -\Gamma^i{}_{jk} X^j dx^k$$

leaving the origin invariant and an infinitesimal translation

$$dX^i = -\Gamma^i{}_k dx^k.$$

It therefore establishes between the vectors V^i of the associated spaces correspondences defined by the differential equations

$$dV^i + \Gamma^i{}_{jk} V^j dx^k = 0. \tag{11.17}$$

Corresponding vectors are said to be parallel. These remarks bring our formulation of generalized affine geometry into relation with the theory of infinitesimal parallelism.

The integrability conditions of the vector displacement are (11.2 (a), (c)).

If $x^i = x^i(t)$ is a curve of A_n the quantity $(dx^i/dt)_q$ may be represented geometrically by a vector $V^i = (dx^i/dt)_q$ in the tangential coördinate system of $B_n(q)$ and by virtue of the local correspondences this vector plays the rôle of tangent vector to the curve at the point q. With this observation before us an autoparallel may be defined as a curve in A_n whose tangent vectors are parallel according to (11.17) along the curve. The differential equations for the family of all such curves are obtained immediately from (11.17) as

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \tag{11.18}$$

These curves are the familiar "paths" of the theory of infinitesimal parallelism and it is to be noted that in general they are not the same as the generalized straight lines defined by (11.14). The necessary and sufficient condition that these two systems be the same is most simply expressed by the condition

$$\overset{0}{\Pi}{}^i{}_{jk} = \Pi^i{}_{jk}$$

where

$$\Pi^i{}_{jk} = \Gamma^i{}_{(jk)} - \frac{1}{n+1} (\delta^i{}_j \Gamma^h{}_{(hk)} + \delta^i{}_k \Gamma^h{}_{(hj)}),$$

and similarly $\overset{0}{\Pi}{}^i{}_{jk}$, are the projective connections * formed from the symmetrical Γ's.

To conclude this section we consider the result of imposing Assumption D (p. 171) upon the displacement parameters in (11.1). This condition is for the case under consideration

$$\Gamma^i{}_j = \delta^i{}_j. \tag{11.19}$$

Integrability conditions (11.2 (b)) become

$$2\Gamma^i{}_{[jk]} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj} = 0, \tag{11.2 (b')}$$

and most important of all

$$\overset{0}{\Gamma}{}^i{}_{jk} = \Gamma^i{}_{jk}. \tag{11.20}$$

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* L. P. Eisenhart, "Non-Riemannian geometry," *American Mathematical Society Colloquium Publications*, 1927, p. 98. The quantities $\Pi^i{}_{jk}$ are due to T. Y. Thomas.

It is interesting to note that the necessary and sufficient conditions for the existence of the relation (11. 20) entirely apart from (11. 19) is the vanishing of the covariant derivative of the tensor $\Gamma^i{}_j$ with respect to $\Gamma^i{}_{jk}$.

12. Generalized projective geometry. The field to which we shall next apply our general methods is that of the generalized projective geometry. This field, like that of the generalized affine geometry discussed in the preceding section has been widely exploited in recent years but there is a notorious lack of uniformity in formulation and method of investigation. This is due to the fact that the passage from the affine to the projective case in non-holonomic geometry involves such an extensive generalization that a certain freedom of choice in the formulation is inescapable. This fact alone shows the desirability of a general non-holonomic theory such as the one we have given here to serve as a norm in the construction of the more involved non-holonomic geometries. The first generalization of projective geometry was made by Weyl * in 1921 and investigation was continued along the same general lines by T. Y. Thomas, O. Veblen and others in the years immediately following. Cartan † in his development of the subject introduced a point of view which has been adopted in its essentials in many of the recent contributions to the subject.‡ The brief development we give here on the basis of our general theory amounts to establishing this point of view on a solid foundation. In the process several new results are obtained.

The fundamental group G has in the projective case the form

$$\bar{X}^{i'} = (a^i{}_j X^j + a^i{}_0)/(a^0{}_k X^k + a^0{}_0) \tag{12.1}$$

where the ratios of the $a^\alpha{}_\beta$ $(\alpha, \beta = 0, 1, \cdots n)$ form $n^2 + 2n$ essential parameters with $\det | a^\alpha{}_\beta | \neq 0$. The classical projective space clearly satisfies axioms B with the above group for G. Hence by theorem (2. 2) every B_n for this G is such a space. Just as in the affine case (§ 11) there is no distinction between preferred and quasi-preferred coördinate systems but on the other hand there is now more than one tangential coördinate system associated with a given allowable system x in the underlying A_n. Indeed if X is one such system in $B_n(x)$ then any system $\bar{X}$ obtained from X by the transformation

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* "Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformem Auffassung," *Gott. Nach.* (1921), pp. 99-112.

† "Sur les variétés a connexion projective," *Bulletin de la Société Mathématique de France*, vol. 52 (1924), pp. 205-241. Cf. Introduction.

‡ Of these contributions the following are representative:

E. Bortolotti, "Sulle connessioni proiettive," *Rendiconti del Circolo Matematico di Palermo*, vol. 56 (1932), pp. 1-57; D. van Dantzig, "Theorie des projektiven Zusammenhanges n-dimensionaler Raüme," *Mathematische Annalen*, vol. 106 (1932), pp. 400-464; J. A. Schouten and D. van Dantzig, "On projective connections and their applications to the general field theory," *Annals of Mathematics*, vol. 34 (1933); O. Veblen, "Projektive Relativitatstheorie," *Erg. der Math.* (Springer), 1933; J. H. C. Whitehead, "The representation of projective spaces," *Annals of Mathematics*, vol. 32 (1931), pp. 327-360.

In the fourth item a fairly complete bibliography will be found.

$$\bar{X}^i = X^i/(1 + b_k X^k) \tag{12.2}$$

is one also, and conversely all the tangential systems are obtained in this way. The invariantive representation (cf. § 9) of a set of differentials at a point x is therefore a straight line through the contact point of $B_n(x)$ with the contact point itself omitted.

A complete set of generators for the group 12.1 is

$$\xi^i{}_{(a)}: \quad \begin{pmatrix} X^1 \cdots X^n & 0 \cdots 0 & \cdots & 0 \cdots 0 & X^1X^1 & X^1X^2 & \cdots & X^1X^n & 1 & 0 & \cdots & 0 \\ 0 \cdots 0 & X^1 \cdots X^n & \cdots & 0 \cdots 0 & X^2X^1 & X^2X^2 & \cdots & X^2X^n & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 \cdots 0 & 0 \cdots 0 & \cdots & X^1 \cdots X^n & X^nX^1 & X^nX^2 & \cdots & X^nX^n & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

$i =$ row; $\alpha =$ column.

The displacement law therefore will have the form

$$dX^i + P^i{}_{jk}(x, dx) X^j dx^k + X^i P_{jk}(x, dx) X^j dx^k + P^i{}_k(x, dx)\, dx^k = 0 \tag{12.4}$$

where the P's are homogeneous of degree zero in dx and of class $\geqq 1$. The condition for the preservation of local correspondences, Assumption D, is

$$P^i{}_k = \delta^i_k. \tag{12.5}$$

In accordance with § 8, for the purposes of invariant theory or questions of equivalence, it will be sufficient to keep to admissible tangential reference schemes and hence to consider only transformations which are combinations of the two types

$$\text{(a)} \quad \bar{x}^i = \bar{x}^i(x) \qquad \text{(b)} \quad \bar{X}^i = X^j \partial\bar{x}^i/\partial x^j \tag{12.6}$$

and

$$\text{(a)} \quad \bar{x}^i = x^i \qquad \text{(b)} \quad \bar{X}^i = X^i/(1 + \phi_j(x) X^j) \tag{12.7}$$

where $\phi_i(x)$ are arbitrary functions of x of class $\geqq 2$.

Under (12.6) the $P^i{}_{jk}$ transform as the components of an affine connection * and P_{jk}, $P^i{}_k$ as tensors of the character indicated by their indices while under (12.7) the transformation equations are

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* That is to say, according to (11.4(a)).

$$(12.8)\qquad \begin{aligned} &(a)\quad \bar{P}^i{}_{jk} = P^i{}_{jk} - P^i{}_k\phi_j - \delta^i{}_j\phi_l P^l{}_k \\ &(b)\quad \bar{P}_{jk} = P_{jk} + \partial\phi_j/\partial x^k - \phi_l P^l{}_{jk} + P^l{}_k\phi_l\phi_j \\ &(c)\quad \bar{P}^i{}_k = P^i{}_k. \end{aligned}$$

The conditions for the holonomity of the geometry are

$$(12.9)\qquad \begin{aligned} &(a)\quad P^i{}_{jkl} \equiv \partial P^i{}_{jk}/\partial x^l - \partial P^i{}_{jl}/\partial x^k + P^r{}_{jk}P^i{}_{rl} - P^r{}_{jl}P^i{}_{rk} \\ &\qquad\qquad + \delta^i{}_j P^r{}_k P_{rl} - \delta^i{}_j P^r{}_l P_{rk} + P_{jl}P^i{}_k - P_{jk}P^i{}_l = 0 \\ &(b)\quad P^0{}_{jkl} \equiv \partial P_{jk}/\partial x^l - \partial P_{jl}/\partial x^k + P_{rl}P^r{}_{jk} - P_{rk}P^r{}_{jl} = 0 \\ &(c)\quad P^i{}_{0kl} \equiv \partial P^i{}_k/\partial x^l - \partial P^i{}_l/\partial x^k + P^r{}_k P^i{}_{rl} - P^r{}_l P^i{}_{rk} = 0 \\ &(d)\quad P^i{}_{jk},\ P_{jk},\ P^i{}_k \text{ independent of } dx. \end{aligned}$$

These conditions are of course independent of the coördinate systems used in A_n and the associated B_n's.

In forming the fundamental equations for the introduction of projective coördinates in A_n we shall use in place of the Mayer-Lie system the following simpler set of equations

$$(12.10)\qquad \frac{\partial^2\bar{X}^i}{\partial X^j\partial X^k} = \frac{\partial\bar{X}^i}{\partial X^j}\frac{\partial\Theta}{\partial X^k} + \frac{\partial\bar{X}^i}{\partial X^k}\frac{\partial\Theta}{\partial X^j},$$

$$\left(\Theta = \frac{1}{n+1}\log\text{const}\left|\frac{\partial\bar{X}}{\partial X}\right|\right)$$

which are known * to have as complete solutions just the group (12.1). System (12.10) is not passive. Its integrability conditions are

$$\frac{\partial^2\Theta}{\partial X^i\partial X^j} - \frac{\partial\Theta}{\partial X^i}\frac{\partial\Theta}{\partial X^j} = 0.$$

Referring the associated spaces to an admissible tangential reference scheme equations (6.5), (6.6) furnish

$$(12.11)\qquad \begin{aligned} &(a)\quad \frac{\partial\bar{X}^i}{\partial q^j} = \frac{\partial\bar{X}^i}{\partial X^l{}_0}P^l{}_j \\ &(b)\quad \frac{\partial^2\bar{X}^i}{\partial q^j\partial q^k} = \frac{\partial^2\bar{X}^i}{\partial X^l{}_0\partial X^m{}_0}P^l{}_j P^m{}_k + \frac{\partial\bar{X}^i}{\partial X^l{}_0}\left(\frac{\partial P^l{}_j}{\partial q^k} + P^m{}_j P^l{}_{mk}\right). \end{aligned}$$

Combining (12.10) and (12.11) the fundamental equations may be written

$$(12.12)\qquad \frac{\partial^2\bar{X}^i}{\partial q^j\partial q^k} = P^r{}_{jk}\frac{\partial X^i}{\partial q^r} + \frac{\partial X^i}{\partial q^j}\frac{\partial\Theta}{\partial q^k} + \frac{\partial X^i}{\partial q^k}\frac{\partial\Theta}{\partial q^j}$$

where

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* O. Veblen and J. M. Thomas, "Projective invariants of affine geometry of paths," *Annals of Mathematics*, vol. 27 (1926), p. 284.

$$(12.13)\qquad \overset{0}{P}{}^i{}_{jk} = p^i{}_r(\partial P^r{}_j/\partial q^k + P^s{}_j P^r{}_{sk}), \qquad p^i{}_r P^r{}_j + \delta^i{}_j.$$

Under (12.6) $\overset{0}{P}{}^i{}_{jk}$ transform as an affine connection and under (12.7)

$$(12.14)\qquad \overset{0}{P}{}^{i\prime}{}_{jk} = \overset{0}{P}{}^i{}_{jk} + \delta^i{}_j P^l{}_k \phi_l + \delta^i{}_k P^l{}_j \phi_l.$$

The quantities $\partial\Theta/\partial x^j$ may be eliminated from (12.12) to give the equivalent equations

$$(12.15)\qquad \frac{\partial^2 X^i}{\partial q^j \partial q^k} - \frac{1}{n+1}\frac{\partial \log|\partial X/\partial x|}{\partial q^j}\frac{\partial X^i}{\partial q^k} - \frac{1}{n+1}\frac{\partial \log|\partial X/\partial x|}{\partial q^k}\frac{\partial X^i}{\partial q^j} = \overset{0}{\Pi}{}^l{}_{jk}\frac{\partial X^i}{\partial q^l}$$

where

$$(12.16)\qquad \overset{0}{\Pi}{}^i{}_{jk} = \overset{0}{P}{}^i{}_{jk} - \frac{1}{n+1}(\delta^i{}_j \overset{0}{P}{}^r{}_{rk} + \delta^i{}_k \overset{0}{P}{}^r{}_{rj}).$$

The quantities $\overset{0}{\Pi}{}^i{}_{jk}$ are invariant under (12.7) and transform under (12.6) like the projective connection of T. Y. Thomas i. e.

$$(12.17)\qquad \overset{\bar{0}}{\Pi}{}^i{}_{jk} = \left(\overset{0}{\Pi}{}^r{}_{st}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k} - \frac{\delta^r{}_j}{n+1}\frac{\partial \log|\partial x/\partial \bar{x}|}{\partial \bar{x}^k}\frac{\delta^r{}_k}{n+1}\frac{\partial \log|\partial x/\partial \bar{x}|}{\partial \bar{x}^j}\right)\frac{\partial \bar{x}^i}{\partial x^r}.$$

The generalized straight lines in A_n may now be defined just as in the affine case. Using a suitable parameter t the differential equations of these lines in projective coördinates are (11.13) and expressing these equations in general coördinates by the aid of (12.12) (cf. 11.13(a)) there results

$$\frac{d^2 x^i}{dt^2} + \left(\overset{0}{P}{}^i{}_{jk} + \delta^i{}_j\frac{\partial\Theta}{\partial x^k} + \delta^i{}_k\frac{\partial\Theta}{\partial x^j}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Through a further change in parameter these equations may be written

$$(12.18)\qquad \frac{d^2 x^i}{dt^2} + \overset{0}{P}{}^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

From the method of derivation of equations (12.18), they continue to represent the same geometrical loci under all transformations (12.6), (12.7). This is also clear from a direct consideration of the law of transformation of $\overset{0}{P}{}^i{}_{jk}$ given above.

We shall merely note in passing that systems of paths associated with any point field $X^i(x)$ may be defined just as in the affine case (p. 176). To each field will correspond a set of parameters $\overset{x}{P}{}^i{}_{jk}$ with properties analogous to those of $\overset{0}{P}{}^i{}_{jk}$.

There is a noteworthy simplification of the equivalence problem as formulated in § 8 which is available in the projective case. It is possible in case $u \geqq 4$ to select in invariantive manner a unique admissible tangential reference scheme to be associated with a given underlying system x, namely the one in which

$$P^i{}_{ik} = 0. \tag{12.19}$$

Under (12.6) and (12.7) respectively

$$\begin{aligned} &\text{(a)} \quad \bar{P}^i{}_{il} = P^i{}_{ik} \frac{\partial x^k}{\partial \bar{x}^l} + \frac{\partial \log \mid \partial \bar{x}/\partial x \mid}{\partial \bar{x}^l}, \\ &\text{(b)} \quad \bar{P}^i{}_{ik} = P^i{}_{ik} - (n+1) P^l{}_k \phi_l. \end{aligned} \tag{12.20}$$

Hence if $\{X\}$ is a set of tangential coördinate systems in which $P^i{}_{ik} \neq 0$ it is only necessary to apply (12.7) with $\phi_l = [1/(n+1)] p^k{}_l P^i{}_{ik}$ to obtain the unique set $\{\bar{X}\}$ in which (12.19) hold true. Then to insure the invariance of (12.19) the allowable transformations must be restricted to the special combination of (12.6), (12.7),

$$\begin{aligned} &\text{(a)} \quad \bar{x}^i = \bar{x}^i(x) \\ &\text{(b)} \quad X^i = \frac{X^j \, \partial \bar{x}^i / \partial x^j}{1 - p^l{}_k \theta_l X^k}, \end{aligned} \tag{12.21}$$

where θ_l is an abbreviation for $\dfrac{1}{n+1} \dfrac{\partial \log \mid \partial \bar{x}/\partial x \mid}{\partial x^l}$.

The transformation induced by (12.21) on the displacement parameters is

$$\begin{aligned} &\text{(a)} \quad \bar{P}^i{}_{jk} = (P^r{}_{st} + P^r{}_t p^l{}_s \theta_l + \delta^r{}_s \theta_t) \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial x^t}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^r} + \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^s}. \\ &\text{(b)} \quad P_{jk} = \left(P_{rs} - p^l{}_r \frac{\partial \theta_l}{\partial x^s} - \theta_l \frac{\partial p^l{}_r}{\partial x^s} + p^l{}_m \theta_l P^m{}_{rs} + p^l{}_r \theta_l \theta_s\right) \frac{\partial x^r}{\partial \bar{x}^j} \frac{\partial x^s}{\partial \bar{x}^k} \\ &\text{(c)} \quad \bar{P}^i{}_k = P^r{}_s \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^r} \qquad p^i{}_l P^l{}_j = \delta^i{}_j. \end{aligned} \tag{12.22}$$

The formal statement of the equivalence problem becomes: Given two non-holonomic projective spaces X_n, $\bar{X}_n$ in which coördinates have been chosen such that $P^i{}_{ik} = \bar{P}^i{}_{ik} = 0$, to find an allowable transformation $\bar{x}^i = \bar{x}^i(x)$ under which (12.22) are identically satisfied.

Even with this simplification the equivalence problem with its concomitant invariant theory is a great deal more complicated than in the case of the affine theory previously discussed. This situation subsists because of the non-linear character of the non-homogeneous projective transformations which renders impossible the concise tensor theoretic treatment of the affine case. By the use of homogeneous projective coördinates, linearity may be regained (at the expense of complete functional determinacy it is true) and a projective tensor calculus introduced which simplifies the formalism of the theory. This idea due originally to T. Y. Thomas was elaborated and given added significance by Veblen. We now turn to this line of development. However the non-homogeneous discussion already given is not to be considered supererogatory. It is of interest because it is the most direct development from the standpoint of our general theory of non-holonomic geometries and furthermore offers a welcome basis for the establishment of the homogeneous theory.

Each tangent space being an ordinary projective space, homogeneous coördinates Z^α $(\alpha = 0, 1, \cdots, n)$ may be introduced. The domain of each homogeneous coördinate system is of course the whole space. The existence of the special class of non-homogeneous projective coördinates which we have called tangential permits the selection of a correspondingly restricted class of homogeneous coördinates to play a like rôle in the new formulation. The simplest procedure is to replace each tangential system X by the homogeneous system Z in which

$$X^i = Z^i/Z^0 \tag{12.24}$$

and each such system Z will be called appropriately a homogeneous tangential coördinate system.

In the new coördinates the restricted transformations (12.6), (12.7) become

$$\text{(12.25)} \qquad \begin{aligned} &\text{(a)} \quad \bar{x}^i = \bar{x}^i(x) \\ &\text{(b)} \quad \bar{Z}^\alpha = \rho v^\alpha{}_\beta Z^\beta \end{aligned}$$

where

$$\text{(12.26)} \qquad v^\alpha{}_\beta : \quad \begin{pmatrix} 1 & \phi_1 \cdots \phi_n \\ 0 & \\ \vdots & \dfrac{\partial \bar{x}^i}{\partial x^j} \\ 0 & \end{pmatrix}$$

and $\rho(x)$, $\phi_i(x)$ are arbitrary functions of class $\geqq 2$.

To obtain an expression for the displacement law in the new coördinates make the substitution (12.24) in (12.4). There results

$$\frac{dZ^i + P^i{}_{jk} Z^j dx^k + P^i{}_j Z^0 dx^j}{Z^i} = \frac{dZ^0 - P_{jk} Z^j dx^k}{Z^0} \qquad (i = 1, \cdots, n).$$

These equations may be written in more convenient form by the following device. Let the common value of each member be written $\tau(x, dx)\, dx^0$ where $\tau(x, dx)$ is an arbitrary function of degree zero in dx, and dx^0 represents an arbitrary differential form, that is to say, an independent parametric differential. The above equations become

$$dZ^\alpha + \Pi^\alpha{}_{\beta\gamma} Z^\beta dx^\gamma = 0 \tag{12.27}$$

with

$$\left\{\begin{array}{ll} \text{(a)} & P^i{}_{jk} = \Pi^i{}_{jk} - \delta^i{}_j \Pi^0{}_{0k} \\ \text{(b)} & P_{jk} = -\Pi^0{}_{jk} \\ \text{(c)} & P^i{}_k = \Pi^i{}_{0k} \\ \text{(d)} & \delta^\alpha{}_\beta\, \tau = \Pi^\alpha{}_{\beta 0}. \end{array}\right. \tag{12.28}$$

Conversely (12.4) may be derived by the use of (12.27), (12.28). (12.28) show that $\Pi^\alpha{}_{\beta\gamma}$ are arbitrary to the extent of a transformation

$$\Pi'^\alpha{}_{\beta\gamma} = \Pi^\alpha{}_{\beta\gamma} + \delta^\alpha{}_\beta \psi_\gamma. \tag{12.29}$$

Clearly the parametric differential dx^0 may be subjected to any transformation

$$dx^{0\prime} = dx^0 + \sigma_i dx^i$$

without affecting the displacement and it is therefore possible to make the convenient formal convention that under (12.25) the differentials $dx^0, dx^1, \cdots, dx^n$ transform by

$$d\bar{x}^\alpha = v^\alpha{}_\beta dx^\beta. \tag{12.25(c)}$$

In discussing the transformation theory of $\Pi^\alpha{}_{\beta\gamma}$ the utility of this convention becomes evident. For under (12.25) the law of transformation of Π may be written

$$\bar{\Pi}^\alpha{}_{\beta\gamma} = (\Pi^\rho{}_{\sigma\tau} u^\sigma{}_\beta u^\tau{}_\gamma + \partial u^\sigma{}_\beta / \partial \bar{x}^\gamma)\, v^\alpha{}_\sigma + \delta^\alpha{}_\beta \psi_\gamma. \tag{12.30}$$

Thus it behaves like an affine connection, (11.4(a)), in $(n+1)$ dimensions under the restricted transformations *

$$\begin{array}{ll} \text{(a)} & \bar{x}^i = \bar{x}^i(x) \\ \text{(b)} & \bar{x}^0 = x^0 + \int v^0{}_i dx^i \end{array} \tag{12.31}$$

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* The importance of these transformations for generalized projective geometry was first shown by O. Veblen, "Generalized projective geometry," *Journal of the London Mathematical Society*, vol. 4 (1929), p. 144. J. H. C. Whitehead (*loc. cit.*) has introduced the convenient term "change of representation" to apply to such a transformation.

and indeterminate to within transformations which preserve parallelism.* The displacement (12.27) therefore can be interpreted as an affine direction displacement in a space of $(n+1)$ dimensions subject to the infinite group (12.31) of coördinate transformations. When (12.31(b)) is not integrable the $(n+1)$-dimensional space is of the extended type studied by Schouten,† Vranceanu and others and called by them non-holonomic. Our use of this term has of course quite a different significance.

The geometric object having as components the parameters Π will be called an indeterminate projective connection.

Especially from the invariant-theoretic viewpoint the indeterminancy in $\Pi^{\alpha}{}_{\beta\gamma}$ is very undesirable. Bortolotti has shown ‡ that the indeterminate parameters of an affine direction displacement may be normalized to give a unique set transforming as an affine connection and has applied the result to the projective case.§ The normalization is effected by selecting that particular set of equivalent Π's for which

$$\Pi^{\alpha}{}_{[\beta\alpha]}=0. \tag{12.32}$$

Calling this set $P^{\alpha}{}_{\beta\gamma}$, its expression in terms of any set of the Π's from which it arises is

$$P^{\alpha}{}_{\beta\gamma}=\Pi^{\alpha}{}_{\beta\gamma}+(2/n)\delta^{\alpha}{}_{\beta}\Pi^{\sigma}{}_{[\gamma\sigma]}. \tag{12.33}$$

From (12.33)

$$P^{\alpha}{}_{\beta 0}=(1/n)\delta^{\alpha}{}_{\beta}P^{i}{}_{0i}$$

and hence for this set of parameters

$$\tau(x,dx)=(1/n)P^{i}{}_{0i}. \tag{12.34}$$

It is of interest to note that if $\Pi^{\alpha}{}_{\beta\gamma}$ is semi-symmetric in the sense of Schouten, $P^{\alpha}{}_{\beta\gamma}$ is itself symmetric.

The parameters $P^{\alpha}{}_{\beta\gamma}$ will be said to form the normalized projective connection. Under (12.31) its law of transformation is of course

$$P^{\alpha}{}_{\beta\gamma}=(P^{\sigma}{}_{\rho\tau}u^{\rho}{}_{\beta}u^{\tau}{}_{\gamma}+\partial u^{\sigma}{}_{\beta}/\partial\bar{x}^{\gamma})v^{\alpha}{}_{\sigma}. \tag{12.35}$$

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* See, for example, Eisenhart, *op. cit.*, p. 30.

† See, for example, Schouten and van Kampen, "Zur Einbettungs- und Krummungstheorie nichtholonomer Gebilde," *Mathematische Annalen*, vol. 103 (1930), pp. 752-783 and the list of references there given.

‡ "Sulla geometria delle varietà a connessione affine," *Ann. Mat. pura appl.*, vol. 8 (1930), p. 78.

§ "Sulla connessione proiettive," *Rendiconti del Circolo Matematico di Palermo*, vol. 56 (1932), p. 21 (6.9). The normalized parameters $\Gamma^{\alpha}{}_{\beta\gamma}$ (6.5) which Bortollotti seems to prefer are not available unless $\Pi^{\alpha}{}_{0\beta}=\delta^{\alpha}{}_{\beta}$.

The existence of the unique parameters P and of the $(n+1)$-dimensional affine representation makes possible a direct application of the familiar affine tensor analysis to the projective case.* However the functional form of the tensors used must be given consideration because of the special nature from the projective standpoint of the parameter x^0. Indeed it is immediately evident that an $(n+1)$ affine tensor, say $T^\alpha{}_{\beta\gamma}$, has n-dimensional projective significance only if it takes the general form

$$(12.36) \qquad T^\alpha{}_{\beta\gamma} = \rho\, t^\alpha{}_{\beta\gamma}\,(x^1, \cdots, x^n).$$

where ρ need not be a point function but only an expression on which the operation of differentiation with respect to x^α is defined. For example an $(n+1)$-dimensional affine contravariant vector $Z^\alpha(x^0, \cdots, x^n)$ represents the homogeneous coördinates of a point field in the projective tangent spaces only if x^0 enters in the guise of a homogeneity factor as above.

Now the covariant derivative of $T^\alpha{}_{\beta\gamma}$ with respect to $P^\alpha{}_{\beta\gamma}$

$$(12.37) \qquad T^\alpha{}_{\beta\gamma,\delta} = \partial T^\alpha{}_{\beta\gamma}/\partial x^\delta + T^\sigma{}_{\beta\gamma}P^\alpha{}_{\sigma\delta} - T^\alpha{}_{\sigma\gamma}P^\sigma{}_{\beta\delta} - T^\alpha{}_{\beta\sigma}P^\sigma{}_{\gamma\delta}$$

is of course also a tensor. Patently such a derived tensor must also be required to have the general form (12.36) if the process of covariant differentiation is to be of projective value. The necessary and sufficient condition for this to be the case as follows directly from (12.37) is

$$\rho = \exp[\textstyle\int \rho_\alpha(x_1 \cdots x_n)\,dx^\alpha] = \exp(\rho_0 x^0 + \textstyle\int \rho_i dx^i).$$

Thus we are led to the following

Definition. An n-dimensional projective tensor is an $(n+1)$-dimensional affine tensor of the functional form

$$T^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} = \exp(\rho_0 x^0 + \textstyle\int \rho_i dx^i)\, t^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}\,(x^1 \cdots x^n)$$

under the special class of transformations (12.31). ρ_0 is called the index of the projective tensor.

Veblen was first to introduce this definition of a projective tensor but in limiting ρ_0 to being constant. We have attempted to arrive at it logically from the initial idea of a projective displacement.

The parameter x^0 has been put to the welcome service of lending functional homogeneity to projective tensors. However it must be remembered that any tensor obtained from a given tensor by multiplication with a function $\rho(x^1, \cdots, x^n)$ is geometrically equivalent to the original. This fact introduces complications into the analytical treatment of the subject which can be avoided only by special normalizing devices.*

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* In this fact lies the principal justification for the introduction of x^0. But one must avoid the error of assuming that certain well-known affine tensors retain their tensor character when $\partial\bar{x}^0/\partial x^i$ is not a gradient. A very simple case in point is the curl of a covariant vector. Compare also the remark under (12.39(a)).

Expressed in the tensor notation the fundamental displacement (12. 27) takes the form

$$Z^\alpha{}_{,\beta} dx^\beta = Z^\alpha \psi_\beta dx^\beta, \qquad \psi_\beta \text{ arb}, \tag{12. 38}$$

and the comma in $Z^\alpha{}_{,\beta}$ indicates covariant differentiation with respect to $P^\alpha{}_{\beta\gamma}$. The integrability conditions of these equations are identical with those of an affine direction displacement in the $(n+1)$-dimensional representation, conditions first explicitly given by Bortolotti,

$$\text{(a)} \quad P^\alpha{}_{\beta\gamma\delta} - \frac{\delta^\alpha{}_\beta}{n+1} P^\sigma{}_{\sigma\gamma\delta} = 0 \tag{12. 39}$$

where

$$P^\alpha{}_{\beta\gamma\delta} = \partial P^\alpha{}_{\beta\gamma}/\partial x^\delta - \partial P^\alpha{}_{\beta\delta}/\partial x^\gamma + P^\sigma{}_{\beta\gamma} P^\alpha{}_{\sigma\delta} - P^\sigma{}_{\beta\delta} P^\alpha{}_{\sigma\gamma}.$$

$P^\alpha{}_{\beta\gamma\delta}$ transforms as a tensor only when $\partial x^0/\partial \bar{x}^i$ is a gradient and may be termed for convenience the curvature quasi-tensor of $P^\alpha{}_{\beta\gamma}$. The left-hand side of (12. 39(a)) is a tensor without restriction.

Since in general P and ψ involve dx, we must add the condition

$$\text{(b)} \quad \frac{\partial}{\partial(dx^\delta)} \left[P^\alpha{}_{\beta\gamma} - \frac{\delta^\alpha{}_\beta}{n+1} P^\sigma{}_{\sigma\gamma} \right] = 0. \tag{12. 39}$$

When (12. 39) are identically satisfied the geometry is holonomic.

Of course, the conditions (12. 39) could have been written in the more immediate form

$$\Pi^\alpha{}_{\beta\gamma\delta} = 0 \quad \frac{\partial \Pi^\alpha{}_{\beta\gamma}}{\partial(dx^\delta)} = 0 \tag{12. 40}$$

where $\Pi^\alpha{}_{\beta\gamma\delta}$ is the curvature quasi-tensor formed from $\Pi^\alpha{}_{\beta\gamma} = P^\alpha{}_{\beta\gamma} - \delta^\alpha{}_\beta \psi_\gamma$ but (12. 39) have the advantage of avoiding the indeterminate ψ_γ.

We are now in a position to construct the fundamental differential equations for the introduction of homogeneous projective coördinates in the underlying space. Formally the n-dimensional projective group as expressed in homogeneous coördinates is identical with the $(n+1)$-dimensional centered affine group.† The Mayer-Lie equations (6. 12) become in this case

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* For example, a contravariant vector Z^α may be normalized by the restriction $| Z^\alpha{}_{,\beta} | = 1$ provided the determinant does not vanish, and a covariant tensor $G_{\alpha\beta}$ may be subjected to the invariant condition $G_{00} = 1$. The literature of classical projective differential geometry (Wilczynski, Fubini, etc.) is replete with like normalizations.

† But of the $(n+1)^2$ parameters only $n^2 + 2n$ are of geometrical significance.

$$(12.41)\qquad \text{(a)}\quad \frac{\partial^2 \bar{Z}^a}{\partial Z^\beta \partial Z^\gamma} = 0$$

$$\text{(b)}\quad \bar{Z}^a - \frac{\partial \bar{Z}^a}{\partial Z^\beta} Z^\beta = 0.$$

The auxiliary relations (12. 41(b)) express the homogeneity of the finite equations of the group and reduce the parameters to $(n+1)^2$.

In applying the theory of § 6 it is convenient to write the displacement equations in terms of the indeterminate connection $\Pi^a{}_{\beta\gamma}$ and to choose the coördinates of the contact point of each tangent space as $(1, 0, 0, \cdots, 0)$.* Equations (6. 5), (6. 6) become respectively

$$\frac{\partial \bar{Z}^a}{\partial q^\beta} = \frac{\partial \bar{Z}^a}{\partial Z^\gamma{}_0} \Pi^\gamma{}_{0\beta}$$

$$\frac{\partial^2 \bar{Z}^a}{\partial q^\beta \partial q^\gamma} = \frac{\partial^2 \bar{Z}^a}{\partial Z^\mu{}_0 \partial Z^\nu{}_0} \Pi^\mu{}_{0\beta} \Pi^\nu{}_{0\gamma} + \frac{\partial \bar{Z}^a}{\partial Z^\mu{}_0} \left(\frac{\partial \Pi^\mu{}_{0\beta}}{\partial q^\gamma} + \Pi^\nu{}_{0\beta} \Pi^\mu{}_{\nu\gamma} \right).$$

Using (12. 41(a)) there results

$$(12.42)\qquad \text{(a)}\quad \frac{\partial^2 Z^a}{\partial q^\beta \partial q^\gamma} - \Pi^\sigma{}_{\beta\gamma} \frac{\partial Z^a}{\partial q^\sigma} = 0$$

with

$$(12.43)\qquad \overset{0}{\Pi}{}^a{}_{\beta\gamma} = \pi^a{}_\sigma (\partial \Pi^\sigma{}_{0\beta}/\partial q^\gamma + \Pi^\mu{}_{0\beta} \Pi^\sigma{}_{\mu\gamma}), \qquad \pi^a{}_\beta \Pi^\beta{}_{0\gamma} = \delta^a{}_\gamma,$$

while (12. 41(b)) furnishes

$$Z^a - \pi^\beta{}_0\, \partial Z^a/\partial q^\beta = 0,$$

that is

$$(12.42)\qquad \text{(b)}\quad \partial Z^a/\partial q^0 = \tau Z^a.$$

From (12. 43)

$$(12.44)\qquad \overset{0}{\Pi}{}^a{}_{\beta 0} = \delta^a{}_\beta \tau, \qquad \overset{0}{\Pi}{}^a{}_{0\beta} = \delta^a{}_\beta \tau + \delta^a{}_0 \frac{\partial \log \tau}{\partial q^\beta}.$$

It is essential to consider the effect upon the fundamental equations (12. 42) of the indeterminacy (12. 29) in Π. Under (12. 29)

$$\pi'^a{}_\beta = \pi^a{}_\beta - (\delta^a{}_0/\tau') \pi^\gamma{}_\beta \psi_\gamma, \qquad \tau' = \tau + \psi_0.$$

This relation and (12. 29) applied to (12. 43) shows that the effect is to replace $\overset{0}{\Pi}{}^a{}_{\beta\gamma}$ by

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* More generally the contact point is $(e^{\lambda x^0} \rho(x), 0, 0, \cdots, 0)$ but the use of this expression instead of the above may be shown to be equivalent to replacing $\Pi_{\beta\gamma}{}^a$ by $\Pi_{\beta\gamma}{}^a + \delta_\beta{}^a [\partial \log e^{\lambda x^0} \rho(x)/\partial x^\gamma]$, a change of no significance because of the indeterminacy in the Π's.

$$(12.45)\quad \overset{0}{\Pi}'^{\alpha}{}_{\beta\gamma} = \overset{0}{\Pi}^{\alpha}{}_{\beta\gamma} + \delta^{\alpha}{}_{\beta}\psi_{\gamma} + \delta^{\alpha}{}_{\gamma}\psi_{\beta} + \frac{\delta^{\alpha}{}_{0}}{\tau'_{0}}\left[\frac{\partial\psi_{\beta}}{\partial q^{\gamma}} - \psi_{\delta}\overset{0}{\Pi}^{\delta}{}_{\beta\gamma} - \psi_{\beta}\psi_{\gamma}\right].$$

The quantities $\overset{0}{\Pi}$ subject to the indeterminacy (12.45) will be called the derived indeterminate projective connection. As a set of normalized $\overset{0}{\Pi}$ there are immediately at hand

$$(12.47)\qquad \overset{0}{P}{}^{\alpha}{}_{\beta\gamma} = p^{\alpha}{}_{\sigma}(\partial P^{\sigma}{}_{0\beta}/\partial q^{\gamma} + P^{\mu}{}_{0\beta}P^{\sigma}{}_{\mu\gamma}), \qquad p^{\alpha}{}_{\sigma}P^{\sigma}{}_{0\beta} = \delta^{\alpha}{}_{\beta}.$$

From the form of equations (12.42(a)) the law of transformation of $\overset{0}{P}$ is identical with that of P. $\overset{0}{P}$ will be called the normalized derived projective connection. Here again as in the affine case the derived connection does not determine the displacement. For from (12.47)

$$(12.48)\qquad P^{\alpha}{}_{\beta\gamma} = p^{\mu}{}_{\beta}(P^{\alpha}{}_{0\sigma}\overset{0}{P}{}^{\sigma}{}_{\mu\gamma} - \partial P^{\alpha}{}_{0\mu}/\partial q^{\gamma})$$

and for given $\overset{0}{P}$ the choice of $P^{\alpha}{}_{0\beta}$ remains arbitrary.

In discussing the integrability of the fundamental equations (12.40) it is necessary to keep in mind the indeterminacy (12.45) in $\overset{0}{\Pi}^{\alpha}{}_{\beta\gamma}$. The most immediate conditions of integrability are

$$(12.49)\qquad (a)\quad \overset{0}{\Pi}^{\alpha}{}_{\beta\gamma\delta} = 0\,^{*} \qquad\qquad (b)\quad \partial\overset{0}{\Pi}^{\alpha}{}_{\beta\gamma}/\partial(dx^{\delta}) = 0$$

where $\overset{0}{\Pi}^{\alpha}{}_{\beta\gamma\delta}$ is the curvature quasi-tensor formed from a suitably chosen determination $\overset{0}{\Pi}^{\alpha}{}_{\beta\gamma}$ of the indeterminate derived connection. The identity (11.12) has as its counterpart in the projective theory †

$$(12.50)\qquad \Pi^{\alpha}{}_{0\sigma}\overset{0}{\Pi}^{\sigma}{}_{\beta\gamma\delta} = \Pi^{\sigma}{}_{0\beta}\Pi^{\alpha}{}_{\sigma\gamma\delta}.$$

From (12.40), (12.48), (12.49), and (12.50) there follows

THEOREM 12.1. *The integrability of* (12.27) *implies the integrability of* (12.42). *Conversely the integrability of* (12.42) *implies the integrability of* (12.27) *provided* $\partial\Pi^{\alpha}{}_{0\beta}/\partial(dx^{\delta}) = 0$.

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* Note that (12.49(a)) implies $\Pi^{\alpha}{}_{[\beta\gamma]} = 0$.

† When quantities $\Pi^{\alpha}{}_{\beta\gamma}$, $\overset{0}{\Pi}^{\alpha}{}_{\beta\gamma}$ occur together in any discussion it is understood that they are related by (12.43).

By means of (12.50) the concise integrability conditions (12.37(a)) may be written in terms of the curvature tensor $\overset{0}{P}{}^{a}{}_{\beta\gamma\delta}$ of $\overset{0}{P}{}^{a}{}_{\beta\gamma}$ to give by virtue of the above theorem a correspondingly concise set of conditions to replace (12.49(a)) namely

$$\overset{0}{P}{}^{a}{}_{\beta\gamma\delta} - \frac{\delta^{a}{}_{\beta}}{n+1} \overset{0}{P}{}^{\mu}{}_{\mu\gamma\delta} = 0. \tag{12.51(a)}$$

A corresponding substitute for (12.49(b)) is from (12.39(b)), (12.47), (12.48), and Theorem (12.1) simply

$$\partial \overset{0}{P}{}^{a}{}_{\beta\gamma} / \partial (dx^{\delta}) = 0. \tag{12.51(b)}$$

Our conditions are now completely independent of ψ.

Any function Z satisfying (12.42) forms a projective scalar as function of q. From (12.42(b)) it must be of index τ. It may be readily verified (cf. p. 174) that as functions of x the particular set of $(n+1)$ solutions Z^{a} which satisfy the initial conditions $(Z^{a})_{q=x} = \delta^{a}{}_{0}$, $(\partial Z^{a}/\partial q^{\beta})_{q=x} = \Pi^{a}{}_{0\beta}(x)$ form a contravariant projective vector of index $-\tau$ and satisfy the displacement equations (12.27).

The remarks already made in § 11 regarding the utility of the fundamental affine equations (11.8) in the non-holonomic case apply equally to the fundamental projective equations (12.42). The generalized straight lines are the curves of A_n along which the projective coördinates satisfy

$$d^2 Z^{a}/dt^2 = 0$$

in terms of a suitable parameter t. The differential equations of these curves obtained by direct application of (12.42) are

$$\frac{d^2 x^{a}}{dt^2} + \overset{0}{P}{}^{a}{}_{\beta\gamma} \frac{dx^{\beta}}{dt} \frac{dx^{\gamma}}{dt} = 0. \tag{12.52}$$

Since integrability is no longer in question the normalized parameters may conveniently be used.

These curves should be identical with those defined by (12.18). That this is indeed the case follows from (12.28(a)) and (12.44).

At the end of the discussion in non-homogeneous coördinates (p. 181) a convenient normalization of the tangential coördinate systems was introduced. In the homogeneous formulation this is accomplished by associating with a given underlying coördinate system the tangential system * in which $P^{a}{}_{ai} = 0$. In general

$$\bar{P}^{a}{}_{ai} = P^{\sigma}{}_{\sigma j} \frac{\partial x^{j}}{\partial \bar{x}^{i}} + P^{\sigma}{}_{\sigma 0} \frac{\partial x^{0}}{\partial \bar{x}^{i}} + \frac{\partial \log | \partial x/\partial \bar{x} |}{\partial \bar{x}^{i}}.$$

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* These are the systems which Weyl (*loc. cit.*) calls semi-osculating.

First leaving the underlying coördinates x^i fixed and allowing only x^0 to vary it is seen that the choice $\partial x^0/\partial \bar{x}^i = -P^\sigma{}_{\sigma i}/(n+1)\tau$ and no other annuls $\bar{P}^a{}_{ai}$, whence we are led to a unique system. Secondly if the vanishing of $\bar{P}^a{}_{ai}$ is to remain invariant the allowable transformations (12.31) must be restricted to

$$(12.53)\qquad \begin{aligned} &(\text{a}) \quad \bar{x}^i = \bar{x}^i(x) \\ &(\text{b}) \quad \bar{x}^0 = x^0 + \frac{1}{(n+1)\tau} \log |\partial \bar{x}/\partial x|. \end{aligned}$$

This is essentially the only way of picking out a unique tangential coördinate system to be associated with a given coördinate system in A_n. For it has been proved * that the Jacobian of (12.53) gives the only $(n+1)$-dimensional representations of (12.53(a)), a particular representation being obtained for each fixed function τ.

(12.53) with $\tau = 1$ are fundamental in T. Y. Thomas' projective theory.† Starting with a system of paths defined by equations of the form (11.18) he constructs in an invariantive way a homogeneous projective connection having these paths as generalized straight lines. Our point of view has been to take the projective displacement as fundamental rather than the paths. There are of course many projective displacements having the same set of paths.

We close this discussion of generalized projective geometry with a remark regarding the possibility of restricting (12.31(b)) to being integrable. In such a case it is customarily written ‡

$$(12.31(\text{c}))\qquad \bar{x}^0 = x^0 + \log \rho\,(x^1 \cdots x^n).$$

In general the question of the isomorphism of two projective spaces cannot be completely answered when this restriction is made unless the families of admissible tangential reference schemes related by means of (12.31 (a), (c)) are chosen in some unique way from the complete family of admissible schemes. The discussion of the normalization $P^\sigma{}_{\sigma i} = 0$ shows that such an invariantive family does indeed exist, namely the set in which $P^\sigma{}_{\sigma i}$ is a gradient.

13. Non-holonomic geometries subordinate to generalized projective geometry. In this final section it will be indicated how in our formulation the generalized non-euclidean, affine and euclidean geometries § issue as specializations of generalized projective geometry. In each case the subordination is accomplished in two steps, (1) introducing the geometry in its classical form into each tangent space by suitably restricting the projective group, (2) determining a projective displacement which is an isomorphism as acting between these newly specialized tangent spaces.

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* H. P. Robertson and H. Weyl, "On a problem in the theory of groups etc.," *Bulletin of the American Mathematical Society* (1929), pp. 686-690.

† "A projective theory of affinely connected manifolds," *Mathematische Zeitschrift*, vol. 25 (1926), pp. 723-733.

‡ This is the change of gauge of Veblen's projective theory.

§ The generalized euclidean *metric* geometry is just the familiar Riemannian geometry. Its position as a specialization of generalized affine geometry is so well known that a discussion here is hardly warranted.

Non-euclidean geometry. In this case the tangent spaces are rendered non-euclidean by introducing in each a non-degenerate quadric

$$G_{\alpha\beta}(x)Z^\alpha Z^\beta = 0 \qquad G_{00} \neq 0 \tag{13.1}$$

to serve as absolute. The quantities $G_{\alpha\beta}(x)$ form a symmetric projective tensor the index of which is conveniently chosen 2τ. Secondly there must be defined a non-euclidean displacement, that is to say a projective displacement mapping the quadrics upon each other, the condition for which is

$$G_{\alpha\beta,\gamma} = G_{\alpha\beta}\psi_\gamma. \tag{13.2}$$

There are of course many distinct displacements satisfying (13.2).

Veblen, who first developed the idea of generalized non-euclidean geometry* for use in his projective theory of relativity, employs a non-euclidean displacement which has claims to being the simplest available. The parameters are

$$\Sigma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta 0 \end{matrix} \right\} \phi_\gamma + \tau\delta^\alpha_\beta\phi_\gamma \tag{13.3}$$

where $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$ are the Christoffel symbols formed from $G_{\alpha\beta}$ and ϕ_γ is the covariant vector $G_{0\gamma}/G_{00}$.† However it is important to note that (13.3) preserves its form only under transformations in which $\partial x^0/\partial \bar{x}^i$ is a gradient. To obtain from (13.3) a formula for the parameters of a non-euclidean displacement which is valid without restriction we first observe that by applying the transformation

$$\begin{aligned} \bar{x}^i &= x^i \\ \bar{x}^0 &= x^0 + \int \phi_i dx^i \end{aligned} \tag{13.4}$$

the components $\bar{G}_{0i}$ are made to vanish and $\bar{\phi}_\alpha$ becomes $(1, 0, \cdot \cdot \cdot, 0)$. In this reference scheme we define the connection by (13.3). Then to find its expression in general it is only necessary to apply the inverse of (13. 4) to (13. 3) using the law of transformation (12. 35). There results

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* "A generalization of the quadratic differential form," *Quarterly Journal of Mathematics*, vol. 1 (1930), pp. 60-76.

† *Projektive Relativitätstheorie, Kap VI,* in particular formula (49). Schouten and van Dantzig in their recent unified field theory have also employed a non-euclidean connection. See, for example, "On projective connections and their application to the general field theory," *Annals of Mathematics*, vol. 34 (1933).

$$(13.5)\qquad \Pi^{\alpha}{}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta 0 \end{matrix} \right\} \phi_\gamma + \delta^{\alpha}{}_{0}\phi_{\beta\gamma} + G^{\alpha\sigma}G_{0\beta}\phi_{\gamma\sigma} + G^{\alpha\sigma}G_{0\gamma}\phi_{\beta\sigma} - G^{\alpha\sigma}G_{00}\phi_{\beta\sigma}\phi_\gamma + \tau\delta^{\alpha}{}_{\beta}\phi_\gamma$$

with $\phi_{\alpha\beta} = \frac{1}{2}(\partial\phi_\alpha/\partial x^\beta - \partial\phi_\beta/\partial x^\alpha)$. (13. 5) is a normalized connection when $\partial \log G_{00}/\partial x^\alpha = 2\tau\phi_\alpha$.

We have mentioned previously the desirability of normalizing tensors whenever possible. In the case under consideration an especially convenient normalization is obtained by choosing that tensor $G_{\alpha\beta}$ from the family of geometrically equivalent ones for which

$$G_{00} = \exp(2\tau\bar{x}^0)$$

in the system $\bar{x}$ above. With this choice (13. 5) reduces to the normalized

$$(13.5')\qquad P^{\alpha}{}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \delta^{\alpha}{}_{0}\phi_{\beta\gamma} + G^{\alpha\sigma}G_{0\beta}\phi_{\gamma\sigma} + G^{\alpha\sigma}G_{0\gamma}\phi_{\beta\sigma}.$$

Affine geometry. Here affine character is given to the projective tangent spaces by selecting in each a hyperplane

$$A_\alpha Z^\alpha = 0$$

to serve as improper locus. A_α is a projective covariant vector which it is convenient to choose of index 0 and to normalize by $A_0 = 1$. If the geometry is to be affine in its entirety the projective displacement must be such as to map these hyperplanes upon each other, that is to say such that

$$(13.6)\qquad A_{\alpha,\beta} = A_\alpha \rho_\beta.$$

Making $\alpha = 0$, $\rho_\beta = A_\sigma P^{\sigma}{}_{0\beta}$ and (13. 6) becomes

$$(13.7)\qquad A_{\alpha,\beta} = A_\alpha A_\sigma P^{\sigma}{}_{0\beta}.$$

When (13. 7) are satisfied for a vector A_α the non-homogeneous coördinate system $X^i = Z^i/A_\alpha Z^\alpha$ serves as unique tangential affine coördinate system. The projective displacement should be affine as acting between these systems. To prove this directly it is only necessary to show that $P^{0}{}_{ij} = 0$ * in the system

$$\bar{Z}^i = Z^i$$
$$\bar{Z}^0 = A_\alpha Z^\alpha$$

and this is easily verified.

These results may be expressed as a

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* Compare (12. 4) and (12. 28(b)).

THEOREM 13. 1. *A necessary and sufficient condition that a generalized projective geometry be interpretable as a generalized affine geometry is that a vector* A_α *exist satisfying* (13. 7).*

Euclidean geometry. The absolute in each tangent space is now an $(n-2)$-dimensional quadric defined analytically as intersection of, say

$$(13.13)\qquad \begin{aligned} &(a)\quad G_{\alpha\beta}Z^\alpha Z^\beta = 0 \\ &(b)\quad A_\alpha Z^\alpha = 0 \end{aligned}$$

the quantities $G_{\alpha\beta}$, A_α having the same general characteristics as heretofore.

To obtain a normalized euclidean displacement first choose coördinates in which $A_\alpha = \delta^0{}_\alpha$; then $P^0{}_{ij}$ must vanish since (13. 13(b)) is to be invariant under the displacement. In these special coördinates the invariance of the intersection (13. 13) leads to the equations

$$(13.14)\qquad \partial G_{ij}/\partial x^k - G_{il}P^l{}_{jk} - G_{jl}P^l{}_{ik} = \sigma_k G_{ij}$$

which for symmetric $P^i{}_{jk}$ have as unique solution

$$(13.15)\qquad P^i{}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta^i{}_j\sigma_k + \delta^i{}_k\sigma_j - G_{jk}G^{il}\sigma_l.$$

$P^0{}_{0k}$ vanishes because of (12. 34) and the symmetry of $P^i{}_{jk}$, while $P^i{}_{0j}$ remains arbitrary. In the tangential affine coördinate system obtained from our special coördinates by (12. 24), the euclidean displacement presents itself as a specialized affine displacement, the affine connection being given by (13. 15), and the tensor $\Gamma^i{}_j$ remaining arbitrary.

It must be remembered that the tensor G may be replaced by any tensor

$$(13.16)\qquad G'_{\alpha\beta} = \exp(\textstyle\int \psi_j dx^j)\, G_{\alpha\beta}$$

without affecting the geometry in the tangent spaces. The effect of this arbitrary multiplicative factor upon the parameters (13. 15) is accounted for by leaving σ_k arbitrary. In order to have at hand a definite generalized euclidean geometry a particular choice from among these possible symmetric displacements or the associated asymmetric ones must be made since they are all geometrically distinct.

The similarity of this geometry with the Weyl geometry is very evident.

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* The above results on affine geometry have been obtained by Bortolotti (*loc. cit.*) in slightly less general form.

DIE BETTI'SCHEN ZAHLEN DER ZYKLISCHEN ÜBERLAGERUNGSRÄUME DER KNOTENAUSSENRÄUME.

Von LEBRECHT GOERITZ.

Einleitung. Überlagert man den Knotenaussenraum eines Knotens im Euklideschen Raum h-blättrig zyklisch, so dass nur der Knoten als Verzweigungslinie auftritt und man beim einmaligen positiven Umlauf um die orientierte Knotenlinie vom i-ten Blatt zum $i+1$-ten $(i = 1, 2, \cdots, h-1)$ und vom h-ten zum 1-ten gelangt, so erhält man durch Hinzunahme der Punkte des Knotens zum Raum eine geschlossene dreidimensionale Mannigfaltigkeit $\mathfrak{M}_h$. Durch Untersuchung dieser Mannigfaltigkeiten bei den Schlauchknoten [†] erhält Herr O. Zariski in seiner Arbeit "On the topology of algebroid singularities" [‡] (neben der erneuten Kennzeichnung der Singularitäten algebraischer Kurven [§] durch Klassifikation der Schlauchknoten) das folgende Resultat:

Für einen Schlauchknoten ist die erste Betti'sche Zahl b_h von $\mathfrak{M}_h$ gleich der Anzahl der Wurzeln des Knotenpolynoms [¶] (L-Polynoms [†]) $f(x)$, die gleichzeitig Wurzeln der Gleichung $x^h - 1 = 0$ sind.

Die von Herrn Zariski aufgeworfene Frage, ob dieser Satz für alle Knoten gilt, wird im folgenden beantwortet, und zwar zeigt sich, dass nur der schwächere Satz 2 allgemein richtig ist. Die mit Hilfe dieses Satzes aus $f(x)$ und h allein berechenbaren Schranken für b_h (Satz 3) werden in nicht trivialen Fällen (obere Schranke verschieden von der unteren) angenommen.

1. *Einige bekannte Bemerkungen.* Ist S ein die Knotenlinie einmal umschlingender Weg, K ein Element der Kommutatorgruppe der Knotengruppe, so führe man durch die Festsetzung $SKS^{-1} = K^x$ den Operator x in die Kommutatorgruppe ein. Dann sei $M(x)$ eine Polynommatrix, deren Zeilen den definierenden Relationen der kommutativen Kommutatorgruppe

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[†] K. Reidemeister, *Knotentheorie*, Springer, Berlin 1933. Diese Note schliesst sich in der Bezeichnung an dieses Buch an. Es wird im folgenden mit "Knotentheorie" zitiert.

[‡] *American Journal of Mathematics*, vol. 54 (1932), p. 453.

[§] Vergl. auch Burau: *Kennzeichnung der Schlauchknoten*, Hamburger Abhandlungen 9 (1932), S. 125.

[¶] J. W. Alexander, "Topological invariants of knots and links," *Transactions of the American Mathematical Society*, vol. 30 (1928), p. 275.

mit Operator $\mathfrak{K}(x)$ entsprechen und deren Spalten den Erzeugenden dieser Gruppe zugeordnet sind. Das Polynom $m_{ik}(x)$ in der i-ten Zeile und k-ten Spalte von $M(x)$ sei der Exponent der k-ten Erzeugenden in der i-ten Relation. Aus der Menge der Exponentenmatrizen sei $M(x)$ als quadratische Matrix so gewählt, dass die Determinante das Knotenpolynom $f(x)$ liefert.†

Eine Exponentenmatrix der definierenden Relationen der kommutativen Fundamentalgruppe der in der Einleitung erklärten Mannigfaltigkeit $\mathfrak{M}_h$ sei M_h. Man erhält eine solche Matrix aus unserer Polynommatrix $M(x)$ (vergl. *Knotentheorie*, III, § 7), indem man jedes Polynom $m_{ik}(x) = \sum_{\nu=0}^{l} a_\nu x^\nu$ durch die folgende Matrix von h zeilen und Spalten ersetzt: E_0 sei die h-reihige Einheitsmatrix, E_1 entstehe aus E_0 durch zyklische Umordnung der Spalten, und zwar gehe die l-te Spalte in die $l+1$-te über $(l = 1, 2, 3, \cdots, h-1)$ und die h-te in die 1-te, E_ν ($\nu \geqq 0$, eine natürliche Zahl) entstehe durch ν-malige Anwendung dieses Schrittes. Ferner bedeute das Produkt einer Zahl mit einer Matrix, etwa $a \cdot M$ die Multiplikation jedes der Elemente von M mit der Zahl a und die Summe zweier Matrizen gleicher Zeilenzahl und Spaltenzahl die Matrix, in der ein Element der ursprünglichen Matrizen durch die Summe der entsprechenden Elemente beider Matrizen ersetzt ist. Dann werde $\sum_{\nu=0}^{l} a_\nu x^\nu$ durch $\sum_{\nu=0}^{l} a_\nu E_\nu$ ersetzt.

Die erste Betti'sche Zahl b_h von $\mathfrak{M}_h$ erhält man demnach, indem man die Zeilenzahl $h \cdot r$ der so erhaltenen Matrix M_h um ihren Rang ρ vermindert. Diese Zahl

$$b_h = h \cdot r - \rho$$

soll aus der Matrix $M(x)$ bestimmt werden.

2. *Reduktion des Problems.* Den Koeffizientenbereich der Elemente von $M(x)$ erweitern wir auf den Körper der rationalen Zahlen R und erlauben die folgenden Abänderungen von $M(x)$:

a) Multiplikation einer Zeile oder Spalte von $M(x)$ mit λ, wenn λ aus R und $\lambda \neq 0$ ist.

b) Vertauschung zweier Zeilen oder zweier Spalten von $M(x)$.

c) Addition des x^ν-fachen (ν sei eine ganze rationale Zahl) der zweiten Zeile von $M(x)$ zur ersten oder der zweiten Spalte zur ersten.

d) Eine beliebige Folge der Operationen *a*, *b*, *c*.

Mittels dieser Operationen kann man bekanntlich $M(x)$ auf die Diagonalform

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† Über die Möglichkeit dieser Wahl vergleiche die in der Einleitung zitierte Arbeit von J. W. Alexander oder *Knotentheorie*, S. 49 u. 50.

$$M^*(x) = \begin{pmatrix} f_1(x) & 0 & 0 & \cdots & 0 \\ 0 & f_2(x) & 0 & \cdots & 0 \\ 0 & 0 & f_3(x) & \cdots & 0 \\ 0 & 0 & 0 & \cdots & f_r(x) \end{pmatrix}$$

bringen, bei der $f_i(x) = \sum_{\nu=0}^{l^{(i)}} a_\nu^{(i)} x^\nu$ $(i = 1, 2, \cdots, r)$ Polynome von x mit rationalen Koeffizienten sind. Dabei ist keines der Polynome $f_i(x)$ identisch Null, da die Determinante von $M(x)$ von Null verschieden ist.

Man erklärt nun analog zum Obigen diejenigen Matrizen zu M_h äquivalent, die aus M_h durch die folgenden Operationen hervorgehen:

a') Multiplikation einer Zeile oder Spalte von M_h mit λ, wenn λ aus R und $\lambda \neq 0$ ist.

b') Vertauschung zweier Zeilen oder zweier Spalten von M_h.

c') Addition der zweiten Zeile von M_h zur ersten oder der zweiten Spalte zur ersten.

d') Eine beliebige Folge der Operationen *a'*, *b'*, *c'*.

Dann erkennt man, dass man eine zu M_h äquivalente Matrix M^*_h erhält, wenn man in $M^*(x)$ die x-Potenzen in der in 1. angegebenen Weise ersetzt. Die Operationen *a*, *b*, *c*, *d* für $M(x)$ übertragen sich nämlich auf die entsprechenden Operationen *a'*, *b'*, *c'*, *d'* für M_h, jeweils angewandt auf die h aus einer Zeile oder Spalte von $M(x)$ und deren äquivalenten Matrizen entspringenden Zeilen oder Spalten von M_h und deren äquivalenten Matrizen. Dabei ist zu beachten, dass die Multiplikation einer Zeile oder Spalte der Polynommatrix mit x^ν nur eine zyklische Umordnung der h daraus entspringenden Zeilen oder Spalten der Zahlmatrix bewirkt.

Da bei Anwendung der Operationen *a'*, *b'*, *c'*, *d'* der Rang von M_h sich nicht ändert, genügt es zur Bestimmung von b_h den Rang der Matrix M^*_h zu bestimmen. Das gelingt nun sehr einfach.

3. *Rangbestimmung.* Den Rang von M^*_h berechnet man, indem man den Rang jedes der $f_i(x)$ von $M^*(x)$ in M^*_h entsprechenden zyklischen h-reihigen Bestandteils M^*_{hi} von M^*_h bestimmt. Und zwar gilt:

SATZ 1. *Die Matrix M^*_{hi} hat den Rang $h - \alpha_i$, wo α_i die Anzahl der verschiedenen Wurzeln von $f_i(x) = 0$ ist, die gleichzeitig auch Wurzeln von $x^h - 1 = 0$, also h-te Einheitswurzeln sind.*

Den einfachen Beweis dieser Tatsache geben wir an: Sei

$$f_i(x) = \sum_{\nu=0}^{l^{(i)}} a_\nu^{(i)} x^\nu, \quad \text{so wird} \quad M^*_{hi} = \sum_{\nu=0}^{l^{(i)}} a_\nu^{(i)} E_\nu$$

eine zyklische Matrix der Form

$$M^*_{hi} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{h-1} \\ a_{h-1} & a_0 & a_1 & \cdots & a_{h-2} \\ a_{h-2} & a_{h-1} & a_0 & \cdots & a_{h-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

wobei

(1) $$a_k = \sum_{l=0,1,\ldots} a^{(i)}_{k+lh}$$

ist; falls dabei $h-1 > l^{(i)}$, so sei $a^{(i)}_{l+1} = a^{(i)}_{l+2} = \cdots = a_{h-1} = 0$ gesetzt. Ferner seien $1, \xi_1, \xi_2, \cdots, \xi_{h-1}$ die h-ten Einheitswurzeln, dann hat die Matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi_1 & \xi_2 & \cdots & \xi_{h-1} \\ 1 & \xi_1^2 & \xi_2^2 & \cdots & \xi^2_{h-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \xi_1^{h-1} & \xi_2^{h-1} & \cdots & \xi^{h-1}_{h-1} \end{pmatrix}$$

eine von Null verschiedene Determinante (als Vandermondesche Determinante). Die Produktmatrix $M^*_{hi} \cdot W$ hat also den gleichen Rang wie M^*_{hi}. Nun ist

$$M^*_{hi} \cdot W = \begin{pmatrix} \Sigma a_k & \Sigma a_k \xi_1{}^k & \Sigma a_k \xi_2{}^k & \cdots & \Sigma a_k \xi^k{}_{h-1} \\ \Sigma a_k & \xi_1{}^{h-1} \cdot \Sigma a_k \xi_1{}^k & \xi_2{}^{h-1}\, \Sigma a_k \xi_2{}^k & \cdots & \xi^{h-1}_{h-1}\, \Sigma a_k \xi^k{}_{h-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Sigma a_k & \xi_1\, \Sigma a_k \xi_1{}^k & \xi_2\, \Sigma a_k \xi_2{}^k & \cdots & \xi_{h-1}\, \Sigma a_k \xi^k{}_{h-1} \end{pmatrix},$$

wobei die Summe immer von $k=0$ bis $k=h-1$ zu erstrecken ist. Die neue Matrix hat soviel Spalten aus lauter Nullen als es Wurzeln ξ_i gibt, die gleichzeitig Wurzeln von $\sum_{k=0}^{h-1} a_k x^k = 0$ und wegen der Erklärung von a_k in (1) Wurzeln der Gleichung $\sum_{\nu=0}^{l^{(i)}} a_\nu{}^{(i)} x^\nu = 0$ sind. Die restlichen l Spalten sind die mit von Null verschiedenen Faktoren multiplizierten Spalten einer Vandermondeschen Matrix. Es gibt also darin nach dem Laplaceschen Entwicklungssatz für Determinaten eine Unterdeterminante vom Grade l, die von Null verschieden ist. Damit ist Satz 1 aber bewiesen.

Daraus folgt unmittelbar, dass der Rang von M^*_h gerade $r \cdot h - \sum_{i=1}^{r} \alpha_i$ und also $b_h = \sum_{i=1}^{r} \alpha_i$ ist. Demnach gilt das folgende Endresultat:

SATZ 2. *Ist α_i die Anzahl der verschiedene h-ten Einheitswurzeln, die* *gleichzeitig Wurzeln des in 2: erkläten i-ten Elementarteilers von $M(x)$ sind, so ist die erste Betti'sche Zahl von* $\mathfrak{M}_h$

$$b_h = \sum_{i=1}^{r} \alpha_i.$$

4. *Einige Folgerungen und Beispiele.* Aus dem letzten Satz des vorigen Abschnittes erschliesst man

SATZ 3. *Die erste Betti'sche Zahl b_h von $\mathfrak{M}_h$ ist kleiner oder gleich der Anzahl der Wurzeln, die das Knotenpolynom $f(x)$ mit $x^h - 1 = 0$ gemeinsam hat, mit ihrer zu $f(x)$ gehörigen Vielfachheit gezählt, und grösser oder gleich jener Anzahl bei Zählung der verschiedenen Wurzeln.*

Dass das Gleichheitszeichen in beiden Fällen angenommen werden kann, sieht man für den ersten Fall an den Schlauchknoten, da das Polynom $f(x)$ dieser Knoten nur einfache Wurzeln hat, besser an den Knoten deren l Bestandteile gleiche Schlauchknoten sind. In diesem Falle hat $M(x)$ die Gestalt

$$M(x) = \begin{pmatrix} M_1(x) & 0 & \cdots & 0 \\ 0 & M_1(x) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & M_1(x) \end{pmatrix},$$

und es ist die Determinante dieser Matrix $f(x) = |M_1(x)|^l$, wo $m_1(x)$ die Matrix eines Bestandteiles ist. Ist α_1 die Zahl der $|M_1(x)| = f_1(x) = 0$ und $x^h - 1 = 0$ gleichzeitig angehörenden Wurzeln, so ist $b_h = l \cdot \alpha_1$ also gleich der Anzahl der mit ihrer Vielfachheit gezählten Wurzeln von $f(x) = 0$, die gleichzeitig Wurzeln von $x^h - 1 = 0$ sind.

2. Für den zweiten Fall an speziellen Knoten, von denen wir ein Beispiel herausgreifen: Der Knoten 8, 10a der Knotentabelle bei Alexander und Briggs † hat als Polynom

$$f(x) = (1 - x + x^2)^3$$

und $f(x)$ als einzigen Elementarteiler. Demnach wird die Betti'sche Zahl $b_3 = 2$.

ROSTOCK, D. 4. XI. 1933.

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† J. W. Alexander und G. B. Briggs, "On types of knoted curves," *Annals of Mathematics*, vol. 28 (1926-27), p. 562.

INVOLUTIONS OF ORDER TWO ASSOCIATED WITH THE SURFACES OF GENERA $p_a = p_g = 0,\ P_2 = 1,\ P_3 = 0$.

By Roberta F. Johnson.

1. *Introduction.* The involutions studied in this paper are all of order two; each has a finite number of fixed points; and, finally, they are all associated with a surface of genus zero and bigenus one with a bicanonical curve of order zero. This association occurs in one of two ways or both. Either the involution exists on the surface, or the image of the involution is such a surface, or both the surface on which the involution exists and the image surface are of genus zero and bigenus one with a bicanonical curve of order zero. When we say that an involution exists on a surface, or belongs to a surface, or that the surface contains an involution we mean that the points of the surface are associated in pairs. These pairs of points form the groups of the involution. The involutions considered here are determined by space Cremona transformations which leave the surface invariant, not point for point, but in such a way that the points of the surface are associated in pairs. A surface is called the image of an involution if to each point of the surface there corresponds a pair of points of the involution.

The purpose of this paper is to organize the work which has been done on the above mentioned involutions and to present certain new results. The method of procedure will be, first, to derive an algebraic formulation for a general surface of genus zero and bigenus one with a bicanonical curve of order zero; secondly, to show how the involutions are associated with certain specializations of this surface. However, we shall first summarize briefly the important properties of a general surface with genera $p_a = p_g = 0$, $P_2 = 1$ and a bicanonical curve of order zero.

2. *The General Surface with Genera* $p_a = p_g = 0$, $P_2 = 1$, $P_3 = 0$. It has been shown that all the multiple genera of a surface F with $p_a = p_g = 0$, $P_2 = 1$ and a bicanonical curve of order zero are definitely determined and have the values $P_3 = P_5 \cdots = 0$, $P_2 = P_4 = P_6 \cdots = 1$. Also, the surface considered is completely characterized by $p_a = p_g = P_3 = 0$, $P_2 = 1$, in other words, any surface with these genera has a bicanonical curve of order zero.* The surfaces F_n, $p_a = p_g = P_3 = 0$, $P_2 = 1$ have the linear genus one, $p^{(1)} = 1$.

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* F. Enriques, "Sopra le superficie algebriche di bigenere uno," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze,* series 3, vol. 14 (1907), p. 332.

If $| C |$ is any linear system of curves cut on F_n then the adjoint system $| C_a |$ possesses the same characteristics as $| C |$, that is, the two systems have the same genus and the same grade. The double of a system and the double of its adjoint are equivalent $| 2C | = | 2C_a |$. Since the surface F_n is neither rational nor ruled it cannot possess an algebraic series of rational curves.* Thus, the only rational curves existing on F_n are isolated curves. Every complete irreducible linear system of genus $\pi \geqq 1$ traced on F_n has the dimension $\pi - 1$ and the grade $2\pi - 2$.† This system cannot contain multiple basis points. In general, it does not have even simple basis points. When the system does possess simple basis points, the curves of the system are hyperelliptic and the number of basis points is two; these points are double points for each of the g'_2 existing on every curve of the system.‡ The surface F_n possesses an infinite discontinuous group of birational transformations into itself.§

The general surface F_n of genera $p_a = p_g = P_3 = 0$, $P_1 = 1$ can always be transformed by a birational transformation into any one of four types of surfaces. One of these is the sextic surface which passes doubly through the edges of a tetrahedron. This surface was first mentioned by Enriques in 1896.¶ A second projective model is a double plane which has a curve of branch points composed of a sextic curve K_6 and two lines p, q.‖ The sextic K_6 has two tacnodes and the two lines p, q are the tacnodal tangents. The intersection of the two lines is a double point of the sextic. The third projective model is a double quartic in a space of four dimensions with a curve of branch points of order eight and four isolated branch points. The latter are conical double points of the surface.** Finally, the fourth type of surface which is birationally equivalent to a general surface with genera $p_a = p_g$

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* Picard and Simart, *Théorie des fonctions algébriques*, vol. 2, p. 512.

† F. Enriques, "Sopra le superficie algebriche di bigenere uno," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*, series 3, vol. 14 (1907), pp. 334-335.

‡ *Ibid.*, p. 335.

§ *Ibid.*, pp. 350-352.

¶ F. Enriques, "Introduzione alla geometria sopra le superficie algebriche," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*, series 3, vol. 10 (1896), p. 66. See also F. Enriques, "Sopra le superficie algebriche di bigenere uno," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*, series 3, vol. 14 (1907), pp. 346-350.

‖ F. Enriques, "Sopra le superficie algebriche di bigenere uno," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*, series 3, vol. 14 (1907), pp. 339-346.

** L. Godeaux, "Recherches sur les surfaces algébriques de genre zero et de bigenre un, Troisième communication," *Academie Royale de Belgique, Classe de Sciences, Bulletins*, series 5, vol. 13 (1927), pp. 114-133.

$= P_3 = 0$, $P_2 = 1$ is a surface F_{10} of order ten in a space of five dimensions which is the image of a congruence of order seven and class three of lines belonging to ∞^1 quadrics of a net.*

3. *Two Analytical Representations of the General Surface F_n with $p_a = p_g = P_3 = 0$, $P_2 = 1$.* Consider the two general quadrics

$$f(x_1, x_2, x_3, x_4) = \Sigma\, a_{ij}x_ix_j = 0,\ a_{ij} = a_{ji},$$
$$\phi(x_1, x_2, x_3, x_4) = \Sigma\, b_{ij}x_ix_j = 0,\ b_{ij} = b_{ji}.$$

The sextic F_6

$$(1) \qquad f(x_2x_3x_4, x_3x_4x_1, x_4x_1x_2, x_1x_2x_3) + x_1x_2x_3x_4\, \phi(x_1, x_2, x_3, x_4) = 0$$

contains the edges of the tetrahedron of reference as double lines. Also, this is the most general sextic surface which passes doubly through these six edges. Hence, any surface F_n, $p_a = p_g = P_3 = 0$, $P_2 = 1$ is birationally equivalent to a sextic surface whose equation is given by (1). These surfaces are ∞^{10} in number.

An analytical formulation of the double plane which has been proved to be a projective model of the surface F_n of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ can be derived by use of equation (1). First, transform F_n into the typical sextic F_6 which has (1) for its equation. Let u_1 and u_2 represent the two lines $x_1 = x_2 = 0$ and $x_3 = x_4 = 0$. From an arbitrary point of space one line can be drawn to meet these two lines. This line will meet F_6 in two residual points. Thus, by means of the bisecants of u_1 and u_2 there is established a (2, 1) correspondence between the points of F_6 and the points of an arbitrary plane, such as the plane $z_2 = z_4$. The coordinates of any point $P \equiv (x_1, x_2, x_3, x_4)$ on a line u meeting the two skew lines are given by the equations $x_1 = \lambda z_1$, $x_2 = \lambda z_2$, $x_3 = \mu z_3$, $x_4 = \mu z_4$. If this point is on F_6 these values of x_i must satisfy equation (1) and we obtain a quadratic expression in λ/μ. If the line u meets F_6 in two coincident points the roots of this quadratic equation are equal. When this is the case the bisecants of the skew lines u_1 and u_2 are tangent to F_6. The points in which these tangents meet the double plane $z_2 = z_4$ are *branch points* of the double plane. Hence, the curve of branch points has the equation

$$\begin{aligned}(2) \quad & z_1^2z_3^2[(a_{13} + b_{24})z_2^2 + (a_{14} + b_{23})z_2z_3 \\ & \quad + (a_{23} + b_{14})z_1z_2 + (a_{24} + b_{13})z_1z_3]^2 \\ & - z_1z_3[z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)]\ [z_3f_{12}(z_2, z_1) + z_1\phi_{34}(z_3, z_2)] = 0,\end{aligned}$$

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* G. Fano, "Nuove ricerche sulle congruenze di rette del 3° ordine," *Memorie della Reale Accademia di Torino*, series 2, vol. 51 (1901).

where

$$
\begin{aligned}
f_{12}(z_2, z_1) &= a_{11}z_2^2 + 2a_{12}z_1z_2 + a_{22}z_1^2, \\
f_{34}(z_2, z_3) &= a_{33}z_2^2 + 2a_{34}z_2z_3 + a_{44}z_3^2, \\
\phi_{12}(z_1, z_2) &= b_{11}z_1^2 + 2b_{12}z_1z_2 + b_{22}z_2^2, \\
\phi_{34}(z_3, z_2) &= b_{33}z_3^2 + 2b_{34}z_3z_2 + b_{44}z_2^2.
\end{aligned}
$$

This consists of the two lines $z_1 = 0$ and $z_3 = 0$ and the sextic K_6'

$$
\begin{aligned}
z_1z_3[(a_{13} + b_{24})z_2^2 + (a_{14} + b_{23})z_2z_3 + (a_{23} + b_{14})z_1z_2 + (a_{24} + b_{13})z_1z_3]^2 \\
- [z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)]\,[z_3f_{12}(z_2, z_1) + z_1\phi_{34}(z_3, z_2)] = 0.
\end{aligned}
$$

This sextic has two tacnodes at $M \equiv (0, 0, 1)$ and $N \equiv (1, 0, 0)$ with the lines $z_1 = 0$, $z_3 = 0$ as the respective tacnodal tangents and a double point at $(0, 1, 0)$.

4. *The Involutions of Order Two with Genera* $p_a = p_g = P_3 = 0$, $P_2 = 1$. (A). *Classification of the Involutions.* By the genera of the involution I_2 of order two which belongs to a surface F_n is meant the genera of the image surface Φ_m. When a pair of the points of the involution I_2 of F_n are coincident, then the corresponding point on the image surface Φ_n is called a *branch point* of the involution. The coincident points of I_2 are also called *the fixed points* of the involution. If an involution of any order exists on an algebraic surface and has only a finite number of fixed points the involution is generated by a group of birational transformations of the surface F_n into itself.*

It is possible to classify completely the involutions of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ according to the surfaces which contain them. If Φ is the image of an involution of order two and genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ belonging to a surface F, then F must be either a surface also of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$, or a surface of genus one, $(p_a = P_4 = 1)$.

Not every surface Φ, $p_a = p_g = P_3 = 0$, $P_2 = 1$ represents (or is the image of) an involution of order two belonging to a surface F of the same genera. However, if Φ is birationally equivalent to a double plane with a curve of branch points of one of two types then Φ represents an involution belonging to a surface F of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. These two types are

(a) a curve of order eight composed of a quartic and a conic which are tangent in two points and the two common tangents. The four branch points are the variable intersections of the conic and the quartic.

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* L. Godeaux, "Sur les involutions douées d'un nombre fini de points unis appartenant à une surface algébrique," *Rendiconti della Reale Accademia dei Lincei*, series 5, vol. 23 (1914), pp. 408-413.

(b) a curve of order eight composed of two cubics which are tangent in two points and the two common tangents. The cubics also intersect in the point common to the two lines. The four variable intersections of the two cubics are the branch points of the involution.* Every surface $p_a = p_g = P_3 = 0$, $P_2 = 1$ is the image of an involution of order two belonging to a surface of genus one. This involution does not have any fixed points.†

4. (B). *Case of the Quartic and the Conic.* Consider now equation (2) which is the equation of the curve of branch points of the double plane which is a general surface with genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. Place $b_{33} = ka_{11}$, $a_{22} = kb_{44}$, $a_{12} = -b_{34}$, $a_{23} + b_{14} = 0$, $a_{14} + b_{23} = 0$, $a_{24} + b_{13} = k(a_{13} + b_{24})$ in equation (1). Then the curve of branch points has the form

$$(2') \quad z_1z_3(z_2^2 + kz_1z_3)\{z_1z_3(a_{13} + b_{24})^2(z_2^2 + kz_1z_3) \\ -(a_{11}z_3 + b_{44}z_1)[z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)]\} = 0.$$

The surfaces with this curve of branch points are images of involutions of order two which belong to surfaces F of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$.

The surface Φ is birationally equivalent to the sextic surface Φ_6 which passes doubly through the edges of the tetrahedron of reference

$$(3) \quad \Phi_6 \equiv x_3x_4(x_2x_4 + kx_1x_3)[a_{11}x_2x_3 + 2(a_{13} + b_{24})x_1x_2 + b_{44}x_1x_4] \\ + x_1x_2[x_1x_2f_{34}(x_4, x_3) + x_3x_4\phi_{12}(x_1, x_2)] = 0.$$

The conics $z_2^2 + kz_1z_3 = 0$ have for images on the surface Φ_6 curves cut out by the quadric surfaces $x_2x_4 + kx_1x_3 = 0$. The quartics

$$z_1z_3(a_{13} + b_{24})^2(z_2^2 + kz_1z_3) - (a_{11}z_3 + b_{44}z_1)[z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)] = 0$$

have for images on the surface Φ_6 curves cut out by surfaces which belong to the linear system of ruled sextics.

$$(4) \quad F_6: \lambda_1x_1x_2^2x_3x_4^2 + \lambda_2x_1^2x_2x_3^2x_4 + \lambda_3x_1x_2^2x_3^2x_4 + \lambda_4x_1x_2^2x_3^3 \\ + \lambda_5x_2^3x_3^2x_4 + \lambda_6x_1^2x_2x_4^3 + \lambda_7x_1^3x_3x_4^2 + \lambda_8x_1^2x_2x_3x_4^2 = 0.$$

These surfaces contain the edges

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* L. Godeaux, "Mémoire sur les surfaces algébriques doubles ayant un nombre fini de points de diramation," *Toulouse Faculté des Sciences, Annales*, series 5, vol. 3 (1913), pp. 289-312.

† F. Enriques, "Un'osservazione relativa alle superficie di bigenere 1," *Rendiconti della Reale Accademia di Bologna* (1908).

$$\begin{aligned} &u_1 \equiv x_1 = x_2 = 0 \text{ and } u_2 \equiv x_3 = x_4 = 0 \text{ as triple lines,}\\ &u_3 \equiv x_1 = x_3 = 0 \text{ and } u_4 \equiv x_2 = x_4 = 0 \text{ as double lines,}\\ &u_5 \equiv x_1 = x_4 = 0 \text{ and } u_6 \equiv x_2 = x_3 = 0 \text{ as simple lines.}\end{aligned}$$

Hence, the intersection of Φ_6 and F_6 consists of these edges counted a proper number of times and a residual curve C_{12} of order twelve. Each of the generators of F_6 is a bisecant of C_{12}. There is a $(2, 1)$ correspondence between the points of C_{12} and the points of any plane section of F_6. The genus of a plane section of F_6 is $p = 2$. Hence, if p' is the genus of C_{12}, Zeuthen's formula for the $(2, 1)$ correspondence gives $2p' - n' = 10$, where n' is the number of generators of F_6 which are tangent to Φ_6. There are 12 such generators. Thus, $p' = 11$. The curves C_{12} cannot form a complete linear system on Φ_6; for every complete system of genus eleven has the dimension ten. However, consider the complete linear system cut on Φ_6 by the surfaces

$$\begin{aligned} &\lambda_1 x_1^2 x_2 x_3 x_4^2 + \lambda_2 x_1^2 x_2 x_3^2 x_4 + \lambda_3 x_1 x_2^2 x_3^2 x_4\\ &+ \lambda_4 x_1 x_2^2 x_3^3 + \lambda_5 x_2^3 x_3^2 x_4 + \lambda_6 x_1^2 x_2 x_4^3 + \lambda_7 x_1^3 x_3 x_4^2\\ &+ \lambda_8 x_1^2 x_2 x_3 x_4^2 + \lambda_9 x_1^4 x_4^2 + \lambda_{10} x_1^3 x_2 x_4^2 + \lambda_{11} x_1^3 x_4^3 = 0. \end{aligned} \tag{5}$$

Within this linear system of dimension ten there exist besides the curves C_{12} two other partial systems, one cut on Φ_6 by the quartic surfaces

$$\lambda'_1 x_1^2 x_3 x_4 + \lambda'_2 x_1 x_2 x_3^2 + \lambda'_3 x_1 x_2 x_4^2 + \lambda'_4 x_2^2 x_3 x_4 + \lambda'_5 x_1 x_2 x_3 x_4 = 0,$$

and the other cut out by the planes

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4 = 0.$$

Consider the transformation

$$\begin{aligned} &X_1 = \rho x_1 x_2^2 x_3 x_4^2,\ X_4 = \rho x_1 x_2^2 x_3^3,\ X_7 = \rho x_1^3 x_3 x_4^2,\quad X_{10} = \rho x_1^3 x_2 x_4^2,\\ &X_2 = \rho x_1^2 x_2 x_3^2 x_4,\ X_5 = \rho x_2^3 x_3^2 x_4,\ X_8 = \rho x_1^2 x_2 x_3 x_4^2,\ X_{11} = \rho x_1^3 x_4^3.\\ &X_3 = \rho x_1 x_2^2 x_3^2 x_4,\ X_6 = \rho x_1^2 x_2 x_4^3,\ X_9 = \rho x_1^4 x_4^2. \end{aligned} \tag{6'}$$

It makes the hyperplanes

$$\lambda'_1 X_2 + \lambda'_2 X_4 + \lambda'_3 X_1 + \lambda'_4 X_5 + \lambda'_5 X_3 = 0$$

correspond to the quartic surfaces

$$\lambda'_1 x_1^2 x_3 x_4 + \lambda'_2 x_1 x_2 x_3^2 + \lambda'_3 x_1 x_2 x_4^2 + \lambda'_4 x_2^2 x_3 x_4 + \lambda'_5 x_1 x_2 x_3 x_4 = 0,$$

and the hyperplanes

$$\mu_1 X_9 + \mu_2 X_{10} + \mu_3 X_7 + \mu_4 X_{11} = 0$$

correspond to the planes

$$\mu_1x_1 + \mu_2x_2 + \mu_3x_3 + \mu_4x_4 = 0.$$

From equations (6) we have the equalities

$$(7)\qquad \begin{aligned} X_4X_{11} &= X_3X_7, & X_6X_7 &= X_8X_{11}, & X_5X_9 &= X_3X_{10}, \\ X_1X_7 &= X_3X_{11}, & X_2X_{10} &= X_3X_9, & X_7X_{10} &= X_8X_9. \end{aligned}$$

Consider the two equations

$$(8)\qquad \psi_1X_8 = \psi_2X_7, \quad \psi_3X_6X_7 = X_9X_{10}X_{11},$$

where

$$\begin{aligned} -\psi_1 &= b_{11}X_2 + b_{22}X_5 + a_{33}X_1 + a_{44}X_4 + 2(a_{34} + b_{12})X_3 \\ \psi_2 &= X_1 + kX_2, \quad \psi_3 = a_{11}X_8 + b_{44}X_{11} + 2(a_{13} + b_{24})X_{10}. \end{aligned}$$

The eight equations given in (7) and (8) determine a surface in the space S_{10} of ten dimensions. If we project this surface F upon the original space S_3 of Φ_6 we obtain the equation of Φ_6. Moreover, the transformation T in S_{10} given by the equations

$$\begin{aligned} T\colon\ X_1/X'_1 &= X_2/X'_2 = X_3/X'_3 = X_4/X'_4 = X_5/X'_5 = X_6/-X'_6 \\ &= X_7/-X'_7 = X_8/-X'_8 = X_9/-X'_9 = X_{10}/-X'_{10} = X_{11}/-X'_{11} \end{aligned}$$

leaves the surface F invariant. Hence, T defines an involution on F. Thus, there exists a $(1, 2)$ correspondence between the points of Φ_6 and the points of F. It is easy to show that the surface F is the intersection of two cones whose vertices are the hyperplanes

$$X_2 = X_5 = X_9 = X_{10} = 0 \text{ and } X_1 = X_4 = X_6 = X_{11} = 0.$$

The plane sections of Φ_6 are images of the hyperplane sections cut on F by the hyperplanes

$$\mu_1X_9 + \mu_2X_{10} + \mu_3X_7 + \mu_4X_{11} = 0.$$

These curves are compounded by means of the involution which the transformation T defines on F.

Also, the curves of the eighth order traced on Φ_6 by the quartics

$$\lambda_1x_1^2x_3x_4 + \lambda_2x_1x_2x_3^2 + \lambda_3x_1x_2x_4^2 + \lambda_4x_2^2x_3x_4 + \lambda_5x_1x_2x_3x_4 = 0$$

are images of the curves cut on F by the hyperplanes

$$\lambda_1X_2 + \lambda_2X_4 + \lambda_3X_1 + \lambda_4X_5 + \lambda_5X_3 = 0,$$

and are compounded by means of the given involution.

The fixed points of F occur in either of the two spaces

$$X_1 = X_2 = X_3 = X_4 = X_5 = 0 \text{ or } X_6 = X_7 = X_8 = X_9 = X_{10} = X_{11} = 0.$$

The four fixed points are given by

$$X_6 = X_7 = X_8 = X_9 = X_{10} = X_{11} = 0,$$
$$\psi_3 = 0, \quad \psi_2 = 0, \quad \psi_1 = 0, \quad X_3{}^2 = X_1X_4 = X_2X_5.$$

The second fixed space does not intersect F. Hence, the above four points are the only fixed points of F. These points correspond to the four variable intersections of the conic and the quartic of the curve of branch points of the double plane.

It remains to determine the genus of the surface F. We know that the surface F is of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ or of genera $p_a = P_2 = 1$.* Thus, it will be sufficient to determine the arithmetic genus of F. This can be obtained from the formula †

$$12p_a = 12p\pi_a + (p-1)(p-5)\alpha - (p^2-1)\beta + 12(p-1),$$

where p_a and π_a are the arithmetic genus of F and Φ, p is the order of the involution, α is the number of perfect fixed points of the involution, and β is the number of non-perfect fixed points. Since an involution of order two on an algebraic surface has only perfect fixed points ‡ $\alpha = 4$, $\beta = 0$. Thus, we have $p_a = 0$. This proves that the surface F has the genera

$$p_a = p_g = P_3 = 0, \; P_2 = 1.$$

4. (C) *Case of the Two Cubics.* We return to the general equation (2). This time we make the specialization

$$a_{13} + b_{24} = 0, \quad a_{14} + b_{23} = 0, \quad a_{23} + b_{14} = 0, \quad a_{24} + b_{13} = 0.$$

Then the curve of branch points reduces to the form

$$(9) \qquad z_1z_3[z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)][z_3f_{12}(z_2, z_1) + z_1\phi_{34}(z_3, z_2)] = 0.$$

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* See L. Godeaux, "Mémoire sur les surfaces algébriques doubles ayant un nombre fini de points de diramation," *Toulouse Faculté des Sciences, Annales,* series 5, vol. 3 (1913), p. 310.

† L. Godeaux, "Recherches sur les involutions douées d'un nombre fini de points de coincidence appartenant à une surface algébrique," *Bulletin de la Société Mathematique de France,* vol. 47 (1919), p. 14.

‡ L. Godeaux, "La théorie des involutions douées d'un nombre fini de points de coincidence appartenant à une surface algébrique," *Revista Matematica Hespano-Americana,* Madrid, 1924.

The surfaces Φ with this curve of branch points are images of involutions of order two which belong to a surface F of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. The surface Φ is birationally equivalent to the sextic surface Φ_6 which has the equation

$$\begin{aligned}(10)\qquad & x_3^2x_4^2f_{12}(x_2, x_1) + x_1^2x_2^2f_{34}(x_4, x_3) \\ & + x_1x_2x_3x_4\,[\phi_{12}(x_1, x_2) + \phi_{34}(x_3, x_4)] = 0.\end{aligned}$$

By means of the transformation

$$(11)\qquad \begin{aligned} & X_1 = \rho x_1,\ X_2 = \rho x_2,\ X_3 = \rho x_3, \\ & X_4 = \rho x_4,\ X_5 = \sigma x_1^2x_3x_4,\ X_6 = \sigma x_1x_2x_3^2, \\ & X_7 = \sigma x_1x_2x_4^2,\ X_8 = \sigma x_2^2x_3x_4,\ X_9 = \sigma x_1x_2x_3x_4\end{aligned}$$

we find that the surface Φ, or its projective equivalent Φ_6, is the image of an involution of order two belonging to the surface F of the space S_8 of eight dimensions. The equations of F are

$$(12)\qquad \begin{aligned} & X_1X_6 = X_2X_9,\ X_2X_5 = X_1X_9,\ X_3X_8 = X_4X_9, \\ & X_4X_7 = X_3X_9,\ X_9\,\psi_1 = X_1X_2,\ X_9\,\psi_2 = X_3X_4,\end{aligned}$$

where

$$\begin{aligned}\psi_1 &= a_{11}X_6 + a_{22}X_5 + b_{33}X_7 + b_{44}X_8 + 2(a_{12} + b_{34})X_9, \\ -\psi_2 &= b_{11}X_5 + b_{22}X_6 + a_{33}X_8 + a_{44}X_7 + 2(b_{12} + a_{34})X_9.\end{aligned}$$

This case of the two cubics has been investigated in detail by Lucien Godeaux.*

4. (D). *Case where the Surface F has the Genera $p_a = p_g = P_4 = 1$.* Since there are only thirteen arbitrary parameters in the equation of the sextic surface passing doubly through the edges of a tetrahedron, the equation (1) can be written in the form

$$\begin{aligned}(13)\quad & a_{11}x_2^2x_3^2x_4^2 + a_{22}x_1^2x_3^2x_4^2 + a_{33}x_1^2x_2^2x_4^2 + a_{44}x_1^2x_2^2x_3^2 + x_1x_2x_3x_4 \\ & \times [a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + 2a_{12}x_3x_4 \\ & + 2a_{13}x_2x_4 + 2a_{23}x_1x_4 + 2a_{24}x_1x_3 + 2a_{34}x_1x_2 + 2a_{14}x_2x_3]\end{aligned}$$

This surface Φ_6 is projectively equivalent to the double quartic Φ_0 of S_4. The base of this double quartic is the surface $X_1X_2 = X_5^2$, $X_3X_4 = X_5^2$. The branch points consist of the curve

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* L. Godeaux, "Recherches sur les surfaces algébriques de genre zéro et de bigenere un, Première Communication," *Académie Royale de Belgique, Classe de Sciences, Bulletins,* series 5, vol. 12 (1926), pp. 730-741.

$$(14)\quad \begin{cases} X_1X_2 = X_5^2, \quad X_3X_4 = X_5^2 \\ F(X_1, X_2, X_3, X_4) \equiv a_{13}^2X_1X_3 + a_{14}^2X_1X_4 + a_{23}^2X_2X_3 \\ \qquad + a_{24}^2X_2X_4 + 2(a_{13}a_{24} + a_{14}a_{23})X_5^2 \\ \qquad + 2(a_{13}a_{14}X_1 + a_{23}a_{24}X_2 + a_{13}a_{23}X_3 + a_{14}a_{24}X_4)X_5 \\ \qquad - (a_{11}X_1 + a_{22}X_2 + a_3X_4 + a_4X_3 + 2a_{12}X_5) \\ \qquad \times (a_{33}X_3 + a_{44}X_4 + a_1X_2 + a_2X_1 + 2a_{34}X_5) = 0 \end{cases}$$

and the four isolated branch points

$$A_1 \equiv (1, 0, 0, 0, 0), \qquad A_2 \equiv (0, 1, 0, 0, 0),$$
$$A_3 \equiv (0, 0, 1, 0, 0), \qquad A_4 \equiv (0, 0, 0, 1, 0).$$

By means of the transformation

$$(15)\qquad X_1/x_1^2 = X_2/x_2^2 = X_3/x_3^2 = X_4/x_4^2 = X_5/x_1x_2 = X_5/x_3x_4$$

we find that the surface Φ_0 or its projective equivalent Φ_6 is the image of an involution of order two which belongs to a double quadric F with genera $p_a = p_g = P_4 = 1$. The equations of F are

$$(16)\qquad x_1x_2 - x_3x_4 = 0. \qquad x_5^2 = \psi(x_1, x_2, x_3, x_4)$$

where

$$\psi(x_1, x_2, x_3, x_4) \equiv (a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4)^2$$
$$- (a_{11}x_1^2 + a_{22}x_2^2 + a_3x_4^2 + a_4x_3^2 + 2a_{12}x_1x_2)$$
$$\times (a_{33}x_3^2 + a_{44}x_4^2 + a_1x_2^2 + a_2x_1^2 + 2a_{34}x_3x_4).$$

The involution (I) existing on the surface F of which Φ_0 is the image is defined by the transformation

$$(\mathrm{I})\qquad x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = -x_3, \quad x'_4 = -x_4, \quad x'_5 = -x_5.$$

Thus, every surface of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ is the image of an involution of order two existing on a surface of genus one. This surface F of genus one possesses two other involutions. One of these is the rational involution

$$(\mathrm{I}')\qquad x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad x'_4 = x_4, \quad x'_5 = -x_5.$$

The image of this involution is the quadric $x_1x_2 - x_3x_4 = 0$. The other involution on F is

$$(\mathrm{I}'')\qquad x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = -x_3, \quad x'_4 = -x_4, \quad x'_5 = x_5,$$

which is the product of (I) and (I′). The involution (I) has no fixed points, whereas, the involution (I″) has eight fixed points. Hence, the image surface of (I″) is a surface of genus one.

This case has also been studied by Godeaux.*

The locus of the conjugate points of the quadrics of a web without basis points is an example of the surface of genera one described above. Consider the web of quadrics

$$\lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3 + \lambda_4\phi_4 = 0, \tag{17}$$

where

$$\phi_1(x_i) = \Sigma a_{ik}x_ix_k, \quad \phi_2 = \Sigma b_{ik}x_ix_k, \quad \phi_3 = \Sigma c_{ik}x_ix_k, \quad \phi_4 = \Sigma d_{ik}x_ix_k.$$

The locus of the conjugate points of the web (17) is the Jacobian surface F_4

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(\phi_1, \phi_2, \phi_3, \phi_4)} = 0. \tag{18}$$

This surface F_4 contains ten lines, i. e., the web of quadrics (17) contains ten pairs of planes. The surface F_4 is of genus one. The congruence of lines determined by the pairs of conjugate points of the web (17) is a congruence of order seven and class three. Fano proved that the image of this congruence is a surface of order ten in a space S_5 which has the genera $p_a = p_g = P_3 = 0$, $P_2 = 1$.† Later Fano stated that the congruence G of order seven and class three is a projective model of a general surface of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$; giving as reference the article written in 1901. However, in 1901 what he proved was that every congruence (7, 3) is birationally equivalent to a surface of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ and not the converse. The problem of determining whether or not a general surface of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ can always be preferred to a congruence (7, 3) has not been solved.‡

5. *The Surfaces F of Genera* $p_a = p_g = P_3 = 0$, $P_2 = 1$ *which Contain Involutions of Order Two.* (A). *Classification of Involutions Existing on F.* Given the surface F of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. Suppose there exists on this surface an involution I_n. Designate by Φ the surface which is the image of this involution and by π_a, π_g, Π_i, the genera of Φ. Then the

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* L. Godeaux, "Recherches sur les surfaces algébriques de genre zero et de bigenre un, Troisième communication," *Academie Royale de Belgique, Classe de Sciences, Bulletins*, series 5, vol. 13 (1927), pp. 114-133.

† G. Fano, "Nuove recherche sulle congruence di rette del 3° ordine," *Memorie della Reale Accademia della Scienza di Torino*, series 2, vol. 51 (1901), pp. 72-78.

‡ G. Fano, "Superficie algébriche di genere zero e bigenere uno e lori casi particolari," *Rendiconti del Circolo Matematico di Palermo*, vol. 29 (1910).

surfaces Φ are either rational or also of genera $\pi_a = \pi_g = \Pi_3 = 0$, $\Pi_2 = 1$. This theorem was stated without proof by Godeaux in 1913.[*] However, it is easy to show that the theorem is true. In the first place, suppose $\pi_a < -1$. Then there exists on the surface Φ, a pencil of rational curves.[†] If p is the genus of the curves of F which correspond to the curves of the pencil, then Zeuthen's formula gives $p = -1$, which is absurd. Hence, $\pi_a \geqq -1$. Moreover, $\pi_a \neq -1$. For if σ is the number of branch points of the involution we have the relation $\rho_a = 2\pi_a + 1 - \sigma/4$ [‡] which gives $\sigma = -4$. Thus, $\pi_a \geqq 0$. Since there exists a $(1, n)$ correspondence between Φ and F, and F is a regular surface, Φ must also be regular and $\pi_a = \pi_g$.[§] Also $\pi_g \leqq p_g$,[¶] but $p_g = 0$. Hence, $\pi_g \leqq 0$. Therefore, $\pi_a \leqq 0$, but $\pi_a \geqq 0$. Hence, $\pi_a = 0$.

The rational involutions were discussed in section 2 of this paper. If the surface Φ of genera $\pi_a = \pi_g = \Pi_3 = 0$, $\Pi_2 = 1$ is the image of an involution of order two existing on F it has four branch points. Moreover, as has already been stated, the surface Φ is birationally equivalent to a double plane with a curve of branch points which is either

1° a curve of order eight composed of a conic and a quartic tangent at two points and the two common tangents, or

2° a curve of order eight composed of two cubics tangent at two points and the two common tangents.

5. (B). *Case of the Conic and the Quartic.* Consider the curve of branch points of the double plane which is a general surface with genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. Its equation is given by (2). Place

$$a_{11} = b_{22},\ a_{22} = b_{11},\ a_{33} = b_{44},\ a_{44} = b_{33},\ a_{34} + b_{12} = a_{12} + b_{34}.$$

Then the curve of branch points takes the particular form

(19)
$$z_1^2z_3^2[(a_{13}+b_{24})z_2^2 + (a_{14}+b_{23})z_2z_3 + (a_{23}+b_{14})z_1z_2 + (a_{24}+b_{13})z_1z_3]^2 \\ - z_1z_3[z_1f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2)]^2 = 0.$$

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[*] L. Godeaux, "Sur les involutions appartenant à une surface de genre zero et de bigenre un," *Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences*, Paris, vol. 156 (1913), 1 sem., p. 1306.

[†] Castelnuovo, "Sulle superficie aventi il genere aritmetico negativo," *Rendiconti del Circolo Matematico di Palermo*, vol. 20 (1905), p. 59.

[‡] See L. Godeaux, "Mémoire sur les surfaces algébriques doubles ayant un nombre fini de points de diramation," *Toulouse Faculté des Sciences, Annales*, series 5, vol. 3 (1913), p. 290.

[§] Castelnuovo, "Alcuni resultati sui sistemi lineari di curve appartenenti ad una superficie algebrica," *Memorie di Matematica e di Fisica della Società Italiana delle Scienze*, series 3, vol. 10 (1896), p. 101.

[¶] F. Enriques, "Richerche di geometria sulle superficie algebriche," *Memorie della Reale Accademia delle Scienze di Torino*, series 2, vol. 44, p. 230.

Consider now the net of cubics $| C |$

$$(20) \qquad \lambda_1\phi + \lambda_2 z_1\psi + \lambda_3 z_3\psi = 0,$$

where

$$\psi \equiv (a_{13} + b_{24})z_2^2 + (a_{14} + b_{23})z_2z_3 + (a_{23} + b_{14})z_1z_2 + (a_{24} + b_{13})z_1z_3$$

and

$$\phi \equiv z_1 f_{34}(z_2, z_3) + z_3\phi_{12}(z_1, z_2).$$

If the cubics of the net (20) are referred to the lines $\mu_1X_1 + \mu_2X_2 + \mu_3X_3 = 0$ of a plane ω', a $(2, 1)$ correspondence is established between the original z-plane ω and the X-plane ω'. The image of the curve (19) is a quartic consisting of the two lines $X_2 = 0$, $X_3 = 0$ and the conic $X_2X_3 - X_1^2 = 0$. The pencil of lines with vertex $A'_1 \equiv (0, 1, 0)$ is transformed into the pencil of lines with vertex $X_2 = X_3 = 0$. Also

$$X_1\psi - X_2 f_{34}(z_2, z_3) - X_3\phi_{12}(z_1, z_2) = 0.$$

Thus, there exists an inversion between the net of conics

$$a_1\psi + a_2 f_{34}(z_2, z_3) + a_3\phi_{12}(z_1, z_2) = 0$$

and the plane ω'. From these facts we see that the coincident points of the involution defined on ω by the net of cubics $| C |$ occur when one of the conics of the above net is tangent to a line through $A'_2 \equiv (0, 1, 0)$. The locus of such coincident points is the quartic curve

$$\begin{aligned} F_4(X_i) \equiv {} & [(k_3X_2 + k_2X_3)X_1 - 2(a_{34} + b_{12})X_2X_3]^2 \\ & + 4k_1(a_{44}X_3 + b_{11}X_2)X_1X_2X_3 - 4X_2X_3(b_{22}X_3 + a_{33}X_2) \\ & \times (a_{44}X_3 + b_{11}X_2 - k_4X_1) - 4k_1k_4X_1^2X_2X_3 = 0, \end{aligned}$$

where

$$k_1 = a_{13} + b_{24},\ k_2 = a_{14} + b_{23},\ k_3 = a_{23} + b_{14},\ k_4 = a_{24} + b_{13}.$$

The tangents to this quartic at the points $A'_2 \equiv (0, 1, 0)$ and $A'_3 \equiv (0, 0, 1)$ respectively are the lines $X_3 = 0$ and $X_2 = 0$.

Let (y_i) and (z_i) constitute a pair of points of the involution I_2 on ω, which is defined by the net of cubics $| C |$. This involution is generated by the quartic transformation

$$(I_2) \qquad z_1 : z_2 : z_3 = y_1f_1(y_i) : f_2(y_i) : y_3f_1(y_i)$$

where

$$\begin{aligned} f_1(y_i) &= \psi(y_i)(b_{22}y_3 + a_{33}y_1) - k_1[y_3\phi_{12}(y_1, y_2) + y_1f_{34}(y_2, y_3)] \\ f_2(y_i) &= (k_2y_3 + k_3y_1)[y_3\phi_{12}(y_1, y_2) + y_1f_{34}(y_2, y_3)] \\ & \quad - \psi(y_i)[y_2(b_{22}y_3 + a_{33}y_1) + 2y_1y_3(b_{12} + a_{34})]. \end{aligned}$$

Consider now the double plane F with $p_a = p_g = P_3 = 0$, $P_2 = 1$ which has the plane ω for base and the curve (19) for branch points. The equation of F is

$$(21) \qquad u^2 = z_1^2 z_3^2 (k_1 z_2^2 + k_2 z_2 z_3 + k_3 z_1 z_2 + k_4 z_1 z_3)^2 - z_1 z_3 [z_1 f_{34}(z_2, z_3) + z_3 \phi_{12}(z_1, z_2)]^2.$$

The surface F contains three involutions, namely,

$$(\mathrm{I}'_2) \qquad z'_1 : z'_2 : z'_3 : u' = z_1 f_1(z_i) : f_2(z_i) : z_3 f_1(z_i) : u$$
$$(\mathrm{I}''_2) \qquad z'_1 : z'_2 : z'_3 : u' = z_1 : z_2 : z_3 : -u$$
$$(\mathrm{I}'''_2) \qquad z'_1 : z'_2 : z'_3 : u' = z_1 f_1(z_i) : f_2(z_i) : z_3 f_1(z_i) : -u.$$

The first two of these involutions (I'_2) and (I''_2) are rational. The third involution (I'''_2) is the product of the first two. The image of (I'''_2) is the double plane Φ which has for base the plane ω'. The branch points of the double plane are the images of points lying of the curve of branch points of the double plane F or images of the coincident points of the involution (I_2) of the plane ω. The latter points are given by the quartic curve $F_4(X_i)$. Hence, the double plane Φ has for its equation

$$(22) \qquad Y^2 = X_2 X_3 (X_2 X_3 - X_1^2) F_4(X_i).$$

The conic $X_2 X_3 - X_1^2 = 0$ and the quartic $F_4(X_i) = 0$ have four variable points of intersection. These are the branch points of the involution (I'''_2). The conic and the quartic are tangent at the points $A'_2 \equiv (0, 1, 0)$ and $A'_3 \equiv (0, 0, 1)$. The common tangents at these points are the lines $X_2 = 0$ and $X_3 = 0$. Thus, the double plane is a surface of genera

$$\pi_a = \pi_g = \Pi_3 = 0, \ \Pi_2 = 1.$$

5. (C). *Case of the Two Cubics.* Theoretically at least, it ought to be possible to state explicitly what restrictions it is necessary to impose upon the surface F of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$ in order that it may contain an involution of order two which has as its image a double plane of genera $\pi_a = \pi_g = \Pi_3 = 0$, $\Pi_2 - 1$ with a curve of branch points composed of two lines and two cubics. However, the algebra involved in such an analytic investigation is extremely complex. The existence of such surfaces F was proved in section 4. (C). There the surface F was obtained as a surface of a space S_8. It was the surface of order sixteen having the equations

$$X_1 X_6 = X_2 X_9, \quad X_2 X_5 = X_1 X_9, \quad X_9 \psi_1 = X_1 X_2,$$
$$X_3 X_8 = X_4 X_9, \quad X_4 X_7 = X_3 X_9, \quad X_9 \psi_2 = X_3 X_4$$

where

$$\psi_1 = a_{11} X_6 + a_{22} X_5 + b_{33} X_7 + b_{44} X_8 + 2(a_{12} + b_{34}) X_9$$
$$-\psi_2 = b_{11} X_5 + b_{22} X_6 + a_{33} X_8 + a_{44} X_7 + 2(b_{12} + a_{34}) X_9.$$

By successive projections from points on the surface the surface F_{16} of S_8 can be projected upon a space of three dimensions S_3. The birational equivalent of F_{16} in S_8 is a surface $F_6(X_2, X_3, X_6, X_9)$.

$$(23)\quad \begin{aligned} F_6 = {} & 2(a_{12} + b_{34})X_9^2(b_{33}\psi - a_{44}\phi)^2 + b_{33}X_6^2(cX_9^2 - b_{33}X_3^2)^2 \\ & - \phi(b_{33}\psi - a_{44}\phi)(cX_9^2 - b_{33}X_3^2) = 0 \end{aligned}$$

where

$$\begin{aligned} c &= a_{33}b_{33} - a_{44}b_{44}, \\ \phi &= a_{22}X_9^2 + a_{11}X_6^2 + 2(a_{12} + b_{34})X_6X_9 - X_2^2 \\ \psi &= b_{11}X_9^2 + b_{22}X_6^2 + 2(b_{12} + a_{34})X_6X_9. \end{aligned}$$

This surface F_6 has a double point at $P \equiv (0, 1, 0, 0)$. It contains the line $X_3 = X_9 = 0$ as a double line and has two double conics C_1 and C_2. The equations of C_1 and C_2 are

$$c^{1/2}X_9 = \pm\, b_{33}{}^{1/2}X_3, \qquad b_{33}\psi - a_{44}\phi = 0.$$

Since the genera of an algebraic surface remain invariant when the surface is subjected to a birational transformation the surface F_6 is, as the surface F_{16}, a surface of genera $p_a = p_g = P_3 = 0$, $P_2 = 1$. However, it appears difficult to discover the birational transformation which will transform F_6 to a sextic surface which contains the six edges of a tetrahedron as double lines.

CORNELL UNIVERSITY,
JUNE, 1933.

SUR QUELQUES TRANSFORMATIONS BIRATIONNELLES INVOLUTIVES ASSOCIÉES À UNE CUBIQUE GAUCHE.

Par LUCIEN GODEAUX.

Dans une note récente,[*] M. Purcell, reprenant une idée de Montesano,[†] a étudié quelques transformations birationnelles de l'espace, que l'on peut définir de la manière suivante:

Soient S, S' deux espaces projectifs à trois dimensions, G une congruence linéaire de droites de S, G' une congruence linéaire de droites de S', θ une correspondance birationnelle entre les droites de G, G', enfin Ω une réciprocité entre les espaces S, S'. Par un point P de S passe une droite g de G, à laquelle θ fait correspondre une droite g' de G'. Au point P, on fait correspondre le point P' intersection de g' et du plan ω que Ω fait correspondre à P. Les points P, P' sont homologues dans une transformation birationnelle T.

L'étude préparatoire de la correspondance birationnelle θ entre les congruences G, G' se fait en remarquant que les droites homologues découpent, sur des plans donnés, des points homologues dans une transformation crémonienne. Nous avons eu l'occasion, voici quelques années, d'étudier systématiquement cette question comme introduction à des recherches sur les transformations de Jonquières de l'espace.[‡] Dans le cas où les congruences G, G' sont toutes deux constituées par les cordes de cubiques gauches, le passage par des plans sécants pour l'étude de la transformation θ n'est d'ailleurs pas nécessaire.

Revenons à la transformation T. Lorsque les congruences G, G' coïncident, les espaces S, S' étant superposés, que θ est une involution et Ω une polarité, la transformation T est involutive.

Nous voudrions, dans cette note, attirer l'attention sur deux transformations birationnelles involutives de l'espace, obtenues par ce procédé, qui ne possèdent qu'un nombre fini de points unis. Nous supposerons que la congruence G est formée par les bisécantes d'une cubique gauche.

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[*] "Involutorial space Cremona transformations determined by non-linear null reciprocities," *American Journal of Mathematics*, vol. 55 (1933), pp. 381-389.

[†] "Sulle reciprocità birazionali nulle dello spazio," *Rendiconti della Reale Accademia dei Lincei*, 1° sem. (1888), pp. 583-590.

[‡] "Sur les transformations birationnelles de Jonquières de l'espace," *Mémoires in-8° de l'Academie royale de Belgique* (1922), pp. 1-75.

1. Soit K une cubique gauche représentée par les équations

$$\left\| \begin{matrix} a_x & b_x & c_x \\ a'_x & b'_x & c'_x \end{matrix} \right\| = 0.$$

Une bisécante de K a pour équations

$$\lambda_1 a_x + \lambda_2 b_x + \lambda_3 c_x = 0, \qquad \lambda_1 a'_x + \lambda_2 b'_x + \lambda_3 c'_x = 0$$

et ces bisécantes correspondent biunivoquement sans exception aux points du champs ternaire $(\lambda_1, \lambda_2, \lambda_3)$. Une transformation birationnelle θ' de ce champs donne naissance à une correspondance birationnelle θ entre les cordes de K et réciproquement.

Si nous considérons la transformation birationnelle d'ordre n

$$\lambda'_1 : \lambda'_2 : \lambda'_3 = \phi_1(\lambda_1, \lambda_2, \lambda_3) : \phi_2(\lambda_1, \lambda_2, \lambda_3) : \phi_3(\lambda_1, \lambda_2, \lambda_3), \qquad (\theta')$$

il lui correspond, entre les cordes de K, une transformation birationnelle θ telle qu'à une quadrique circonscrite à K corresponde une surface d'ordre $2n$ passant n fois par K. De plus, à chaque point du champs (λ) fondamental d'ordre s pour θ', correspond une corde de K multiple d'ordre s pour ces surfaces d'ordre $2n$.

2. Considérons une transformation birationnelle θ entre les cordes de K et une polarité Ω de l'espace, dont nous désignerons la quadrique fondamentale par F. A un point P de l'espace, faisons correspondre le point P' conjugué de P par rapport à Ω et situé sur la corde de K que θ fait correspondre à celle qui passe par P. Les points P, P' se correspondent dans une transformation birationnelle T.

Désignons par $r_1, r_2, \cdots, r_\nu$ les cordes de K fondamentales pour la transformation θ et soient $s_1, s_2, \cdots, s_\nu$ les multiplicités de ces droites pour les surfaces ϕ d'ordre $2n$, passant n fois par K, que θ fait correspondre aux quadriques circonscrites à K. On a d'ailleurs, d'après la théorie des transformations birationnelles du plan, les relations

$$s_1^2 + s_2^2 + \cdots + s_\nu^2 = n^2 - 1, \qquad s_1 + s_2 + \cdots + s_\nu = 3(n-1).$$

Il est aisé de voir qu'aux plans de l'espace, T fait correspondre des surfaces Φ d'ordre $4n+1$, passant $2n$ fois par K et respectivement $2s_1, 2s_2, \cdots, 2s_\nu$ fois par $r_1, r_2, \cdots, r_\nu$.

La courbe K et les droites $r_1, r_2, \cdots, r_\nu$ sont des éléments fondamentaux de la transformation T. Supposons qu'un point P, n'appartenant pas à ces courbes, soit fondamental pour T. Son homologue P' doit être indéterminé. Si ω est le plan que θ fait correspondre à P, g la corde de K passant par P, g' la corde que θ lui fait correspondre, le point de rencontre de g' et de ω doit être indéterminé; cela exige que le plan ω contienne la droite g'. Nous désignerons par Δ la courbe fondamentale de T lieu du point P; cette courbe est simple pour les surfaces Φ. Observons que Ω fait correspondre aux points de g' des plans formant un faisceau dont l'axe passe par P et est en général distinct de g. L'un de ces plans passe par g et par suite il existe un point P' de g' dont le point homologue est indéterminé, ce point appartient à la courbe Δ.

La courbe Δ est la seule courbe fondamentale de T en dehors de K et des droites r; comme T est involutive, elle fait correspondre à une droite de l'espace une courbe d'ordre $4n+1$. L'ordre de Δ est donc

$$\begin{aligned}(4n+1)^2 - 12n^2 - 4(s_1^2+s_2^2+\cdots+s_\nu^2) - (4n+1)\\ = (4n+1)^2 - 12n^2 - 4(n^2-1) - (4n+1) = 4(n+1).\end{aligned}$$

Il n'est pas difficile de rechercher les surfaces fondamentales de la transformation T, nous ne nous y arrêterons pas. Considérons plutôt une bisécante g de K qui soit unie pour la transformation θ. Cette droite g sera transformée en elle-même par T. Aux points de g, Ω fait correspondre les plans passant par une droite g_1, qui est en général distincte de g. Supposons que la droite g_1 ne rencontre pas g, ce qui est le cas général. Les couples de points de g homologues dans T forment une involution qui possède deux points unis, points de rencontre de g et de la quadrique fondamentale F de Ω.

Si la droite g_1 s'appuie sur g, ce qui ne se présentera en général que si θ possède une infinité de droites unies, il existe un point de g auquel correspondent tous les points de cette droite; ce point appartient à la courbe Δ et est uni pour T.

On voit que T possède une courbe unie ou un nombre fini de points unis selon que θ possède une infinité ou un nombre fini de droites unies.

3. Envisageons un cas particulier, celui où la transformation θ' est donnée par

$$\lambda'_1 : \lambda'_2 : \lambda'_3 = \lambda_2\lambda_3 : \lambda_3\lambda_1 : \lambda_1\lambda_2.$$

La transformation θ fait correspondre aux quadriques circonscrites à K des surfaces du quatrième ordre passant doublement par K et simplement par les droites

$$a_x = a'_x = 0, \quad b_x = b'_x = 0, \quad c_x = c'_x = 0,$$

que nous désignerons par r_1, r_2, r_3. La transformation T est actuellement du neuvième ordre et les surfaces Φ qu'elle fait correspondre aux plans de l'espace passent quatre fois par K, deux fois par chacune des droites r_1, r_2, r_3 et une fois par la courbe Δ, qui est du douzième ordre. Aux droites de l'espace, T fait correspondre des courbes du neuvième ordre s'appuyant en 14 points sur K, en deux points sur chacune des droites r_1, r_2, r_3 et en 12 points sur la courbe Δ.

La transformation θ possède quatre droites unies l_1, l_2, l_3, l_4, d'équations

$$a_x + b_x + c_x = 0, \qquad a'_x + b'_x + c'_x = 0, \tag{l_1}$$
$$-a_x + b_x + c_x = 0, \qquad -a'_x + b'_x + c'_x = 0, \tag{l_2}$$
$$a_x - b_x + c_x = 0, \qquad a'_x - b'_x + c'_x = 0, \tag{l_3}$$
$$a_x + b_x - c_x = 0, \qquad a'_x + b'_x - c'_x = 0. \tag{l_4}$$

Sur chacune de ces droites se trouvent deux points unis pour T; ce sont les points où ces droites coupent la quadrique fondamentale F de Ω. L'involution d'ordre deux engendrée par T possède donc huit points unis distincts.

4. Un autre cas particulier intéressant, qui est d'ailleurs un cas limite du précédent, s'obtient en prenant pour θ' la transformation

$$\lambda'_1 : \lambda'_2 : \lambda'_3 = \lambda_1\lambda_2 : \lambda_1^2 : \lambda_2(\lambda_1 - \lambda_3).$$

Aux quadriques circonscrites à K, θ fait correspondre des surfaces du quatrième ordre ϕ passant deux fois par K et une fois par chacune des droites r_1, r_2 respectivement d'équations

$$b_x = b'_x = 0, \qquad c_x = c'_x = 0.$$

De plus, les surfaces ϕ touchent, le long de la droite r_2, la quadrique Q, d'équation

$$a_x c'_x - a'_x c_x = 0. \tag{Q}$$

La transformation T est encore du neuvième ordre et fait correspondre aux plans de l'espace des surfaces Φ passant quatre fois par K, deux fois par chacune des droites r_1, r_2 et une fois par la courbe Δ d'ordre 12. De plus, la droite r_2 est tacnodale pour les surfaces Φ; en d'autre termes, la génératrice de la quadrique Q, infiniment voisine de r_2, est double pour ces surfaces. Aux droites de l'espace, T fait correspondre des courbes du neuvième ordre s'appuyant en 14 points sur K, en deux points sur chacune des droites r_1, r_2 et en 12 points sur la courbe Δ. De plus, ces courbes touchent la quadrique Q aux deux points d'appui sur r_2.

La transformation θ possède trois droites unies: la droite r_2 et les droites l_1, l_2 d'équations respectives

$$2a_x + 2b_x + c_x = 0, \qquad 2a'_x + 2b'_x + c'_x = 0, \qquad (l_1)$$
$$2a_x - 2b_x + c_x = 0, \qquad 2a'_x - 2b'_x + c'_x = 0. \qquad (l_2)$$

D'une manière plus précise, si nous désignons par Q_1, Q_2 les quadriques circonscrites à K et passant la première par les droites r_2, l_1, la seconde par les droites r_2, l_2, les génératrices de ces quadriques, bisécantes de K, infiniment voisines de r_2, sont unies pour θ.

La transformation T possède six points unis distincts, deux sur chacune des droites l_1, l_2, r_2. Mais les deux points unis situés sur cette dernière droite sont d'une nature particulière. En général, les points unis d'une involution du second ordre de l'espace, supposés en nombre fini, sont des points unis parfaits, c'est à-dire qu'à une courbe passant par un de ces points, correspond une courbe touchant la première au point considéré. Au contraire, si A est un des points unis situés sur la droite r_2, les points infiniment voisins de A situés sur la quadrique fondamentale F de Ω et respectivement sur les quadriques Q_1, Q_2 sont unis pour T, mais aux autres points du domaine de A, T fait correspondre des points distincts du même domaine. Cette particularité provient du fait que le point A est à la fois uni et fondamental (comme appartenant à la droite r_2) pour T.

On peut encore dire que l'involution engendrée par T possède huit points unis, mais ces points ne sont plus distincts.*

Liége (Université),
24 Octobre, 1933.

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* Pour les propriétés de la transformation considérée ici, voir notre note "Sur une transformation quadratique involutive," *Mathesis* (1926), pp. 353-360.

ON A CERTAIN RATIONAL V_n^{2n+1} IN S_{2n+1}.

By B. C. Wong.

The n-dimensional variety of the lowest order in a $(2n+1)$-space S_{2n+1} which is not the locus of planes or of spaces of higher dimensions is a V_n^{2n+1} of order $2n+1$. One known characteristic property of this variety is that it has one and only one apparent double point,[*] that is, of its ∞^{2n} secant lines one and only one passes through a general given point of S_{2n+1}. Mr. Babbage [†] has recently made a study of this V_n^{2n+1} and has shown that it is rational and is representable upon an S_n by means of the ∞^{2n+1} V^3_{n-1}'s through the complete intersection V^4_{n-2} of two given V^2_{n-1}'s. It contains ∞^{2n-4} lines of which ∞^{n-3} pass through each of its points. On it lie ∞^1 quadric varieties of $n-1$ dimensions whose containing n-spaces form a V_{n+1}^{n+1}. Another interesting property is that it contains $\infty^{(m+1)(n-m)}$ V_m^{2m+1}'s each being contained in an S_{2m+1} and of the same nature as V_n^{2n+1} itself for $n=m$.

Now V_n^{2n+1} (or any n-dimensional variety) has very numerous characteristics. Projecting V_n^{2n+1} from a general point of S_{2n+1} upon an S_{2n}, we obtain for projection a ${}_0V_n^{2n+1}$ with one improper double point. If we project V_n^{2n+1} from a general line of S_{2n+1} upon an S_{2n-1}, the projection is a ${}_1V_n^{2n+1}$ with a double curve upon which lie a finite number of pinch points. In general, the projection ${}_kV_n^{2n+1}$ of V_n^{2n+1} from a general S_k upon an S_{2n-k} contains a number of multiple varieties. If a multiple variety is of multiplicity s, its dimension is $t=(s-1)k-(s-2)n$. On each of these s-fold varieties are loci of higher multiplicities and lower dimensions and also pinch loci of lower dimensions. These pinch loci may have themselves multiple loci and they have also intersections with the other multiple loci of the s-fold variety. The number of these varieties of different multiplicities is indeed enormous. As none of these, except the fact that V_n^{2n+1} has one apparent double point, has been mentioned by Mr. Babbage in his work above referred to, we here propose to determine the orders of all the multiple varieties on the projection ${}_kV_n^{2n+1}$ in S_{2n-k}.

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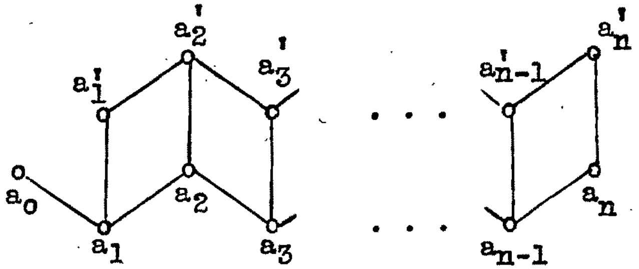
[*] There are other V_n's in S_{2n+1} with one apparent double point. The order m of such a V_n is such that $2n+1 \leq m \leq 2^n+1$ if the variety is not the locus of ∞^h $(n-h)$-spaces where $1 \leq h \leq n-2$. The lowest possible order of a V_n in S_{2n+1} with one apparent double point is $n+2$, but in this case V_n^{n+2} is the rational locus of ∞^1 $(n-1)$-spaces. There are also V_n's which have no apparent double points, as for example, all the complete intersections of $n+1$ $2n$-dimensional varieties.

[†] "A series of rational loci with one apparent double point," *Proceedings of the Cambridge Philosophical Society*, vol. 27 (1931), pp. 399-403.

We shall also determine the orders of the pinch varieties on the double varieties only as the pinch varieties on the loci of higher multiplicities offer problems which require further investigation.

Let $b_t^{(s)}$ where $t = (s-1)k - (s-2)n$ denote the order of the s-fold variety $V_t^{b_t^{(s)}}$ on the projection ${}_kV_n^{2n+1}$ in S_{2n-k}. For $s = 1$, $b_t^{(1)} = 2t + 1$ is the order of V_t^{2t+1} in S_{2t+1} or of V_n^{2n+1} for $n = t$. If $s = 2$, we have the double varieties $V_t^{b_t^{(s)}}$'s for all the values of t from 0 to $n-1$. Now let j_{t-1} denote the order of the pinch locus $V_{t-1}^{j_{t-1}}$ lying on the double variety $V_t^{b_t^{(2)}}$. Thus, if $t = 1$, then j_0 means the number of pinch points on the double curve $C^{b_1^{(2)}}$ on the projection ${}_1V_n^{2n+1}$ in S_{2n-1}; and if $t = 2$, j_1 is the order of the pinch curve C^{j_1} on the double surface $F^{b_2^{(2)}}$ contained in the projection ${}_2V_n^{2n+1}$ in S_{2n-2}; etc. Now we proceed to the determination of $b_t^{(s)}$ and j_{t-1}.

The method we employ here is a modification of the so-called method of symbolic representation.* We have often used this method with success for problems of similar nature. It consists in completely degenerating a given variety V_n^ν into ν n-spaces and symbolizing these n-spaces and their relations with a set of ν symbols chosen in a certain manner or representing them diagrammatically with a set of ν dots having certain relative positions. Every general type of varieties has its own symbolic representation. Without going into any further details of explanation, we give below the diagrammatic representation of the V_n^{2n+1} we are studying:



Each dot in this diagram represents an S_n of the degenerate V_n^{2n+1}. Two consecutive dots or dots joined by a segment as a_0, a_1 or a_1, a'_1 or a'_1, a'_2 represent two S_n's in an S_{n+1}. Two dots separated by a third or dots through which pass two segments intersecting in a third dot, as a_0, a_2 or a_1, a'_2, mean two S_n's in an S_{n+2}. In general, any group of q dots in the diagram represents a group of q S_n's in the degenerate V_n^{2n+1} and the dimensions of the space containing these S_n's depends upon the manner in which the dots are chosen.

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* B. C. Wong, "On the number of apparent multiple points of varieties in hyperspace," *Bulletin of the American Mathematical Society*, vol. 36, pp. 102-106; "On surfaces in spaces of four and five dimensions," *Bulletin of the American Mathematical Society*, vol. 36, pp. 861-866; and "On certain characteristics of k-dimensional varieties in r-space," *Bulletin of the American Mathematical Society*, vol. 38, pp. 725-730.

Now consider the case $n = 1$. The diagram consists of three symbols a_0, a_1, a'_1 and it represents a cubic curve C^3 in an S_3 which has degenerated into three lines. The line a_0 has a point in common with the line a_1 which in turn has a point in common with the third line a'_1, but a_0 and a'_1 are skew. We say, therefore, that C^3 has $b_0^{(2)} = 1$ apparent double point.

Now for $n = 2$, we have the Del Pezzo quintic surface F^5 in S_5 which has degenerated into five planes symbolized by a_0, a_1, a'_1, a_2, a'_2. The four planes a_1, a'_1, a_2, a'_2 are contained in an S_4; the plane a_0 lies in an S_3 with a_1 and in an S_4 with each of the two planes a'_1, a_2 and is skew to a'_2. Since from a general given point of S_5 we can construct only one line incident with two skew planes, we say that F^5 has $b_0^{(2)} = 1$ apparent double point. Projecting F^5 from a line of S_5 upon an S_3, we obtain for projection a quintic surface ${}_1F^5$ with a double curve $C^{b_1^{(2)}}$ upon which lie $b_0^{(3)}$ triple points and j_0 pinch points. To find $b_1^{(2)}$ we count the number of pairs of dots such that the dots of each pair are separated by a third dot. There are five such pairs. Hence, the order of the double curve on ${}_1F^5$ is $b_1^{(2)} = 5$. To determine $b_0^{(3)}$ we notice that there is only one triple of symbols a_0, a'_1, a_2 representing three non-concurrent planes of the degenerate F^5. Hence, $b_0^{(3)} = 1$; that is, the projection ${}_1F^5$ in S_3 has only one triple point. To find j_0, we see that every pair of dots separated by just one other dot counts as two pinch points. For example, consider the two dots a_0, a'_1 and the dot a_1 separating them. The three dots represent the three planes of a degenerate cubic surface in S_4. Since the projection in S_3 of a non-degenerate cubic surface in S_4 has two pinch points, every pair of dots separated by a third in the diagrammatic representation of any surface accounts for two pinch points on the projection in S_3. In the case in question, the number of such pairs is 4 and hence ${}_1F^5$ has $j_0 = 8$ pinch points.

Consider one more case and let $n = 3$. We have now a V_3^7 in S_7. Degenerate V_3^7 into 7 S_3's and represent them by the 7 dots or symbols a_0, a_1, a'_1, a_2, a'_2, a_3, a'_3 in the diagram. It is easy to see that V_3^7 has only one apparent double point given symbolically by the pair of dots a_0, a'_3. Thus, $b_0^{(2)} = 1$. It is also easy to see that the order of the double curve $C^{b_1^{(2)}}$ on the projection ${}_1V_3^7$ in S_5 is $b_1^{(2)} = 5$, for there are just five pairs of dots, the dots of each pair being separated by at least two other dots. The number of pinch points on this double curve is $j_0 = 8$, for the diagram contains only four pairs of dots such that the dots of each pair are separated by just two other dots.

If we project V_3^7 from a general plane of S_7 upon an S_4, the projection

${}_2V_3{}^7$ has now a double surface $F^{b_2^{(2)}}$ of order $b_2{}^{(2)}$, a triple curve $C^{b_1^{(3)}}$ of order $b_1{}^{(3)}$ and a finite number, $b_0{}^{(4)}$, of quadruple points. Upon the double surface lies a pinch curve C^{j_1} of order j_1. There are also a finite number of pinch points on the triple curve, but we have already excluded these and other pinch loci on multiple varieties of multiplicities higher than 2 from the present work. It is not difficult to see that $b_2{}^{(2)}$ is given by the number of pairs of dots each separated by at least one dot and is equal to 13; that j_1 is given by twice the number of pairs each separated by just one single dot and is equal to 16. There are in the diagram 7 triples of non-consecutive dots, and, hence, we conclude that the triple curve is of order $b_1{}^{(3)} = 7$. Finally, ${}_2V_3{}^7$ in S_4 has $b_0{}^{(4)} = 1$ quadruple point given by the only quadruple of four non-consecutive dots a_0, a'_1, a_2, a'_3.

Reasoning in exactly the same manner for the determination of $b_t{}^{(s)}$ and j_{t-1}, we find, remembering that $b_i{}^{(1)} = 2i + 1$ and letting $b_i{}^{(0)} = 1$, the following results:

$$b_t{}^{(2)} = \sum_{i=0}^{t} b_i{}^{(1)} b_{t-i}^{(1)} - \sum_{i=0}^{t-1} b_i{}^{(1)} b_{t-1-i}^{(1)} = 2 \sum_{i=0}^{t-1} b_i{}^{(1)} + b_t{}^{(1)} = t^2 + (t+1)^2;$$

$$\begin{aligned} b_t{}^{(3)} &= \sum_{i+j=0}^{t} b_i{}^{(1)} b_j{}^{(1)} b_{t-i-j}^{(1)} - \sum_{i+j=0}^{t-1} b_i{}^{(1)} b_j{}^{(1)} b_{t-1-i-j}^{(1)} + \sum_{i+j=0}^{t-2} b_i{}^{(1)} b_j{}^{(1)} b_{t-2-i-j}^{(1)} \\ &= \sum_{i=0}^{t} b_i{}^{(1)} b_{t-i}^{(2)} - \sum_{i=0}^{t-1} b_i{}^{(1)} b_{t-1-i}^{(2)} = 2 \sum_{i=0}^{t-1} b_i{}^{(2)} + b_t{}^{(2)} \\ &= \tfrac{1}{3}(2t+1)(2t^2+2t+3); \end{aligned}$$

. . . .

$$b_t{}^{(s)} = \sum_{j=0}^{s-h} (-1)^j \binom{s-h}{j} \sum^{t-j} b_{i_1}{}^{(\sigma_1)} b_{i_2}{}^{(\sigma_2)} \cdots b_{i_{s-h+1}}^{(\sigma_{s-h+1})},$$

where

$$\sigma_1 + \sigma_2 + \cdots + \sigma_{s-h+1} = s, \qquad i_1 + i_2 + \cdots + i_{s-h+1} = t - j.$$

We also find that

$$j_{t-1} = 2b_t{}^{(2)} - 2b_{t-1}^{(2)} = 8t.$$

The above recursion formula for $b_t{}^{(s)}$ may take on various forms for various values of h and of $\sigma_1, \sigma_2, \cdots, \sigma_{s-h+1}$. For the purpose of calculation we may write

$$b_t{}^{(s)} = \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \sum^{t-j} b_{i_1}{}^{(1)} b_{i_2}{}^{(1)} \cdots b_{i_s}{}^{(1)}$$

obtained by putting $h = 1$ and consequently $\sigma_1 = \sigma_2 = \cdots = \sigma_s = 1$; or

$$b_t{}^{(s)} = \sum_{i=0}^{t} b_i{}^{(\sigma)} b_{t-i}^{(s-\sigma)} - \sum_{i=0}^{t-1} b_i{}^{(\sigma)} b_{t-1-i}^{(s-\sigma)}$$

obtained by putting $h = s - 1$ and $\sigma_1 = \sigma$, $\sigma_2 = s - \sigma$. If we now let $\sigma = 1$ in this last formula, we have

$$b_t{}^{(s)} = \sum_{i=0}^{t} b_i{}^{(1)} b^{\,s-1}_{\,t-i} - \sum_{i=0}^{t-1} b_i{}^{(1)} b^{\,(s-1)}_{\,t-1-i} = b_t{}^{(s-1)} + 2\sum_{i=0}^{t-1} b^{\,s-1}_{\,t-1-i} \,.$$

Finally, we may write

$$b_t{}^{(s)} = b^{\,(s)}_{\,t-1} + b_t{}^{(s-1)} + b^{\,(s-1)}_{\,t-1} \,.$$

We now notice a few properties of these b's and j's and the corresponding properties of the $V_n{}^{2n+1}$ in S_{2n+1}. First, $b_t{}^{(s)}$ is independent of n. Thus, the projection ${}_0V_n{}^{2n+1}$ in S_{2n} has one improper double point, that is, $b_0{}^{(2)} = 1$, for all values of n. The double curve on the projection ${}_1V_n{}^{2n+1}$ in S_{2n-1} is always $b_1{}^{(2)} = 5$ for $n > 1$ and the double surface on the projection ${}_2V_n{}^{2n+1}$ in S_{2n-2} is always $b_2{}^{(2)} = 13$ for $n > 2$. We also conclude that the curve in which $V_n{}^{2n+1}$ is met by any general S_{n+2} of S_{2n+1} is of deficiency $n - 1$.

j_{t-1} is also independent of n. This means that the pinch variety of dimension $t - 1 = (s-1)k - (s-2)n - 1$ on the t-dimensional double variety $V_t{}^{b_t{}^{(2)}}$ on the projection ${}_kV_n{}^{2n+1}$ in S_{2n-k} is always $j_{t-1} = 8t$ for all values of $n > 1$.

Another interesting property is that

$$b_t{}^{(s)} = b_s{}^{(t)};$$

that is, the number of groups of s dots in the diagrammatic representation of $V_n{}^{2n+1}$ such that any two dots of each group are separated by at least $t - 1$ other dots is equal to the number of groups of t dots such that any two dots of each group are separated by at least $s - 1$ other dots. The corresponding property on $V_n{}^{2n+1}$ is that the order of the s-fold variety of dimension $t = (s-1)k - (s-2)n$ on the projection ${}_kV_n{}^{2n+1}$ is equal to that of the t-fold variety of dimension s on the projection of $V_n{}^{2n+1}$ in a space of an appropriate number of dimensions. Thus, the double variety $V_3{}^{b_3{}^{(2)}}$ on the projection ${}_3V_6{}^{13}$ in S_9 and the triple surface $V_2{}^{b_2{}^{(3)}}$ on the projection ${}_4V_n{}^{13}$ in S_8 are of the same order $b_3{}^{(2)} = b_2{}^{(3)} = 25$; the double variety $V_5{}^{b_5{}^{(2)}}$ and the 5-fold surface $V_2{}^{b_2{}^{(5)}}$ on the projection ${}_5V_6{}^{13}$ in S_7 have the same order $b_5{}^{(2)} = b_2{}^{(5)} = 61$. This projection ${}_5V_6{}^{13}$ has a triple variety $V_4{}^{b_4{}^{(3)}}$ of order $b_4{}^{(3)} = 129$ and a quadruple variety $V_3{}^{b_3{}^{(4)}}$ of order $b_3{}^{(4)}$ which is also equal to 129. The projection ${}_{n-1}V_n{}^{2n+1}$ in S_{n+1} has always $b_0{}^{(n+1)} = 1$ $(n+1)$-fold point.

Other properties may be inferred. It suffices to add that the relation

$$j_{t-1} = 2b_t{}^{(2)} - 2b^{\,(2)}_{\,t-1} \qquad 0 \leqq t \leqq n - 1$$

hold for all varieties in general. In our case, if we let j_{n-1} denote the rank of the curve in which V_n^{2n+1} is met by a general S_{n+2} or the order of the locus of tangent lines to the projection ${}_{n-1}V_n^{2n+1}$ in S_{n+1} from a general point of S_{n+1}, then j_{n-1} is given by twice the number of pairs of consecutive points in the symbolic representation of V_n^{2n+1}, that is,

$$j_{n-1} = 2(3n - 1).$$

Thus, the rank of the twisted cubic curve in S_3 is 4 and the order of the tangent cone to the projection in S_3 of the Del Pezzo quintic surface in S_5 is 10; etc. We have the following relations:

$$\begin{aligned} 2b_1^{(2)} &= 2b_0^{(2)} + j_0, \\ 2b_2^{(2)} &= 2b_1^{(2)} + j_1 = 2b_0^{(2)} + j_0 + j_1, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ 2b_{n-1}^{(2)} &= 2b_0^{(2)} + j_0 + j_1 + \cdots + j_{n-2}, \\ 2n(2n+1) &= 2b_0^{(2)} + j_0 + j_1 + \cdots + j_{n-2} + j_{n-1}, \end{aligned}$$

which are known * to be true for all n-dimensional varieties of any order m in a space of any dimension r. For our V_n^{2n+1} in S_{2n+1}, $m = r = 2n + 1$.

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* B. C. Wong, "On certain characteristics of k-dimensional varieties in r-space," *Bulletin of the American Mathematical Society,* vol. 38, pp. 725-730.

A CHAIN OF DETERMINANT THEOREMS ARISING FROM THE CHARACTERIZATION OF PSEUDO r-SPHERIC (S_n) SETS.*

By LEONARD M. BLUMENTHAL.†

1. *Introduction.* A set S of undefined elements is called a semi-metric set provided that to each two elements p, q of S there corresponds a non-negative real number pq such that $pq = qp$, while $pq = 0$ if and only if the elements p, q are identical. If two semi-metric sets S, S' may be mapped isometrically upon each other, the sets are called congruent.

The set C of points forming the circumference of a circle of positive radius r evidently forms a semi-metric set if to each pair of distinct points is made correspond the length of the shorter arc of the circle joining them, while the number zero is attached to each pair of identical points. A semi-metric set is called d-cyclic provided the set is congruent to a subset of C, where $d = \pi r$; and a semi-metric set is pseudo d-cyclic provided the set is not congruent to a subset of C, though each three of its elements is congruent to three points of C.

In recent papers both d-cyclic and pseudo d-cyclic sets have been characterized by the writer.‡ It has been shown that pseudo d-cyclic quadruples are of three types, according as the quadruple contains no linear triples, exactly three, or four linear triples. Such quadruples are referred to as being of the first, second, or third kinds, respectively.

The principal theorem characterizing pseudo d-cyclic sets containing more than four elements we state as follows:

THEOREM I_G. *A pseudo d-cyclic set containing more than four elements is equilateral, with* $p_ip_j = 2d/3$ $(i, j = 1, 2, \cdots, n)$, $i \neq j$, *provided no four of the elements form a pseudo d-cyclic quadruple of the second kind.*

This theorem, first proved without algebraic formalism was shown in a recent paper § to be equivalent to the following determinant theorem:

THEOREM I_A. *If the determinant*

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* Presented at the Christmas meeting of the American Mathematical Society, December, 1933.

† National Research Fellow.

‡ L. M. Blumenthal, *American Journal of Mathematics*, vol. 54 (1932), pp. 387-396; pp. 729-738.

§ L. M. Blumenthal and G. A. Garrett, "Characterization of spherical and pseudo-spherical sets of points," *American Journal of Mathematics*, vol. 55 (1933), pp. 619-640.

$$\Delta_n = \begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{13} & \cdots & \cos\alpha_{1n} \\ \cos\alpha_{21} & 1 & \cos\alpha_{23} & \cdots & \cos\alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos\alpha_{n1} & \cos\alpha_{n2} & \cos\alpha_{n3} & \cdots & 1 \end{vmatrix}$$

$0 < \alpha_{ij} \leq \pi$, $\alpha_{ij} = \alpha_{ji}$ $(i, j = 1, 2, \cdots, n)$, $i \neq j, n > 4$, *is such that*

(i) *each third-order principal minor vanishes,*

(ii) *at least one fourth-order principal minor does not vanish,*

(iii) *if* $\Delta(i, j, k, l)$ *is any non-vanishing fourth-order principal minor, it is possible so to arrange the labeling of the elements of the minor that* $\cos\alpha_{ij} + \cos\alpha_{kl} \neq 0$.

Then each angle contained in the determinant has the value $2\pi/3$ *and the determinant has the value* $-1/2(3/2)^{n-1}(n-3)$.

A determinant of order n, having all the elements of its principal diagonal unity and the remaining elements equal to $-1/2$, we denote by $\Delta_n(-1/2)$. Then Theorem I_A states that any determinant satisfying the hypotheses of the theorem is a determinant of the form $\Delta_n(-1/2)$.

The hypothesis (iii), expressing the condition that no pseudo d-cyclic quadruple contained in the set of n points $p_1, p_2, \cdots, p_n$ for which the determined Δ_n is formed* be of the second kind, though simple and natural when viewed geometrically, is seen to be both complicated and artificial when stated algebraically. This suggests that interesting conclusions might follow when this hypothesis is suppressed.† This indeed turns out to be the case, and the purpose of this paper is to present the theorem so obtained. It is a remarkable fact that though the suppression of this hypothesis produces a slight weakening of the conclusion of Theorem I_G, the exclusion of the hypothesis has, as we shall see, only a trivial effect upon Theorem I_A.

2. We consider first the determinant with real elements of order five:

$$\Delta_5 = \begin{vmatrix} 1 & r_{12} & \cdots & r_{15} \\ r_{21} & 1 & \cdots & r_{25} \\ \cdot & \cdot & \cdot & \cdot \\ r_{51} & r_{52} & \cdots & 1 \end{vmatrix},$$

where $r_{ij} = r_{ji}$, $r_{ii} = 1$ $(i, j = 1, 2, \cdots, 5)$, and we prove the following theorem:

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* In the determinant Δ_n, $\alpha_{ij} = p_ip_j/r$ radians.

† It is to be noted that in Theorem I_A we have $0 < \alpha_{ij} \leq \pi$, while in what follows we assume $0 < \alpha_{ij} < \pi$.

THEOREM II$'_A$. *If the symmetric determinant* Δ_5, *is such that* (i) $|r_{ij}| < 1$ $(i, j = 1, 2, \cdots, 5)$ $i \neq j$, (ii) *every third-order principal minor vanishes,* (iii) *at least one fourth-order principal minor does not vanish, then* (a) *no fourth-order principal minor vanishes,* (b) *the absolute value of each element of* Δ_5 *outside of the principal diagonal is* 1/2.

Proof of (*a*). By hypothesis (i), we may write $r_{ij} = \cos \alpha_{ij}$, $0 < \alpha_{ij} < \pi$, $(i, j = 1, 2, \cdots, 5)$, $i \neq j$. Consider now the set consisting of five elements $p_1, p_2, \cdots, p_5$ and to each pair p_i, p_j let (1) $p_i p_j = r\alpha_{ij}$, $r > 0$, $(i, j = 1, 2, \cdots, 5)$, $i \neq j$, while $p_i p_i = 0$. The set $p_1, p_2, \cdots, p_5$ is evidently a semi-metric set. By hypothesis (ii) and (1) each triple of points contained in this set is congruent to three points of a circle of radius r, while by hypothesis (iii) the five points are not d-cyclic since they contain at least one quadruple that is not *d-cyclic*.[*] We may assume the labeling so that the non-vanishing fourth-order principal minor assured by (iii) is formed for the points p_1, p_2, p_3, p_4. These four points are, then, pseudo d-cyclic. We distinguish two cases: Case A. The quadruple is of the first or third kinds; Case B. The quadruple is of the second kind.

We prove first that Δ_5 *contains at least one other non-vanishing fourth-order principal minor.* Suppose that the four remaining principal fourth-order minors all vanish, and let us consider Case A.[†] It has been shown that in this case $\cos \alpha_{12} = \cos \alpha_{34} = a$, $\cos \alpha_{13} = \cos \alpha_{24} = b$, $\cos \alpha_{23} = \cos \alpha_{14} = c$. Expanding each of the four minors we obtain, using the above relations,

$$(\mathrm{I}) \quad \begin{array}{rrrrl}
(ac-b)\cos\alpha_{15} & +\,(ab-c)\cos\alpha_{25} & +\,(1-a^2)\cos\alpha_{35} & & = 0 \\
(ab-c)\cos\alpha_{15} & +\,(ac-b)\cos\alpha_{25} & & +\,(1-a^2)\cos\alpha_{45} & = 0 \\
(ab-c)\cos\alpha_{15} & & +\,(bc-a)\cos\alpha_{35} & +\,(1-b^2)\cos\alpha_{45} & = 0 \\
 & (ac-b)\cos\alpha_{25} & +\,(bc-a)\cos\alpha_{35} & +\,(1-c^2)\cos\alpha_{45} & = 0
\end{array}$$

The angles $\alpha_{15}, \alpha_{25}, \alpha_{35}, \alpha_{45}$ are not all $\pi/2$; in fact we get a contradiction if we suppose that any two of them are equal to $\pi/2$. Suppose that $\alpha_{15} = \alpha_{35} = \pi/2$, and consider $\Delta(p_1, p_3, p_5) = 0$, by (ii). Then α_{13} must have the value 0 or π which is impossible.

The equations (I) cannot then be satisfied unless the determinant of the coefficients vanishes. Since $\Delta(p_1, p_2, p_3) = 0$ we have either

$$\alpha_{12} + \alpha_{23} + \alpha_{13} = 2\pi$$

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[*] That hypotheses (ii) and (iii) have this geometrical interpretation is shown in the paper by Blumenthal and Garrett referred to above.

[†] The treatment of this case is given in the Blumenthal and Garrett paper and is reproduced here for the sake of completeness. We shall refer to this paper as B-G.

or the angle is the sum of the other two. In either case, the vanishing of the determinant of the coefficients leads to

$$\text{(II)} \qquad \begin{vmatrix} 0 & \sin\alpha_{12} & \sin\alpha_{13} & \sin\alpha_{23} \\ \sin\alpha_{12} & 0 & \sin\alpha_{23} & \sin\alpha_{13} \\ \sin\alpha_{13} & \sin\alpha_{23} & 0 & \sin\alpha_{12} \\ \sin\alpha_{23} & \sin\alpha_{13} & \sin\alpha_{12} & 0 \end{vmatrix} = 0$$

and evaluation of the determinant yields

$$(\sin\alpha_{12} + \sin\alpha_{23} + \sin\alpha_{13})(\sin\alpha_{12} + \sin\alpha_{23} - \sin\alpha_{13}) \\ \times (\sin\alpha_{12} - \sin\alpha_{23} + \sin\alpha_{13})(-\sin\alpha_{12} + \sin\alpha_{23} + \sin\alpha_{13}) = 0.$$

But it is readily shown that no one of the factors in this expression can vanish (since $\Delta(p_1, p_2, p_3) = 0$) and the desired contradiction is obtained for Case A.

Consider now Case B. In this case we have $a = \cos\alpha_{12} = -\cos\alpha_{34}$, $b = \cos\alpha_{13} = -\cos\alpha_{24}$, $c = \cos\alpha_{23} = -\cos\alpha_{14}$. Using these relations in the expansion of the four fourth-order principal minors assumed to vanish, we find that the first equation of the new set (I′) is identical with the first equation of the set (I), while the last three equations of (I′) differ from the last three equations of (I) only in the sign of the coefficient of $\cos\alpha_{45}$. Such a change leaves the condition (II) unaltered and the contradiction is obtained as in Case A.

Thus, we have shown that the determinant Δ_5 contains at least one other non-vanishing fourth-order principal minor. We may assume the labeling so that $\Delta(p_1, p_2, p_3, p_4)$, $\Delta(p_1, p_2, p_3, p_5)$ are these minors. We show now that none of the three remaining principal fourth-order minors can vanish. There are three cases to be considered.

Case 1. *Both quadruples* p_1, p_2, p_3, p_4 and p_1, p_2, p_3, p_5 *are of the first, or third kinds.*

This case is handled in B-G, page 626.

Case 2. *One quadruple, say* p_1, p_2, p_3, p_4, *is of the second kind, while* p_1, p_2, p_3, p_5 *is of the second or third kinds.* We have then

$$\cos\alpha_{12} = -\cos\alpha_{34} = \cos\alpha_{35}, \qquad \cos\alpha_{13} = -\cos\alpha_{24} = \cos\alpha_{25}, \\ \cos\alpha_{14} = -\cos\alpha_{23} = -\cos\alpha_{15}.$$

Suppose, now, that any other fourth-order principal minor, say $\Delta(p_1, p_2, p_4, p_5)$, vanishes. We obtain

$$\begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{14} \\ \cos\alpha_{12} & 1 & \cos\alpha_{24} \\ \cos\alpha_{15} & \cos\alpha_{25} & \cos\alpha_{45} \end{vmatrix} = 0.$$

Applying the above relations, we have

$$\begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{14} \\ \cos\alpha_{12} & 1 & \cos\alpha_{24} \\ \cos\alpha_{14} & \cos\alpha_{24} & -\cos\alpha_{45} \end{vmatrix} = 0.$$

Consider the function $\phi(x)$ defined as follows:

$$\phi(x) \equiv \begin{vmatrix} 1 & \cos\alpha_{12} & \cos\alpha_{14} \\ \cos\alpha_{12} & 1 & \cos\alpha_{24} \\ \cos\alpha_{14} & \cos\alpha_{24} & x \end{vmatrix}$$

Since $\Delta(p_1, p_2, p_4) = 0$, one root of $\phi(x) = 0$ is $x = 1$. Since the coefficient of x in the equation does not vanish, $x = 1$ is the only root of the equation. Hence, we must have for the vanishing of $\Delta(p_1, p_2, p_4, p_5)$, $\cos\alpha_{45} = -1$, which is impossible by hypothesis (i). This method is applied to show that none of the remaining fourth-order principal minors can vanish.

CASE 3. *Both quadruples are of the second kind.* This case is treated in the same manner as Case 1, and the proof of (a) is complete.

Proof of (*b*). Since, from (a), none of the fourth-order principal minors of Δ_5 vanish, each quadruple contained in the five points $p_1, p_2, \cdots, p_5$ is pseudo d-cyclic. If none of these pseudo d-cyclic quadruples is of the second kind, then the five points form an equilateral set with $p_ip_j = 2d/3$; $(i, j = 1, 2, \cdots, 5)$, $i \neq j$, by the theorem stated in the *Introduction* of this paper. Then $\cos\alpha_{ij} = \cos p_ip_j/r = -1/2$ and (b) is proved.

Let us determine how many pseudo d-cyclic quadruples of the second kind a pseudo d-cyclic quintuple can contain. It has been shown* that if p_i, p_j, p_k, p_l form a pseudo d-cyclic quadruple of the second kind then $\cos\alpha_{ij} = -\cos\alpha_{kl}$, $\cos\alpha_{ik} = -\cos\alpha_{jl}$, $\cos\alpha_{il} = -\cos\alpha_{jk}$. Using these relations, it is easily seen that a pseudo d-cyclic quintuple *cannot contain five, exactly three, or exactly one pseudo d-cyclic quadruple of the second kind.* It may, however, contain *exactly four* or *exactly two* of these quadruples. In the first case, however, it is not difficult to show that *the labeling may always be so selected* that

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* See B-G, p. 623.

$$\text{(A)}\quad \begin{aligned}&\cos\alpha_{12}=\cos\alpha_{13}=\cos\alpha_{14}=\cos\alpha_{15}=1/2;\ \text{and}\\ &\cos\alpha_{23}=\cos\alpha_{24}=\cos\alpha_{25}=\cos\alpha_{34}=\cos\alpha_{35}=\cos\alpha_{45}=-1/2;\end{aligned}$$

while in the second case we can always select the labeling so that

$$\text{(B)}\quad \begin{aligned}&\cos\alpha_{12}=\cos\alpha_{14}=\cos\alpha_{24}=\cos\alpha_{35}=-1/2,\\ &\cos\alpha_{13}=\cos\alpha_{15}=\cos\alpha_{23}=\cos\alpha_{25}=\cos\alpha_{34}=\cos\alpha_{45}=1/2.\end{aligned}$$

In either case the absolute value of each element outside of the principal diagonal is 1/2 and the theorem is proved.

3. The elementary operation of multiplying k rows and the *corresponding* k columns of a determinant by minus one we denote by E_k. Evidently a determinant Δ_5 that satisfies the hypotheses of Theorem II'_A will continue to do so when subjected to the operation E_k, $k \leq 5$.

THEOREM II_A. *If the symmetric determinant* Δ_5 *is such that* (i) $|r_{ij}| < 1$, $(i, j = 1, 2, \cdots, 5)$, $i \neq j$, (ii) *every third order principal minor vanishes,* (iii) *at least one fourth-order principal minor does not vanish, then* Δ_5 *may be transformed into* $\Delta_5(-1/2)$ *by the operation* E_k, $k \leq 2$, *and* $\Delta_5 = -(3/2)^4$.

Proof. If the five points contain no pseudo d-cyclic quadruple of the second kind, then the theorem is true, by B-G, page 627, Theorem 9. From the proof of part (b) of Theorem II'_A, we have seen that the quintuple can contain exactly four or exactly two pseudo d-cyclic quadruples of the second kind, and that in either case the labeling can be so chosen that the relations (A), (B), respectively, are valid. Forming Δ_5 for these two cases, we note that in Case A we transform Δ_5 into $\Delta_5(-1/2)$ by applying E_1 to the first row and first column, while in Case B we apply E_2 to the third and fifth rows and columns. We easily compute Δ_5 to complete the proof of the theorem.

The Theorem II'_A is equivalent to the following theorem characterizing pseudo d-cyclic quintuples.

THEOREM II_G. *If* p_1, p_2, p_3, p_4, p_5 *form a pseudo d-cyclic quintuple such that no two points have a distance equal to d, then* $p_ip_j = 2d/3$ or $d/3$, $(i, j = 1, 2, \cdots, 5)$, $i \neq j$, *and at least four of the numbers* p_ip_j *equal* $2d/3$.*

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* Theorem II_G is the geometric analogue of Theorem II'_A rather than of Theorem II_A which, indeed, has no interpretation in the geometry under consideration. The above makes clear, at least for the case of five points, the effect of suppressing the hypothesis that no four points form a pseudo d-cyclic guadruple of the second kind. It is seen that while the conclusion of Theorem I_G (stated for five points) is somewhat stronger than the conclusion of Theorem II_G, the difference between Theorem I_A for Δ_5 and Theorem II_A is trivial.

4. We prove now by the induction* the theorem characterizing pseudo d-cyclic sets containing n points, $n > 4$, with no two points *diametral* (that is, $p_ip_j \neq d$).

THEOREM III$_G$. *A pseudo d-cyclic set* $p_1, p_2, \cdots, p_n$ *containing more than four points and having no two points diametral is such that* $p_ip_j = 2d/3$ *or* $d/3$ $(i, j = 1, 2, \cdots, n)$, $i \neq j$.†

We have proved the theorem for the case $n = 5$. We assume the theorem is true for $n = k$, $k > 4$, and show that this implies the validity of the theorem for $n = k + 1$.

Let $p_1, p_2, \cdots, p_k, p_{k+1}$ be a pseudo d-cyclic set of $k + 1$ points, without diametral points. At least one of the sets of k points contained in these $k + 1$ points is pseudo d-cyclic, for otherwise since $k > 4$ and the circle has the congruence order 4, the $k + 1$ points would be d-cyclic, contrary to hypothesis. We assume the labeling so that $p_1, p_2, \cdots, p_k$ is pseudo d-cyclic. Since no pair of these k points is diametral, by hypothesis the k points are such that $p_ip_j = 2d/3$ or $d/3$ $(i, j = 1, 2, \cdots, k)$, $i \neq j$.

Evidently, the $k + 1$ points must contain at least one other pseudo d-cyclic set of k points. For, in the contrary case it is easily seen that every quadruple contained in the set $p_1, p_2, \cdots, p_k$ is d-cyclic since it is contained in a d-cyclic set of k points, and hence $p_1, p_2, \cdots, p_k$ would be d-cyclic. We may suppose, then, that the set $p_2, p_3, \cdots, p_{k+1}$ is pseudo d-cyclic and hence $p_ip_j = 2d/3$ or $d/3$ $(i, j = 2, 3, \cdots, k + 1)$, $i \neq j$. Hence of the $(1/2)k(k + 1)$ distances determined by the $k + 1$ points $p_1, p_2, \cdots, p_k, p_{k+1}$, each is seen to equal $2d/3$ or $d/3$ except the distance p_1, p_{k+1} which does not enter into these two sets. To determine this distance, consider any triple of points, say p_1, p_2, p_{k+1}, containing this distance. Since this triple is d-cyclic, we must have either

$$(p_1p_2 + p_2p_{k+1} - p_1p_{k+1})(p_1p_2 - p_2p_{k+1} + p_1p_{k+1})(p_1p_2 - p_2p_{k+1} - p_1p_{k+1}) = 0$$

or

$$p_1p_2 + p_2p_{k+1} + p_1p_{k+1} = 2d.$$

Since p_1p_2 and p_2p_{k+1} must have one of the values $2d/3$ or $d/3$, it is readily seen that in either of the above cases, $p_1p_{k+1} = 2d/3$ or $d/3$. Hence the theorem is proved.

We are now in a position to prove the principal theorem of this paper.

THEOREM III$_A$. *If the determinant of real elements*

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* We prefer to make this inductive proof geometrical. It may, of course, be given algebraically.

† Of course, *all* of the distances p_ip_j cannot equal $d/3$. A certain minimum number μ of these distances must equal $2d/3$; thus, when $n = 5$, $\mu = 4$.

$$\Delta_n = \begin{vmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & 1 & r_{23} & \cdots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & r_{n3} & \cdots & 1 \end{vmatrix}$$

$n > 4$, $r_{ij} = r_{ji}$ $(i, j = 1, 2, \cdots, n)$, $i \neq j$ *is such that*

(i) $|r_{ij}| < 1$ $(i, j = 1, 2, \cdots, n)$, $i \neq j$,

(ii) *each third-order principal minor vanishes,*

(iii) *at least one fourth-order principal minor does not vanish.*

Then Δ_n *may be transformed into* $\Delta_n(-1/2)$ *by* E_k, *and hence*

$$\Delta_n = -1/2(3/2)^{n-1}(n-3).$$

Proof. By Theorem III_G each element of Δ_n outside of the principal diagonal has the absolute value 1/2, while from Theorem II_A we know that there exists an elementary operation $I_{k'}$ $(k' \leq 2)$ transforming the principal fifth-order minor Δ_5 in the upper left-hand corner of Δ_n into $\Delta_5(-1/2)$. Suppose this done. It is then evident that the principal sixth-order minor Δ_6 occupying the upper left-hand corner of Δ_n has all the elements of its last column (with the possible exception of 1) of the same sign. This is clear upon examining the third-order principal minors of Δ_6 that contain two elements (exclusive of 1) from the last column. The vanishing of these minors together with the facts that Δ_5 has been transformed into $\Delta_5(-1/2)$ and the square of each element of Δ_n outside of the principal diagonal equals 1/4, yields the desired result. Hence Δ_6 may be transformed into $\Delta_6(-1/2)$ by $E_{k'+1}$. Repeating the above argument, we see that Δ_n can be transformed into $\Delta_n(-1/2)$ by E_k.

The determinant $\Delta_n(-1/2)$ is readily evaluated to yield

$$-1/2(3/2)^{n-1}(n-3).$$

5. There is reason to believe that Theorem III_A is the first link in a chain of theorems concerning determinants of type Δ, the k-th link of which is the following theorem:

THEOREM. *If the determinant of real elements* Δ_n *is of order* $n > k+3$ *and* (i) *every principal minor of order less than* $k+2$ *is positive,* (ii) *every principal minor of order* $k+2$ *vanishes,* (iii) *at least one principal minor of order* $k+3$ *does not vanish, then* Δ_n *may be transformed by the elementary operation* E_p *into a determinant with the elements of its principal diagonal unity and all other elements equal to* $-(1+k)^{-1}$. *Then the value of the determinant is* $-(1+k)^{-1}\left(\frac{k+2}{k+1}\right)^{n-1} \cdot (n-k-2)$.

THE INSTITUTE FOR ADVANCED STUDY.

CONCERNING THE GEOMETRY OF ACYCLIC SETS.

By C. H. Harry.

Introduction. In general, when sets of order relations are proposed for point collections the purpose is to construct linearly ordered arrays. One method of doing this is to take as undefined the concept of *betweenness,* where the condition "The point Y lies between the points X and Z" is symbolized by the triadic expression XYZ. The problem of the present paper is to formulate a set of axioms for this triadic expression which will lead to sets which are homeomorphic with subsets of general acyclic Peano spaces.* The work has been divided into two parts, the first contains an analysis of the space in terms of the betweenness relation alone, while the second deals with a discussion of the connection between the limit points and this betweenness relation.

Part I.

For the purpose of analysis the following five axioms will be employed:

Axiom 0. *ABC implies that A, B and C are distinct.*
Axiom 1. *ABC implies CBA.*
Axiom 2. *If A, B and C are distinct, then ABC or BCA or CAB is true.*
Axiom 3. *ABC together with BCA implies CAB.*

Definition. $\overline{UVW}$ means that UVW is true while VWU is false.

Axiom 4. *$\overline{AXB}$ together with either XCB or XBC implies $\overline{AXC}$.*†

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* A Peano space is one which is the continuous image of the unit interval. A necessary and sufficient condition for a set to be a Peano space is that it be metric, self compact, connected and locally connected. Cf. Hahn, *Wiener Berichte,* vol. 123 (1914), p. 2433 and Mazurkiewicz, *Fundamenta Mathematica,* vol. 1 (1920), p. 166. An acyclic Peano space is one which contains no simple closed curve.

† This collection differs from the assumptions used for linearly ordered arrays in Axiom 3. For linear sets this axiom would have to be replaced by a condition like:

Axiom 3′. *ABC implies the falsity of both BCA and CAB.*

E. V. Huntington and J. R. Kline, *Transactions of the American Mathematical Society,* vol. 18 (1917), pp. 301-325, have listed twelve axioms which form the complete set of possible triadic relations among three and four points. These twelve axioms are either contained in or deducible from Axioms 0-2, 3′, 4. Also, in this same paper, examples are given which, with very slight modifications, show Axioms 0-4 to be independent.

The choice of this particular set of axioms was influenced by the following properties of the cut points * of a connected space P. If a point Y separates two points X and Z in the space P they could be said to stand in the relations XYZ and ZYX and only these two. On the other hand, if no one of the three points X, Y and Z separates the other two they could be given all six possible betweenness relations. This definition satisfies Axioms 0-3 provided it is agreed to compare three points when and only when they are distinct. Since at most one of three points can separate the other two, then the relation $\overline{XYZ}$ will occur when and only when Y separates X and Z. To demonstrate Axiom 4 suppose X separates A and B. Then $P - X$ may be expressed as the sum of two sets M_a and M_b which have no point or limit point in common and which contain A and B respectively. If XCB or XBC is true, then C must lie in M_b, else $\overline{CXB}$ must hold. Hence, the point X separates A and C, i. e., $\overline{AXC}$ is true. Thus, all the properties deduced in Part I will be properties of the cut points of a connected set. In particular, cyclic chains and cyclic elements analogous to those of G. T. Whyburn † will be defined in terms of the betweenness relation; and, at the end of Part I, it will be shown that the above definition of betweenness for connected and locally connected sets makes the cyclic chains and elements defined by means of betweenness identical with those of the *Structure* paper. Thus, many of the properties of these chains and elements which are deduced by G. T. Whyburn may be shown without the use of connectivity.

Definition. By the *segment* AB is meant $A + B +$ all points X for which $\overline{AXB}$ is true. Also, in speaking of a segment AB it is always assumed that $A \neq B$.

Definition. By the *cyclic chain* $C(AB)$ is meant $A + B +$ all points X for which:

1° AXB is true, and 2° there is no point Y giving both $\overline{AYX}$ and $\overline{XYB}$, where again it is always assumed that $A \neq B$.

1. In this section properties of segments and cyclic chains will be deduced. The cyclic elements will be defined later.

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* The point X is said to separate or cut a connected set P between two points A and B provided $P - X$ may be expressed as the sum of two sets M_a and M_b neither of which contains a point or limit point of the other and which contain A and B respectively. From the fact that $M_a + X$ is a connected subset of $P - B$ it follows that B does not separate P between A and X. Similarly, A does not separate P between B and X.

† "Concerning the structure of a continuous curve," *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194. Hereafter this paper will be referred to as the *Structure* paper.

1.0 $\overline{ABC}$ *implies* $\overline{CBA}$.

Proof. ABC *implies* CBA, Axiom 1. If BAC were true, then CBA together with BAC would give ACB, Axiom 3. But ACB, by Axiom 1, implies BCA, contrary to the definition of $\overline{ABC}$. Hence, $\overline{CBA}$ holds.

1.01 $\overline{ABC}$ *implies the falsity of* CAB.

1.02 *Given* AXB, *then either* $\overline{AXB}$ *is true or both* XBA *and* BAX *are true.*

1.1 AB *lies in* $C(AB)$.

Proof. Suppose $X \varepsilon AB - A - B$.* Now $\overline{BXA}$ and XYA imply $\overline{BXY}$, by Axiom 4. Therefore, there is no point for which XYA and XYB could be true simultaneously. Hence, both conditions that X belong to $C(AB)$ are fulfilled.

1.2 *If* $D \varepsilon C(AB)$, *then* $C(AB) = C(AD) + C(DB)$.

Proof. The demonstration is accomplished by showing that $C(AB)$ contains $C(AD) + C(DB)$ and that $C(AD) + C(DB)$ contains $C(AB)$. Proceeding with the proof of the former, suppose $X \varepsilon C(AD) - A - D$. It will first be shown that AXB is true by proving the falsity of $\overline{ABX}$ and XAB. Now $\overline{BAX}$ and AXD, by Axiom 4, would imply $\overline{BAD}$, contrary to the condition that D lie in $C(AB)$. On the other hand, $\overline{XBA}$ and BDA, by the same axiom, imply $\overline{XBD}$, which, together with $\overline{ABX}$, contradicts the fact that $X \varepsilon C(AD)$. Since AXB must be true the only thing which would prevent X from lying in $C(AB)$ would be the existence of a point Y giving $\overline{AYX}$ and $\overline{XYB}$. The existence of such a point will now be shown to be impossible. If Y did exist it would be distinct from D, for $\overline{ADX}$ contradicts the condition that X lie in $C(AD)$. As Y, D and X are distinct, YXD or YDX or DYX is true. An application of Axiom 4 shows that $\overline{AYX}$ and $\overline{BYX}$, together with either YXD or YDX, imply both $\overline{AYD}$ and $\overline{DYB}$, contrary to the fact that $D \varepsilon C(AB)$. Thus, as both YXD and YDX are false, $\overline{DYX}$ is true. However, $\overline{AYX}$ and $\overline{XYD}$ prevent X from being in $C(AD)$. Thus, the point Y can not exist and $X \varepsilon C(AB)$. By symmetry $C(AB)$ also contains $C(DB)$.

It remains to show that $C(AB)$ lies in $C(AD) + C(DB)$. In case $\overline{ABD}$ is true $B \varepsilon AD$. Thus, by 1.1, $B \varepsilon C(AD)$, so that by the above paragraph $C(AB)$ lies in $C(AD)$. A similar argument in the case of $\overline{BAD}$ will show that $C(AB) = C(BA)$ lies in $C(DB)$. If these two possibilities are excluded either $\overline{ADB}$ is true or all six relations hold.

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* $X, Y, \ldots \varepsilon N$ means that $X, Y, \ldots$ are points of the set N.

Case 1. $\overline{ADB}$.

Suppose $X \varepsilon C(AB) - A - B - D$. If $\overline{XBD}$ were true, then $\overline{XBD}$, together with BDA, would give $\overline{XBA}$, contrary to the fact that $X \varepsilon C(AB)$. Since $\overline{XBD}$ is false, then either BXD or XBD must be true. Likewise, either XDA or AXD is true. If both AXD and DXB were false the relations would be $\overline{ADX}$ and $\overline{XDB}$, contrary to the fact that $X \varepsilon C(AB)$. Thus, it may be assumed without loss that AXD is true. Hence, from the definition, if X were not in $C(AD)$ there would be a point Y giving the relations $\overline{AYX}$ and $\overline{XYD}$. As $X \varepsilon C(AB)$, $\overline{ABX}$ is false, so that $B \neq Y$. Now $\overline{AYX}$ and $\overline{XYB}$ imply that X is not in $C(AB)$, so that $\overline{XYB}$ must be false, i. e., either YBX or BXY must be true. However, $\overline{DYX}$, together with either YBX or BXY, implies, by Axiom 4, $\overline{DYB}$. But $\overline{ADB}$ and DYB give $\overline{ADY}$, and this, together with $\overline{DYX}$, gives $\overline{ADX}$, contrary to AXD. Thus, Y can not exist and X must lie in $C(AD)$.

Case 2. *ADB, DBA and BAD are all three true.*

Again assume that X is a point of $C(AB) - A - B - D$. In a manner similar to that of Case 1, it can be shown that either AXD or DXB is true. Assume AXD. If X is not in $C(AD)$ there must be a point Y giving $\overline{AYX}$ and $\overline{XYD}$, where again $Y \neq B$. Now $\overline{XYD}$ and either YBD or YDB imply $\overline{XYB}$, which, together with $\overline{AYX}$, contradicts the fact that $X \varepsilon C(AB)$. Hence, as both YBD and YDB are false, $\overline{BYD}$ must hold; i. e., $Y \varepsilon DB$. As Y lies in BD, then by 1. 1 it lies in $C(DB)$. Hence, $C(YB)$ lies in $C(DB)$. It will next be shown that X lies in $C(YB)$. It is first necessary to show $\overline{AYB}$. Now $\overline{AYX}$ and $\overline{XYB}$ would imply that X is not in $C(AB)$. But if $\overline{XYB}$ is false either YXB or YBX must be true, and either one of these relations, together with $\overline{AYX}$ gives, by Axiom 4, $\overline{AYB}$. Thus, as $Y \varepsilon AB$, $Y \varepsilon C(AB)$ and, by Case 1 above, $C(AB)$ lies in $C(AY) + C(YB)$. Since $\overline{AYX}$ prevents X from lying in $C(AY)$, then $X \varepsilon C(YB)$. That is, $X \varepsilon C(DB)$, for $C(YB)$ lies in $C(DB)$. The above proof of $\overline{AYB}$ gives the following corollaries:

1. 21 *If $D \varepsilon C(AB)$ and $Y \varepsilon AD - D$, then $Y \varepsilon AB$.*

1. 22 *If $X \varepsilon AB - A - B$, then $C(AX) \cdot C(XB) = X$.*

1. 3 *If A, B and C are distinct points then $C(AB)$ lies in $C(AC)$ $+ C(CB)$.*

Proof. If $\overline{ABC}$, $\overline{BCA}$ or $\overline{CAB}$ is true the theorem is a direct consequence of 1. 1 and 1. 2. If these three possibilities are excluded then all six order relations are true for A, B and C. It will first be shown that AXC is true for any point X of $C(AB)$ which is different from A, B and C. If $\overline{ACX}$ were true, then, by 1. 21, C would be a point of AB, contrary to the fact that $\overline{ACB}$ is excluded. On the other hand, $\overline{XAC}$ and ACB, by Axiom 4, give $\overline{XAB}$, contrary to the fact that $X \varepsilon C(AB)$. Thus, as $\overline{ACX}$ and $\overline{XAC}$ are both false, AXC must hold. If X is not an element of $C(AC)$, then there exists a point Y such that $\overline{AYX}$ and $\overline{XYC}$ are both true. By 1. 21 $Y \varepsilon AB$, and, by 1. 2, $C(AB)$ lies in $C(AY) + C(YB)$. Since $\overline{AYX}$ prevents X from lying in $C(AY)$, then $X \varepsilon C(YB)$. But if $X \varepsilon C(YB)$, YXB is true. By Axiom 4, $\overline{CYX}$ and YXB give $\overline{CYB}$, i. e., $Y \varepsilon C(CB)$. Thus, by 1. 1 and 1. 2, $C(YB)$ lies in $C(CB)$. Hence, X, a point of $C(YB)$, lies in $C(CB)$.

1. 4 *If X, $Y \varepsilon C(AB)$, where $X \neq Y$, then either $X \varepsilon C(AY)$ or $Y \varepsilon C(AX)$.*

Proof. By Axiom 4, $\overline{XAY}$ and AYB give $\overline{XAB}$, contrary to the fact that $X \varepsilon C(AB)$. Thus, as $\overline{XAY}$ is false, either AXY or AYX is true. Suppose AXY. If X were not in $C(AY)$ there would be a point P giving $\overline{APY}$ and $\overline{XPY}$. By 1. 3 the chain $C(AB)$ lies in $C(AP) + C(PB)$. As $\overline{APY}$ prevents Y from lying in $C(AP)$ it lies in $C(PB)$, i. e., PYB is true. But $\overline{XPY}$ and PYB give $\overline{XPB}$, which, together with $\overline{APX}$, contradicts the fact that $X \varepsilon C(AB)$. The assumption AYX makes Y lies in $C(AX)$. Also, if X, Y, A and B are not distinct the theorem is trivial.

1. 41 *If X and Y are distinct points of $C(AB)$, then $XY - X - Y$ lies in AB.*

1. 5 *If X and Y are distinct elements of $AB - A$, then either $\overline{AXY}$ or $\overline{AYX}$ is true.*

Proof. As the theorem is trivial if A, B, X and Y are not distinct, assume them to be different points. If YAX were true, then $\overline{BYA}$ and YAX would imply $\overline{BYX}$, while $\overline{BXA}$ and XAY would imply $\overline{BXY}$, a contradiction. Since XAY is impossible, then either $\overline{AXY}$ or $\overline{AYX}$ must hold.

1. 51 *If X, Y and Z are distinct points of $AB - A - B$, then $\overline{XYZ}$ or $\overline{YZX}$ or $\overline{ZXY}$ is true.*

Proof. By 1. 5 either $\overline{AXY}$ or $\overline{AYX}$ is true. Assume $\overline{AXY}$. Likewise, either $\overline{AZX}$ or $\overline{AXZ}$ holds. Now $\overline{YXA}$ and $\overline{XZA}$, by Axiom 4, give the desired result. Suppose, then, that the relation is $\overline{AXZ}$. By Axiom 4, $\overline{BYA}$

and YXA give $\overline{BYX}$, while $\overline{BZA}$ and ZXA give $\overline{BZX}$. Applying the results of 1. 5 to the segment XB, it follows that either $\overline{XYZ}$ or $\overline{XZY}$ is true.

2. The notion of cyclic element will now be introduced.

Definition. Two points X and Y are said to be *conjugate* provided there is no point P giving the relation $\overline{XPY}$.

Definition. By a *cut point* X of the space is meant a point X for which there exists two points U and V giving the relation $\overline{UXV}$.

Definition. By a *cyclic element* $C(X)$ is meant any maximal set $C(X)$ which contains X and is such that every pair of its points are conjugate.

2. 1 *If $X \varepsilon C(AB) - AB$ and $D(X)$ is the set of all points of $C(AB)$ which are conjugate to X, then either X is a cut point of the space or $D(X) + X$ is a cyclic element.*

Proof. The proof will be divided into three parts.

1° *Every two points U and V of $D(X)$ are conjugate.*

The assumption that U and V are not conjugate entails the existence of a point W giving $\overline{UWV}$. By definition U, V and X are distinct. Also, as $W \varepsilon AB$ by 1. 41, $X \neq W$. Since U is conjugate to X, $\overline{UWX}$ is false, i. e., either WUX or WXU is true. But, by Axiom 4, $\overline{VWU}$, together with either WUX or WXU, implies $\overline{VWX}$, contrary to the fact that W is conjugate to U. Hence, U and V must be conjugate, i. e., all points of $D(X)$ lie in the same cyclic element $C(X)$.

2° *If a point Q lies in $D(X) - X$, then $D(X) + X$ is a cyclic element.*

Suppose P is a point conjugate to every point of $D(X) + X$. The supposition that $P \cdot C(AB) = 0$ leads to the following contradiction. Either APB must be false or there must exist a point Y giving $\overline{AYP}$ and $\overline{PYB}$. Under the assumption that APB is false it follows that either $\overline{BAP}$ or $\overline{ABP}$ must hold. The relation AXB follows from the fact that $X \varepsilon C(AB) - AB$. But $\overline{PAB}$ and AXB give $\overline{PAX}$, while $\overline{PBA}$ and BXA imply $\overline{PBX}$, both of which contradict the fact that P and X are conjugate. Since APB is true and it is supposed that P does not lie in $C(AB)$, then there is some point Y giving $\overline{AYP}$ and $\overline{PYB}$. As $X \neq Q$, either X or Q, say Q, is distinct from Y. Also, as P is not in $C(AB)$ and Q is, then $P \neq Q$. Thus, one of the three PYQ, PQY or QPY must hold. Since $\overline{PYQ}$ is impossible, as P is conjugate to Q, then either YQP or YPQ is true. But $\overline{AYP}$ and $\overline{BYP}$, together with either YQP or YPQ, imply $\overline{AYQ}$ and $\overline{QYB}$, contrary to the fact that $Q \varepsilon C(AB)$. Hence, Y does not exist and $P \varepsilon C(AB)$. But as P is conjugate to X, then $P \varepsilon D(X)$, i. e., $D(X) + X$ is maximal.

3° *If $D(X) = 0$ and there is a cyclic element $C(X) \neq X$, then for any point P of $C(X) - X$ the relations $\overline{AXP}$ and $\overline{PXB}$ are valid.*

Just as in the proof of 2° above the relation APB must hold. Thus, as $P \cdot C(AB) = 0$, there exists a point Y giving $\overline{AYP}$ and $\overline{PYB}$. If the point X were different from the point Y, then reasoning identical with the latter part of 2° above would lead to the contradiction that $X \cdot C(AB) = 0$, where X replaces the letter Q used above. Thus, if $D(X) = 0$ and X is not a cyclic element $C(X)$ it is a cut point.

2.2 *A cyclic element has at most two points in common with any segment.*

Proof. The theorem is a direct consequence of 1.51 and the definition of cyclic element.

2.3 *If a cyclic element $C(X)$ has two distinct points U and V in common with a cyclic chain $C(AB)$, then $C(X)$ lies in $C(AB)$.*

Proof. Assume P is a point of $C(X)$ not in $C(AB)$. First, APB is true, for either $\overline{PAB}$ or $\overline{PBA}$, together with the fact that U and V are distinct points of $C(AB)$, would contradict the condition that P be conjugate to both U and V. Thus, there is a point Y giving $\overline{AYP}$ and $\overline{PYB}$. As U and V are distinct points of $C(AB)$, one of them, say U, is different from P and Y. Since P is conjugate to U, then $\overline{UYP}$ is false, i. e., either YUP or YPU must hold. But $\overline{AYP}$ and $\overline{BYP}$, together with either YUP or YPU imply $\overline{AYU}$ and $\overline{BYU}$, contrary to the fact that $U \varepsilon C(AB)$. Hence, $P \varepsilon C(AB)$.

2.31 *If a cyclic element has two points in common with AB it lies in $C(AB)$.*

2.4 *If $C(X)$ and $C(Y)$ are distinct cyclic elements they have at most one point P in common. Further, if U and V are any points of $C(X) - P$ and $C(Y) - P$ respectively, then $\overline{UPV}$ is true.*

Proof. As $C(X) \neq C(Y)$, one contains a point not in the other. Suppose $Q \varepsilon C(Y) - C(X)$. Thus, for some point R of $C(X)$ there is a point S

giving $\overline{QSR}$. As every point A of $C(X) - S - R$ is conjugate to R, then $\overline{ASR}$ is false, i. e., either SAR or SRA is true. But $\overline{QSR}$, together with either SAR or SRA, implies $\overline{QSA}$. Thus, for any point U of $C(X) - S$ the relation $\overline{QSU}$ is valid. If B is any point of $C(Y) - S - Q$, then a similar argument gives either SQB or SBQ. But $\overline{USQ}$, together with either of these, implies $\overline{USB}$. Thus, for any point U of $C(X) - S$ and any point V of $C(Y) - S$ the relation $\overline{USV}$ is true. Hence, $C(X)$ and $C(Y)$ have either one point in common or no point in common according to whether or not S lies in both $C(X)$ and $C(Y)$.

2.41 *If X is not a cut point of the space it lies in one and only one cyclic element.*

2.42 *If X and Y are distinct points of $C(AB) - AB$ and X is conjugate to Y, then $D(X) + X = D(Y) + Y$.*

2.43 *If X and Y are distinct points of $C(AB) - AB$ and $D(X) + X$ is not identical with $D(Y) + Y$, then there is a point P of AB giving $\overline{XPY}$.*

2.44 *If X and Y are points of $C(AB) - AB$ and $D(X) + X \neq D(Y) + Y$, then these sets have at most one point P in common.* Further *P must belong to AB and give the relation $\overline{XPY}$.*

Definition. A segment UV is said to be the *minimum segment containing a set N* provided N lies in UV but in no segment XY which is a proper subset of UV.

Definition. The space M is said to have *Property C* provided that given any three distinct points A, X and Y for which $AX \cdot AY - A \neq 0$ there is a point P lying in $AX \cdot AY - A$ such that AP is the minimum segment containing $AX \cdot AY$.

Although nothing has been said about the limit points of the space and the relation between the ordering and the limit points, it can be shown (Cf. Part II) that Property C is not only a consequence of the assumption that every segment is closed and compact, but it is a definitely weaker condition. The next five theorems will be deduced under the assumption that the set M has Property C in addition to satisfying Axioms 0-4.

2.5 *If the space M has Property C, then any cyclic chain $C(AB)$ is identical with the set consisting of AB together with all cyclic elements having exactly two points in common with AB.*

Proof. It will be shown that for any point X of $C(AB) - AB$ the set $D(X)$ has exactly two points in common with AB. The remainder of the theorem will then follow from 2. 1, 2. 2 and 2. 21. Let $N(A)$ be the set $AX \cdot AB$. Suppose first that $N(A) = A$. If X were not conjugate to A there would be a point P giving $\overline{APX}$. But, by 1. 51, P must then be a point of $AB - A - B$. This is impossible as $N(A) = A$. Hence, A is conjugate to X, i. e., $A \varepsilon D(X)$. On the other hand, if $N(A) - A \neq 0$, then by Property C there is a point A' of $N(A) - A$ such that AA' is the minimum segment containing $N(A)$. As $A' \varepsilon AB$ and $X \varepsilon C(AB) - AB$, the points A' and X are distinct. Any point Y giving $\overline{A'YX}$ will, by 1. 51, lie in both AX and AB, i. e., $Y \varepsilon N(A)$. But $\overline{AA'X}$ and $A'YX$ imply $\overline{AA'Y}$, contrary to the fact that Y, being a point of $N(A)$, must lie in AA'. Hence, as such a point Y can not exist, the points A' and X are conjugate, i. e., $A' \varepsilon D(X)$. Since $\overline{AA'X}$ is true and $X \varepsilon C(AB)$, $A' \neq B$. By symmetry either $B \varepsilon D(X)$ or there is a point B' distinct from A and B which lies in both $D(X)$ and AB. If either A' or B' fails to exist then it follows at once that the set $D(X)$ has two and only two points in common with AB. If A' and B' both exist, then $A' \neq B'$, for $\overline{AA'X}$ and $\overline{XB'B}$ are both true and $X \varepsilon C(AB)$.

2. 6 *If M has Property C and A, B and P are any three points for which APB is true but $P \cdot C(AB) = 0$, then there is a point Y of $C(AB)$ such that $C(AP)$ lies in $C(AB) + C(YP)$ while for any point X of $C(AB) - Y$ the relation $\overline{XYP}$ is true.*

Proof. As APB is true but P does not belong to $C(AB)$, there is a point Z giving $\overline{AZP}$ and $\overline{PZB}$. Hence, as $PA \cdot PB - P \neq 0$, Property C may be applied. Let Y be that point of $PA \cdot PB - P$ such that PY is the minimum segment containing $PA \cdot PB$. Either $\overline{ABY}$ or $\overline{BAY}$, together with BYP and AYP, contradicts the condition APB. Hence, as both $\overline{ABY}$ and $\overline{BAY}$ are false, AYB must hold. If Y did not belong to $C(AB)$, then there would have to be a point W giving $\overline{AWY}$ and $\overline{YWB}$, for AYB is true. But, by 1. 41, W lies in both AP and PB, i. e., $W \varepsilon AP \cdot PB - P$. However, $\overline{PYA}$ and YWA imply $\overline{PYW}$, contrary to the fact that PY must contain W. Hence, Y is a point of $C(AB)$. By 1. 2, $C(AP) = C(PB)$ lies in $C(AY) + C(YP)$ and $C(AY)$ lies in $C(AB)$. Also, by the same theorem, $C(AB)$ lies in $C(AY) + C(YB)$. Thus, for any point X of $C(AB) - Y$, the relation $\overline{PYX}$ follows from $\overline{PYA}$ and $\overline{PYB}$.

2. 61 *Under the same hypothesis as 2. 6, the chain $C(YP)$ has exactly the point Y in common with $C(AB)$.*

2.62 *If M has Property C and A, B and P are any three distinct points such that P does not lie in $C(AB)$, then there is a point Y of $C(AB)$ for which $C(AP)$ lies in $C(AB) + C(YP)$. Further, if $X \varepsilon C(AB) - Y$, then $\overline{PYX}$ is true. Finally, $C(YP)$ has exactly the point Y in common with $C(AB)$.*

2.7 *If (M_i) is any countably infinite collection of points M_i and M is a space having Property C, then there is a collection $(C(X_iP_i))$ of chains $C(X_iP_i)$ having the following properties:*

1° $\sum_1^\infty C(X_iP_i)$ contains $\sum_1^\infty M_j$;

2° *For $k > 1$, $C(X_kP_k) \cdot \sum_1^{k-1} C(X_iP_i) = X_k$; and*

3° *If $r > s$, X any point of $C(X_rP_r) - X_r$ and Y any point of $C(X_sP_s) - X_r$, then $\overline{XX_rY}$ is true.*

Proof. Let $X_1 = M_1$ and $P_1 = M_2$. Let P_2 be the first M_i not in $C(X_1P_1)$. Let P_3 be the first M_i not in $C(X_1P_1) + C(X_1P_2)$. Continue in this manner. In general, having chosen $P_1, \cdots, P_{n-1}$, let P_n be the first M_i which is not a point of the set $C(X_1P_1) + \cdots + C(X_1P_{n-1})$. Clearly $\sum_i C(X_1P_i)$ contains $\sum_i M_i$. Since 2.6 makes it possible to assume the existence of a point X_2 such that $C(X_1P_1)$ and $C(X_2P_2)$ have properties 2° and 3°, suppose that for $n-1 > 1$ the points $X_2, \cdots, X_{n-1}$ have been so chosen that $C(X_1P_1), \cdots, C(X_{n-1}P_{n-1})$ satisfy conditions 2° and 3°. Pick a point X^2 in $C(X_1P_1)$ such that $\overline{P_nX^2X}$ is true for every point X of $C(X_1P_1) - X^2$. Let k_2 be the first $i \leqq n-1$ such that $\overline{P_nX^2Y_{k_2}}$ is false for some point Y_{k_2} of $C(X_{k_2}P_{k_2}) - X^2$. Now $X^2 = X_{k_2}$ as follows. Suppose, on the other hand, that X^2 and X_{k_2} are distinct. Since X^2 lies in a $C(X_iP_i)$ for which $i < k_2$, then by the construction of $C(X_{k_2}P_{k_2})$ the relation $\overline{P_{k_2}X_{k_2}X^2}$ holds. But X_{k_2} lies in some $C(X_jP_j)$ where $j < k_2$, so from the definition of k_2 the relation $\overline{P_nX^2X_{k_2}}$ must be valid. Applying Axiom 4, $\overline{P_{k_2}X_{k_2}X^2}$ and $X_{k_2}X^2P_n$ give $\overline{P_{k_2}X_{k_2}P_m}$, while $\overline{P_nX^2X_{k_2}}$ and $X^2X_{k_2}P_{k_2}$ imply $\overline{P_nX^2P_{k_2}}$. Thus, Y_{k_2} is distinct from P_{k_2} and X_{k_2}. That is, $P_{k_2}Y_{k_2}X_{k_2}$ must be true. However, $\overline{X^2X_{k_2}P_{k_2}}$ and $X_{k_2}Y_{k_2}P_{k_2}$ imply $\overline{X^2X_{k_2}Y_{k_2}}$, while $\overline{P_nX^2X_{k_2}}$ and $X^2X_{k_2}Y_{k_2}$ give $\overline{P_nX^2Y_{k_2}}$, contrary to hypothesis. Since X^2 and X_{k_2} are the same point, then 2.6 may be applied to find a point X^3 of $C(X_{k_2}P_{k_2}) - X_{k_2}$ such that $\overline{P_nX^3X}$ is true for each point X of $C(X_{k_2}P_{k_2}) - X^3$. From the choice of X_{k_2} the relation $\overline{X^3X_{k_2}Y}$ holds for each point Y of $\sum_1^{k_2-1} C(X_iP_i) - X_{k_2}$. Thus, for each point U of $\sum_1^{k_2} C(X_jP_j) - X^3$, $\overline{P_nX^3U}$ follows from $\overline{P_nX^3X_{k_2}}$ and Axiom 4.

If k_3 is the first $k < n$ such that some point Y_{k_3} of $C(X_{k_3}P_{k_3}) - X^3$ does not stand in the relation $\overline{P_nX^3Y_{k_3}}$, then $k_3 > k_2$. Just as before, $X^3 = X_{k_3}$. Continuing in this manner there will eventually be a point X^m such that $\overline{P_nX^mX}$ is true for every X of $\sum_1^{n-1} C(X_iP_i) - X^m$, where $X^m \varepsilon \sum_1^{n-1} C(X_iP_i)$. Let $X_n = X^m$. It follows from 1. 1-1. 2 that $\sum_1^n C(X_1P_i) = \sum_1^n C(X_jP_j)$.

2. 8 *Axioms* 0-2, 3′, 4 *imply Property C.**

Proof. Axiom 3′ implies $\overline{UVW}$ whenever UVW is true, so that Property C is trivial.

2. 9 *Axioms* 0-4 *imply that every segment AB possesses a linear order.*

Proof. Let the symbol $X \propto Y$ mean that "X precedes Y." For distinct points X and Y of AB let $X \propto Y$ when:

1° $X = A$ or $B = Y$; and 2° When $\overline{AXY}$ is true.

Thus, by 1. 6, if X and Y are distinct points of AB, then $X \propto Y$ or $Y \propto X$ but not both. If $X \propto Y$ and $Y \infty Z$, then $X \propto Z$ as follows. If $X = A$ or $Z = B$ the theorem is true immediately, so assume X and Z to be points of $AB - A - B$. As $X \propto Y$ and $Y \propto Z$, then Y also lies in $AZ - A - Z$. Now $\overline{ZYA}$ and YXA imply $\overline{ZYX}$, while $\overline{AXY}$ and XYZ give $\overline{AXZ}$, i. e., $X \propto Z$.

3. The notion of end point will now be introduced.

Definition. A point P is said to be an *end point* of the set M provided there exists a collection G of points of $M - P$ having the following properties:

1° $\overline{ZZ'P}$ or $\overline{Z'ZP}$ is true for any two distinct points Z and Z' of G; and

2° If Q is any point of $M - P$, then some point Z of G gives $\overline{QZP}$.

Definition. A cyclic element is said to be *non-degenerate* provided it consists of more than one point.

3. 1 *If M has Property C, then M consists of the non-degenerate cyclic elements, the cut points and the end points.*

Proof. Suppose X is neither a cut point nor contained in any cyclic element consisting of more than one point. Let Z be any point of $M - X$. It will be shown that the set $G = ZX - X$ has properties 1° and 2° listed above. Condition 1° is a consequence of 1. 6. Let Q be any point of $M - X$.

Case 1. $Q \varepsilon C(ZX)$.

As Q is not conjugate to X, for the cyclic element $C(X) = X$, there is a point P giving $\overline{QPX}$. By 1.1 and 1.21, the point P lies in $ZX - X$ and is therefore a point Z' of G.

Case 2. $Q \varepsilon M - C(ZX)$.

Since $\overline{ZXQ}$ is impossible as X is not a cut point of the space, then either ZQX or QZX is true. As $\overline{QZX}$ gives the desired result at once, assume $\overline{QZX}$ to be false. As $Q \varepsilon M - C(ZX)$, then by 1.1, QZX, QXZ and XQZ are all three true. Thus, from 2.6, there is a point Y lying in $C(ZX) - Z - X$ which gives the relation $\overline{QYZ'}$ for every point Z' of $ZX - Y$. From Case 1 it follows that some point Z' of $ZX - X$ stands in the relation $\overline{YZ'X}$. But $\overline{XZ'Y}$ and $Z'YQ$ imply $\overline{XZ'Q}$, so that $G = ZX - X$ is a set satisfying conditions 1^0 and 2^0 above. Therefore, X is an end point of the space.

3.2 *If M is a connected and locally connected space in which betweenness has been defined in terms of separating point as was done on page* 234, *then Axioms* 0-4 *and Property C are a consequence of the definition.*

Proof. It was seen earlier that $\overline{UVW}$ occurs when and only when V separates the points U and W in the set M. Thus, any segment AB consists of $A + B +$ all points X which separate A and B in M. G. T. Whyburn has shown * that AB is closed and compact in any connected and locally connected set. Suppose, then, that A, B and P are any three distinct points for which APB, ABP and BAP are all true. Assume further that $AP \cdot AB - A \neq 0$. Let $N = AP \cdot AB$ and order AB in the manner of 2.9. As AB and AP are closed and compact sets, then N has a last point K which lies in N. From the hypothesis K is distinct from A, B and P. If X is any point giving $\overline{AXP}$ and $\overline{AXB}$ then X lies in N. Since K is the last point of N in AB, either $X = K$ or $\overline{AXK}$ is true. Thus, AK is the minimum segment containing N.

3.3 *If M is any connected and locally connected space in which betweenness has been defined in terms of separating point as above, then the cyclic chains, non-degenerate cyclic elements, cut points and end points defined by means of the betweenness relations are identical with those defined in the Structure paper.*

Proof. In the Structure paper a non-degenerate cyclic element was characterized as being a maximal set $C(X)$ such that no two points U and V of $C(X)$ are separated in M by any point. Since $\overline{UXV}$ means X separates U

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* "On the structure of connected and connected im kleinen point sets," *Transactions of the American Mathematical Society*, vol. 32 (1929), no. 4, p. 927.

and V, then the non-degenerate cyclic elements are the same. The cyclic chains of the *Structure* paper are shown to be identical with the set AB together with all cyclic elements which have exactly two points in common with AB. Hence, by 2. 5, the cyclic chains defined by means of the betweenness relations are identical with those of G. T. Whyburn. The definition of end point in the *Structure* paper is the following: "The point P of a continuum M is an end point of M provided that if N is any subcontinuum of M containing P, then P is not a limit point of any connected subset of $M - N$." An equivalent definition for locally connected continua is the following:* P is an end point when and only when there exists a countable collection of open sets O_i such that P is the only point common to every O_i, the boundary of each O_i is exactly one point Z_i, and O_i contains $O_{i+1} + Z_{i+1}$ for every i. Since Z_{i+1} separates Z_i and P for every i, then the definition of end point in terms of betweenness gives the same points as those defined by Kuratowski and Whyburn.

PART II.

In this part it will be assumed that the set M is a separable metric space. The number $\rho(X, Y)$, which has the usual metric properties, will be used to denote the distance from X to Y. By the *diameter* $\delta(K)$ of a set K is meant the least upper bound of the numbers $\rho(X, Y)$ for all pairs of points X and Y of K. A set of betweenness axioms will now be formulated which will make M homeomorphic with a subset of an acyclic Peano space. In addition to Axioms 0-4 it will be supposed that M has been so ordered that the following four axioms are also true:

AXIOM 5. *If A is a limit point of a set N there is no point P such that $\overline{APX}$ is true for every point X of N different from A.*

AXIOM 6. *M has Property C.*†

AXIOM 7. *If $(C(X_iY_i))$ is a collection of cyclic chains $C(X_iY_i)$ and P a point such that:*

1° *For each $k > 1$ the set $C(X_kY_k) \cdot \sum_{1}^{k-1} C(X_iY_i)$ is X_k or zero, and*

2° *For every number $\varepsilon > 0$ all but a finite number of the sets $C(X_iY_i)$ have a point as close to P as ε, then the numbers $\delta(C(X_iY_i))$ converge to zero with $1/i$.*

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* Cf. C. Kuratowski and G. T. Whyburn, "Sur les elements cyclique et leurs applications," *Fundamenta Mathematicae*, vol. 16 (1930), p. 307.

† Cf. p. 240.

AXIOM 8. *If U and V are any two points of a cyclic element $C(X)$, then it is possible to send $C(X)$ into a subset of the unit interval by a homeomorphism T for which $T(U)$ and $T(V)$ are the ends of the interval.*

The connection between the betweenness relations and the limit points is given by Axiom 5. However, it is to be noted that Axiom 5 is only a necessary condition for a point to be a limit point of a set.

Axiom 7 is used to prevent oscillation. For example, if the space M consists of the graph of $y = \sin 1/x$ for $1 \geqq x > 0$ together with the origin, then M is connected. If betweenness is defined in M in terms of cut point as was done earlier, then Axioms 0-6 and 8 are true. However, such a set is not homeomorphic with the unit interval. It is evident that this example is not locally connected. Also, in the previously mentioned paper by C. Kuratowski and G. T. Whyburn,* it is shown for compact, connected and locally connected metric spaces that any collection of cyclic chains satisfying condition 1° of Axiom 7 is such that the diameters converge to zero. Thus, Axiom 7 stands as a substitute for the stronger property of local connectivity.

Up to the present time no assumption has been made concerning the cyclic elements. Many spaces, for example, any Peano space, may be so ordered that Axioms 0-7 are valid while Axiom 8 might or might not be true. In general, the definition of betweenness in terms of cut point for connected sets will not lead to Axiom 8 unless the cyclic elements are all degenerate. However, to restrict M to a space which is homeomorphic with a subset of an acyclic Peano space it is necessary to restrict the cyclic elements. Axiom 8, which at first sight seems to assume part of the desired result, is really the weakest condition that can be imposed.

4.0 *Axioms 0-5 imply that the cyclic chains and cyclic elements are closed sets.*

Proof. The cyclic elements will be treated first. Let P be a point which does not lie in the cyclic element $C(X)$. Thus, there is a point R of $C(X)$ and some point S of M such that $\overline{PSR}$ is true. If Y is any other point of $C(X)$ either SYR or SRY must be true, for $\overline{RSY}$ contradicts the condition that R and Y be conjugate. However, $\overline{PSR}$, together with either SRY or SYR, implies $\overline{PSY}$. Thus, for any point Y of $C(X) - S$, the relation $\overline{PSY}$ holds. Hence, by Axiom 5, P is not a limit point of $C(X)$. Therefore, each cyclic element is a closed set.

Suppose next that P is not a point of the cyclic chain $C(AB)$. Thus, by definition, either $\overline{PAB}$ or $\overline{PBA}$ is true or there exists a point Q giving

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* *Loc. cit.*

$\overline{AQP}$ and $\overline{PQB}$. Since $C(AB)$ would lie in $C(AQ) + C(QB)$ if Q existed, then in any of the possibilities there would always, by Axiom 4, be a point Z giving $\overline{PZX}$ for every X of $C(AB) - Z$. Thus, by Axiom 5, $C(AB)$ is closed.

That each segment AB is closed does not follow from Axioms 0-5. However, it is a consequence of Axioms 0-6, as follows. If P is a limit point of AB, then by Axioms 0-5 both $\overline{ABP}$ and $\overline{BAP}$ are false. Hence, APB must be true. By 1. 1 and 1. 3 the set AB lies in $AP + PB$, where it is assumed that P is distinct from A and B. As P must then be a limit point of either AP or PB, assume it to be a limit point of AP. Thus, $AP - P$ is not zero. Now $AP - P$ lies in AB as follows. If $X \varepsilon AP - A - P$ the relation $\overline{AXP}$ is true by definition. But, by Axiom 5, for some point Y of AB the relation $\overline{YXP}$ is false, where Y can be chosen distinct from A, B, P and X. But $\overline{AXP}$ and either XPY or XYP give $\overline{AXY}$, so that by 1. 1 and 1. 5 $X \varepsilon AB$. Since $AP - P$ lies in AB and is different from zero, then Axiom 6 may be applied to find a point Z of $AB \cdot AP$ such that AZ contains $AP - P$. If Z and P were not the same point then $\overline{XZP}$ would be true for every point X of $AP - X$, contrary to Axiom 5. Thus, $Z = P$ lies in AB. The following theorem can now be stated.

4. 1 *If M satisfies Axioms* 0-6, *then each segment is a closed set.*

4. 11 *If M satisfies Axioms* 0-5, *then any point P of a segment AB which is a limit point of the segment UV, where U, V* ε *AB lies in UV.*

4. 2 *If M satisfies Axioms* 0-5, 7, *then for each AB there is a homeomorphism T which sends AB into a subset of the unit interval. Further, A and B are transformed into the ends of the interval and T preserves the betweenness relations.*

Proof. For convenience in notation the segment will be ordered in the manner given in 2. 9. The proof will then be stated in terms of the symbol "α". The demonstration follows along lines for similar proofs, the only difference occurring in the following situation. A point P may not be a limit point of both the set of points before P and the set of points after it. In this case the transformation must be so defined that open spaces are left in the unit interval to the right and left of the transform of P. For a separable metric space which has been linearly ordered in such a manner that the order preserves the limit points there are at most a countable number of points which are not limit points of both the points preceding and following them.* Hence, in AB there is a set $G = \sum_i P_i$ which is everywhere dense in AB and which contains every point P of AB which is not a limit point of both AP and PB. Suppose further that $P_1 = A$ and $P_2 = B$. If P_1 is not a limit point of P_1P_2 drop from the interval $(0, 1)$ the open interval $(0, 1/8)$. If P_2 is not a limit point of P_1P_2 drop the open interval $(1 - 1/8, 1)$. Call the resulting set K_1 and define $T(P_1)$ as $p_1^1 = 0$ and $T(P_2)$ as $p_2^1 = 1$. Between p_1^1 and p_2^1 there is one and only one interval belonging to K_1. Let P_3, if it exists, correspond to the midpoint p'^1_1 of this interval. If P_3 is not a limit point of P_1P_3 drop from K_1 the open interval $(p'^1_1 - 1/3^3, p'^1_1)$. If P_3 is not a limit point of P_3P_1 drop from K_1 the open interval $(p'^1_1, p'^1_1 + 1/3^3)$. Let the resulting set be K_2 and relable the points p_1^1, p_2^1, p'^1_1 from left to right as p_1^2, p_2^2, p_3^2. Continue in this manner. In general, having chosen $p_1^n, \cdots, p_{v_n}^n$ and the set K_n, consider the pairs p_i^n and p^n_{i+1}, where $i = 1, 2, \cdots, v_n - 1$. Suppose, for each k, that P_k^n is the point of G which corresponds to p_k^n. Let P_{m_i}, provided it exists, be the first P_m of G which is such that $P_i^n \propto P_{m_i}$ and $P_{m_i} \propto P^n_{i+1}$. Between p_i^n and p^n_{i+1} there is one and only one interval of K_n. If p'^n_i designates the midpoint of this interval, then let $T(P_{m_i}) = p'^n_i$. If P_{m_i} is not a limit point of $P_1P_{m_i}$ drop from K_n the open interval $(p'^n_i - 1/(1+n)^3, p'^n_i)$; and if P_{m_i} is not a limit point of $P_{m_i}P_2$ drop from K_n the open interval $(p'^n_i, p'^n_i + 1/(1+n)^3)$. Do this for each i and let the remaining set of points of K_n be K_{n+1}. Label the points $\sum_i p_i^n$ together with the set $\sum_i p'^n_i$ from left to right as $\sum_{i=1}^{i=v_{n+1}} p_i^{n+1}$.

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* Zarankiewicz, *Fundamenta Mathematicae*, vol. 12 (1928), p. 119.

The proof that T is a homeomorphism which preserves the order and the proof that T can be extended to the whole set AB are rather tedious. Since they follow the lines of similar demonstrations it has been decided for brevity to omit them.

4. 3 *Axioms* 0-6 *imply that there are at most a countable number of cyclic elements in any cyclic chain.*

Proof. Let the cyclic chain be $C(AB)$. Since any two cyclic elements $C(X)$ and $C(X')$ of $C(AB)$ have at most one point in common if they are distinct, then the supposition that there are an uncountable number of cyclic elements in $C(AB)$ implies the existence of an uncountable set G which lies in $C(AB) - AB$ and is such that distinct points X and X' of G lie in cyclic elements $C(X)$ and $C(X')$ which are distinct. By 2. 5 each of these cyclic elements has exactly two points U and V in common with AB. Suppose U and V are so named that $U \propto V$. It is well known that any infinite subset of a linearly ordered array contains an infinite set which is monotonic with respect to this order. Since no three cyclic elements can have a point in common, then for any infinite subset H of G there is an infinite subset H_0 of H having the following property:

If X and X' are any distinct points of H_0 and U and U' the corresponding points then either U always precedes U' or U' always precedes U.

In particular, since M is separable, some point X^* of G must be a limit point of a subset H of G. Without loss it may be assumed that it is possible to choose the set H_0 such that $H_0 = \sum_1^\infty X_i$ and the corresponding set $\sum_1^\infty U_i$ has the property that $U_i \propto U_{i+1}$. As $U_i \propto V_i$, then $U_i V_i \cdot U_{i+t+1} V_{i+t+1}$ is at most U_{i+t+1} if $t \geqq 0$. As $C(X_k)$ lies in $C(U_k V_k)$, then it follows from the definition of the symbol "$\propto$" that the collection of chains $C(U_i V_i)$ not only contains the set H_0 but satisfies the conditions of Axiom 7. Hence, their diameters converge to zero. Thus, X^* is a limit point of AB. By 4.0, X^* must then be a point of AB, contrary to the fact that X^* lies in $C(AB) - AB$. Hence, the collection of cyclic elements belonging to $C(AB)$ is countable. This same theorem may be proved by means of Axioms 0-5, 7 and the assumption that the segments AB are closed. However, the demonstration is much longer than the above. Using this theorem together with Axiom 8 it is possible to state the following:

4.4 *Axioms* 0-8 *imply that any $C(AB)$ may be sent into a subset of the unit interval by a homeomorphism S which sends A and B into the end points, preserves the order of AB and is such that the relation $S(X)S(Y)S(Z)$ on the interval implies XYZ.*

Proof. Let X be any point of $C(AB) - AB$ and let the elements $C(X)$ have the points U and V in common with AB. As $UV = U + V$, then the transformation T of 4.2 was such that no point W of AB was transformed by T into a point $T(W)$ lying between $T(U)$ and $T(V)$. By 4.3 the cyclic elements C of $C(AB)$ may be labeled $C_1, C_2, \cdots$. Let U_i and V_i denote the points which C_i has in common with AB. By Axiom 8 there is a homeomorphism T_i sending C_i into a subset of $(T(U_i), T(V_i))$ in such a manner that $T(U_i) = T_i(U_i)$ and $T(V_i) = T_i(V_i)$. Thus, the transformation S defined as follows:

$$S(P) = T(P) \text{ if } P \varepsilon AB \text{ and } S(P) = T_i(P) \text{ if } P \varepsilon C_i,$$

gives a $1-1$ correspondence between $C(AB)$ and a subset of the unit interval. That S is a homeomorphism with the desired properties follows from 4.0 and Axiom 7. Again, this theorem may be demonstrated by means of Axioms 0-5, 7 provided it is assumed that AB is closed.

The following theorem on the approximation of the space by means of cyclic chains is a direct consequence of 2.7 and the condition that M be separable.

4.5 *Axioms 0-7 imply the existence of a collection of cyclic chains* $R_i = C(X_iP_i)$ *for which*

1° $X_i \varepsilon X_1P_i$; 2° *Every point of* M *is a point or limit point of the set* $\sum_i R_i$; *and* 3° $R_k \cdot \sum_1^{k-1} R_i = X_k$.

This theorem is analogous to the one announced in the previously mentioned paper by C. Kuratowski and G. T. Whyburn.* Such an approximation of the space will be used in constructing a homeomorphism between M and a subset of an acyclic Peano space, the method being to send the sets R_i into subsets of appropriate arcs in the acyclic Peano space.

5. A *universal* acyclic Peano space is such that every acyclic Peano space is homeomorphic with a subset of it. If (t_i) is an infinite collection of arcs t_i of an acyclic space, where $t_i \cdot t_j$ is a fixed point x, $i \neq j$, then x is called an *infinite branch point* of the space. T. Ważewski has shown † that an acyclic Peano space will be a universal acyclic space P provided that on each arc of P there is an everywhere dense subset of points which are infinite branch points of P. In fact, such an acyclic space has been shown to exist in the plane. Such a universal space will be employed in the demonstration of the following theorem.

5.1 *If* M *is a separable metric space the points of which have been so ordered that Axioms 0-8 are valid, then* M *is homeomorphic with a subset of an acyclic Peano space.*

Proof. Choose in M a collection of cyclic chains R_i having the properties announced in 4.5. Let P_0 be any universal acyclic Peano space and x_1 and p_1 any two infinite branch points of P_0. Let r_1 be the arc from p_1 to x_1 and F_1 the set of infinite branch points which lie on r_1. Let T_1 be the following homeomorphism:

1° T_1 sends the set $R_1 = C(X_1P_1)$ into r_1 in such a manner that $T_1(X_1) = x_1$ and $T_1(P_1) = p_1$; and

2° T_1 transforms $R_1 \cdot R_i$, where $i \neq 1$, into a point of F_1. Theorem 4.4

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* *Loc. cit.*

† *Annales Société Polonaise Mathematicae Cracovie*, vol. 2 (1923), p. 49; also, see Menger, *Fundamenta Mathematicae*, vol. 10, p. 108 and Gehman, *Transactions of the American Mathematical Society*, vol. 29 (1927), no. 3, p. 553.

gives a transformation T_{01} satisfying condition 1°, while a theorem due to Fréchet and Urysohn * states that T_{01} may be deformed into a transformation T_1 satisfying both 1° and 2°. Suppose that $T_1(X_2) = x_2$. Since x_2 is an infinite branch point it is possible to find an arc $x_2p'_2$ having just x_2 in common with r_1. Choose on this arc a point p_2 different from x_2 which is an infinite branch point and let r_2 be the arc from x_2 to p_2. If the set of infinite branch points lying on r_2 is F_2, then let T_2 be a homeomorphism sending $R_2 = C(X_2P_2)$ into a subset of r_2 in such a manner that $T_2(X_2) = x_2$, $T_2(P_2) = p_2$ and $T_2(R_2 \cdot R_i)$ is a point of F_2 for each $i \neq 2$. Continue in this manner. In general, having chosen $T_1, \cdots, T_{n-1}$ and the arcs $r_1, \cdots, r_{n-1}$ in such a manner that $r_k \cdot \sum_1^{k-1} r_i = x_k$, let R_n be the cyclic chain $C(X_nP_n)$. Now X_n may lie in more than one of the sets R_i, where $n > i$. However, if it is assumed that the collection (R_i) was constructed as was done in the proof of 2.7 (Part I), it follows that if X_n lies in R_i and R_j, $n > i > j$, the points X_n, X_i and X_j are identical. Hence, for every i for which $T_i(X_n)$ has a meaning the point $T_i(X_n) = x_n$ is the same. Thus, as X_n lies in $\sum_1^{n-1} R_i$ there is some point x_n to which X_n corresponds, and only one such point. Let p'_n be a point for which the arc $x_np'_n$ has just the point x_n in common with $\sum_1^{n-1} r_i$. On this arc choose an infinite branch point p_n different from x_n and let r_n be the arc from x_n to p_n. If F_n is the set of infinite branch points lying on r_n, then let T_n be a homeomorphism which sends R_n into a subset of r_n in such a manner that $T_n(X_n) = x_n$, $T_n(P_n) = p_n$, and $T_n(R_n \cdot R_i)$ is a point of F_n, where $i \neq n$.

For a point X of $\sum_i R_i$ define $T(X)$ as $T_n(X)$ if $X \varepsilon R_n$. It follows at once that T is a $1-1$ correspondence. In order to extend T to the limit points of $\sum_i R_i$ which do not lie in $\sum_i R_i$ use has been made of the following statement:

For each point Z of $M - \sum_i R_i$ there is one and only one collection (R_{t_i}) of sets R_i such that: 1° $t_1 = 1$; 2° $t_i \geqq i$; 3° R_{t_i} and R_{t_j} have no point in common if $|i - j| > 1$; 4° R_{t_i} and $R_{t_{i+1}}$ have exactly the point $X_{t_{i+1}}$ in common; and 5° Z is a sequential limit point of $\sum_i X_{t_i}$. Thus, from the construction of T the corresponding arcs r_{t_i} would have properties 1°-4°. As the space P_0 is compact and locally connected, then there is one and only one point z which is a limit point of $\sum_i r_{t_i}$. Since the space is acyclic the

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* M. Fréchet, *Mathematische Annalen*, vol. 68 (1910), p. 159 and P. Urysohn, *Fundamenta Mathematicae*, vol. 7 (1925), p. 83.

point z does not lie in $\sum_{i} r_{t_i}$. If $T(Z)$ is defined as z, then the extended transformation will also be a $1-1$ correspondence. However, the proof of the above statement about (R_{t_i}) and the proof that the extended transformation is a homeomorphism are both long and detailed. Since they follow along lines for similar proofs it has been decided to omit them in order to conserve space. A demonstration analogous to the one needed can be found in a paper by G. T. Whyburn entitled "On the set of all cut points of a continuous curve." *

6. The converse question on the types of subset of an acyclic Peano space which can be so ordered that Axioms 0-8 are valid will be treated next.

Definition. A subset M of an acyclic Peano space is said to have *Property C'* provided that for any three distinct points a, b and x of M for which the arcs ab and ax contain a point of M different from a, there is a point p of M which lies on $ab \cdot ax$ and is such that ap is the smallest arc containing the set $ab \cdot ax \cdot M$.

In particular, it can be shown that any closed subset M has Property C'. Also, if the points of the acyclic space are ordered in terms of cut point and the points of the subset M given the same order relations as they have in the whole space, then it can be shown that Property C' implies Property C announced earlier.

Suppose then that M is a subset of an acyclic Peano space P in which betweenness has been defined in terms of cut point. If X and Y are any two points of M which are not separated in P by any point of M, then from the definition of betweenness they lie in the same cyclic element. Let $\overline{XY}$ be the arc from X to Y and Z any point which is conjugate to both X and Y. Thus, there is a point W such that the arc $\overline{WZ}$ has just the point W in common with $\overline{XY} - X - Y$. If, for each point Z of the cyclic element $C(X)$ containing X and Y such an arc $\overline{WZ}$ is chosen, then the set $H(XY) = \overline{XY}$ together with the arcs $\overline{WZ}$ will be an acyclic Peano space in which the end points are the points of $C(X)$. The theorem that the end points of such a space are totally disconnected may then be applied. Thus, by the Fréchet-Urysohn theorem mentioned earlier, each $C(X)$ satisfies Axiom 8.

A set $G(AB)$ consisting of the arc $\overline{AB}$ together with arcs $\overline{WX}$ having just the point W in common with $AB - A - B$ and just the point X in common with the cyclic chain $C(AB)$ may be formed for each $C(AB)$. Further, it may be shown that $G(AB) \cdot G(UV)$ lies in $C(AB) \cdot C(UV)$ if $C(AB) \cdot C(UV)$ consists of at most one point. Thus, any collection of cyclic chains satisfying the conditions of Axiom 7 must satisfy the conclusion, for P is compact and locally connected.

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* *Fundamenta Mathematicae*, vol. 14 (1930), p. 185.

It was seen earlier that the definition of betweenness in terms of cut point satisfies Axioms 0-5 if the space P is locally connected. It is now possible to state the following theorem:

6.1. *If P is an acyclic Peano space and M any subset of P which has Property C', then the points of M may be so ordered that Axioms* 0-8 *are valid.*

Up to the present time it has not been shown that Axioms 0-8 make the transform of M have Property C'. However, it would appear that this is so.

Conclusion. It can be shown that the set of all cut points of a Peano space may be so ordered that Axioms 0-8 are true. It is only necessary to order the points of the Peano space in terms of cut point as was done earlier. Thus, the following theorem, due to G. T. Whyburn,* is a consequence of Part II:

The set of cut points of any Peano space is homeomorphic with a subset of an acyclic Peano space.

Also, if Axiom 6 is replaced by the stronger

Axiom 6′. If A, B and C are distinct then $AB \cdot BC \cdot CA \neq 0$,

it will follow that Axiom 8 is trivial. For if ABC is true and $\overline{ABC}$ false, then by Axiom 6′ there is a point Z giving $\overline{AZB}$, $\overline{BZC}$ and $\overline{CZA}$. Thus, any cyclic element consists of at most two points. Also, each cyclic chain $C(AB)$ consists of just the segment AB. Hence, Axioms 0-5, 6′, 7 imply that M is homeomorphic with a subset of a Peano space which is acyclic. In a paper which follows the present one an arbitrary collection N is considered the points of which are so ordered that Axioms 0-4, 6′ are true. In addition to these restrictions is imposed the condition that any collection of segments such that any two of them have at most an end point in common is countable. With these conditions it is shown that a metric may be so defined for N that it is separable. Further, Axioms 5 and 7 are a consequence of the definition. Hence, such a set N may be transformed into a subset of an acyclic Peano space.

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* *Fundamenta Mathematicae*, vol. 14 (1930), p. 185.

NOTE ON THE COOLING OF A RADIOACTIVE SPHERE.

By Arnold N. Lowan.

In a previous paper * the cooling of a radioactive sphere has been investigated in the special case where the initial temperature distribution is a function of r only. The object of the present paper is to apply the method of the previous paper to the case of any arbitrary initial temperature distribution.

The temperature $T(r, \theta, \phi; t)$ must satisfy the following differential equation, initial and boundary conditions:

$$(1) \qquad (\partial/\partial t)T(r, \theta, \phi; t) - k\Delta T(r, \theta, \phi; t) = \phi(r, \theta, \phi; t), \qquad t > 0,$$

$$(2) \qquad \lim_{t\to 0} T(r, \theta, \phi; t) = f(r, \theta, \phi),$$

$$(3) \qquad (\partial/\partial r)T(r, \theta, \phi; t) = 0, \qquad r = 0,$$

$$(4) \qquad (\partial/\partial r)T(r, \theta, \phi; t) + \sigma T(r, \theta, \phi; t) = \sigma F(t), \qquad r = a,$$

where $k =$ ratio between the thermal conductivity K and the product of the density δ and the specific heat γ; $\sigma =$ coefficient of heat transfer by radiation at the surface of the sphere into the medium at temperature $F(t)$ and $\phi(r, \theta, \phi; t) = (1/\gamma\delta) \times$ heat generated per unit time per unit volume. It will be convenient to make the substitution

$$(5) \qquad T(r, \theta, \phi; t) = (1 + \tfrac{1}{2}a^3 - \tfrac{3}{2}ar^2 + r^3)F(t) + u(r, \theta, \phi; t).$$

The function $u(r, \theta, \phi; t)$ must then satisfy the system

$$(6) \qquad \partial u/\partial t - k\Delta u = \phi(r, \theta, \phi; t) - (1 + \tfrac{1}{2}a^3 - \tfrac{3}{2}ar^2 + r^3)(\partial/\partial t)F(t) + 6k(2r - a)F(t) = \psi(r, \theta, \phi; t), \qquad \text{(say)}$$

$$(7) \qquad \lim_{t\to 0} u(r, \theta, \phi; t) = f(r, \theta, \phi) - (1 + \tfrac{1}{2}a^3 - \tfrac{3}{2}ar^2 + r^3)F(0) = g(r, \theta, \phi), \qquad \text{(say)}$$

$$(8) \qquad (\partial/\partial r)u(r, \theta, \phi; t) = 0, \qquad r = 0,$$

$$(9) \qquad (\partial/\partial r)u(r, \theta, \phi; t) + \sigma u(r, \theta, \phi; t) = 0, \qquad r = a.$$

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* "On the cooling of a radioactive sphere," *Physics Review*, November 1, 1933. This paper will be referred to as A. N. L.

The system (6) to (9) is formally analogous to the system (1) to (4), and defines the temperature history in a radioactive sphere (the function ψ playing the rôle of the function ϕ), radiating into a medium at 0°. Let

$$(10) \qquad L\{u(r,\theta,\phi;t)\} = \int_0^\infty e^{-\lambda t}\, u(r,\theta,\phi,t)\,dt = y(r,\theta,\phi;\lambda),$$

$$(11) \qquad L\{\psi(r,\theta,\phi;t)\} = \int_0^\infty e^{-\lambda t}\, \psi(r,\theta,\phi;t)\,dt = Q(r,\theta,\phi;\lambda),$$

where we assume the convergence of the integrals. If we operate on (6), (8) and (9) by the operator L and make use of the identity (see A. N. L.)

$$(12) \qquad \begin{aligned} L\{(\partial/\partial t)u(r,\theta,\phi;t)\} &= \lambda L\{u(r,\theta,\phi;t) - u(r,\theta,\phi;0)\} \\ &= \lambda y(r,\theta,\phi;\lambda) - g(r,\theta,\phi), \end{aligned}$$

the function $y(r,\theta,\phi;\lambda)$ must satisfy the system

$$(13) \qquad k\Delta y - \lambda y = -g(r,\theta,\phi) - Q(r,\theta,\phi;\lambda),$$

$$(14) \qquad \partial y/\partial r = 0, \qquad r = 0,$$

$$(15) \qquad \partial y/\partial r + \sigma y = 0, \qquad r = a.$$

The general solution of (13) to (15) may be written in the form

$$(16) \qquad \begin{aligned} &y(r,\theta,\phi;\lambda) \\ &= \iiint_\Omega G(r,\theta,\phi;r',\theta',\phi';\lambda)\{g(r',\theta',\phi') + Q(r',\theta',\phi';\lambda)\}\,d\tau, \end{aligned}$$

where $d\tau = r'^2 \sin\theta' dr' d\theta' d\phi'$ and the integration is carried out over the volume Ω of the sphere. Further, G is the Green function of the problem defined in the usual manner as the solution of the differential equation

$$(17) \qquad k\Delta G - \lambda G = -\phi_\epsilon(r,\theta,\phi)$$

satisfying the given boundary conditions, the function $\phi_\epsilon(r,\theta,\phi)$ satisfying the condition

$$(18) \qquad \lim_{\rho\to 0} \iiint_\omega \phi_\epsilon(r',\theta',\phi')\,d\tau = 1,$$

where ω is a little sphere of radius ρ with the center at the point (r',θ',ϕ'). From (16) we obtain in a purely formal manner

$$(19) \qquad \begin{aligned} u(r,\theta,\phi;t) = &\iiint_\Omega g(r',\theta',\phi') \cdot L^{-1}\{G(r,\theta,\phi;r',\theta',\phi';\lambda)\}\,d\tau \\ &+ \iiint_\Omega L^{-1}\{G(r,\theta,\phi;r',\theta',\phi';\lambda) \cdot Q(r',\theta',\phi';\lambda)\}\,d\tau, \end{aligned}$$

where L^{-1} is the inverse Laplace operator. Let us put

$$L^{-1}\{G(r,\theta,\phi;r',\theta',\phi';\lambda)\}=\Gamma(r,\theta,\phi;r',\theta',\phi';t). \tag{20}$$

Further, from (11) it follows

$$L^{-1}\{Q(r,\theta,\phi,\lambda)\}=\psi(r,\theta,\phi,t). \tag{11'}$$

It is then known (see A. N. L.)

$$L^{-1}\{GQ\}=\int_0^t \psi(r',\theta',\phi';\eta)\cdot\Gamma(r,\theta,\phi;r',\theta',\phi';t-\eta)\,d\eta. \tag{21}$$

In view of (20) and (21), our solution (19) may formally be written

$$\begin{aligned} u(r,\theta,\phi) &= \iiint_{\Omega} g(r',\theta',\phi')\cdot\Gamma(r,\theta,\phi;r',\theta',\phi';t)\,d\tau \\ &\quad+\iiint_{\Omega} d\tau\int_0^t \psi(r',\theta',\phi';\eta)\cdot\Gamma(r,\theta,\phi;r',\theta',\phi';t-\eta)\,d\eta. \end{aligned} \tag{22}$$

If we assume the validity of the bilinear expansion formula, we have

$$G=\sum\lambda\frac{y_{m,n,i}(r,\theta,\phi)\cdot y_{m,n,i}(r',\theta',\phi')}{k\lambda^2_{n,i}+\lambda}, \tag{23}$$

where the $\lambda_{n,i}$'s are the characteristic values corresponding to the homogeneous differential equation

$$\Delta y+\lambda^2 y=0 \tag{24}$$

in conjunction with the given boundary conditions, and the y's are the corresponding normalized characteristic functions. As is well known, we have *

$$y_{m,n,i}=N(m,n)R(r)P(\mu)\,\begin{matrix}\cos\\ \sin\end{matrix}\,m\phi, \tag{25}$$

where

$$R(r)=r^{-1/2}J_{n+1/2}(\lambda_{n,i}r),\qquad P(\mu)=(1-\mu^2)^{m/2}(\partial^m/\partial\mu^m)P_n(\mu)$$

and, in addition, $\mu=\cos\theta$, n is a positive integer, $m=0,1,2,3\cdots n$, and P_n is a zonal harmonic.

The function (25) is finite for $r=0$, satisfies (8), and will satisfy the

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* H. Carslaw, *The Conduction of Heat* (1921), p. 143.

boundary condition (9), provided the $\lambda_{n,i}$'s are the roots of the transcendental equation

$$(26)\qquad (\sigma - 1/2a) J_{n+1/2}(a\lambda) + \lambda J'_{n+1/2}(a\lambda) = 0,$$

where
$$J'_{n+1/2}(a\lambda) = \{(\partial/\partial x) J_{n+1/2}(x)\}_{x=a\lambda}.$$

Furthermore, $N(m, n)$ is the normalizing factor defined by *

$$(27)\qquad \frac{1}{N(m,n)} = \frac{\pi a^2 \lambda^{-1/2}(m+n)!}{(2n+1)(n-m)!} \cdot \{J'_{n+1/2}(a\lambda)\}^2$$

$$(28)\qquad \frac{1}{N(0,n)} = \frac{2\pi a^2 \lambda^{-1/2}}{2n+1} \cdot \{J'_{n+1/2}(a\lambda)\}^2.$$

With this significance of the symbols $N(m, n)$, $R(r)$ and $P(\mu)$, (23) becomes

$$(23')\qquad \begin{aligned} G = &\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\{N(m,n)\}^2 R(r)P(\mu)\cos m\phi \cdot R(r')P(\mu')\cos m\phi'}{k\lambda^2_{n,i} + \lambda} \\ &+ \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\{N(m,n)\}^2 R(r)P(\mu)\sin m\phi \cdot R(r')P(\mu')\sin m\phi'}{k\lambda^2_{n,i} + \lambda}. \end{aligned}$$

Since we evidently have

$$(29)\qquad \int_0^{\infty} e^{-k\lambda^2_{n,i}t} \cdot e^{-\lambda t}\, dt = \frac{1}{k\lambda^2_{n,i} + \lambda},$$

it follows at once that

$$(30)\qquad \begin{aligned} &\Gamma(r, \theta, \phi; r', \theta', \phi'; t) \\ &= \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \{N(m,n)\}^2 e^{-k\lambda^2_{n,i}t} R(r)P(\mu)\cos m\phi \cdot R(r')P(\mu')\cos m\phi' \\ &\quad + \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \{N(m,n)\}^2 e^{-k\lambda^2_{n,i}t} R(r)P(\mu)\sin m\phi \cdot R(r')P(\mu')\sin m\phi'. \end{aligned}$$

In view of (30) our solution (22) finally becomes

$$\begin{aligned} u(r, \theta, \phi, t) = &\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \{N(m,n)\}^2 e^{-k\lambda^2_{n,i}t} R(r)P(\mu)\cos m\phi \\ &\cdot \iiint_{\Omega} g(r', \theta', \phi') R(r')P(\mu')\cos m\phi'\, d\tau \\ &+ \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \{N(m,n)\}^2 e^{-k\lambda^2_{n,i}t} R(r)P(\mu)\sin m\phi \\ &\cdot \iiint_{\Omega} g(r', \theta', \phi') R(r')P(\mu')\sin m\phi'\, d\tau \end{aligned}$$

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* H. Carslaw, *The Conduction of Heat* (1921), p. 143.

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\begin{aligned}
(31) \qquad & + \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{N(m,n)\}^2 R(r) P(\mu) \cos m\phi \\
& \cdot \int\int_{\Omega}\int d\tau \int_0^t R(r') P(\mu') \cos m\phi \cdot \psi(r', \theta', \phi'; \eta) e^{-k\lambda^2_{n,i}(t-\eta)} d\eta \\
& + \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{N(m,n)\}^2 R(r) P(\mu) \sin m\phi \\
& \cdot \int\int_{\Omega}\int d\tau \int_0^t R(r') P(\mu') \sin m\phi' \cdot \psi(r', \theta', \phi'; \eta) e^{-k\lambda^2_{n,i}(t-\eta)} d\eta.
\end{aligned}
$$

The solution of our problem is thus given by (5), where $u(r, \theta, \phi, t)$ is given by (31).

If, as is obvious from physical considerations, we may assume the boundedness of the functions $f(r, \theta, \phi)$, $F(t)$ and $\phi(r, \theta, \phi, t)$, it may be easily shown that the triple sums in (31) are convergent, and that the derivatives appearing in our differential equations may be obtained by termwise differentiation of the triple sums.

It can then further be shown that the function $T(r, \theta, \phi, t)$, defined by (5) and (31), satisfies the conditions (1) to (4), and thus actually represents the complete solution of our problem.

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OPTICAL PATHS IN THE IONOSPHERE.

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The analysis of optical paths in isotropic media has been developed by several writers to a first approximation,[*] for constant permeability but variable dielectric constant, and Gans has considered also the second approximation in certain cases. An extension of this analysis is required in the calculation of the approximate path of a short wave beam in the ionized region above the earth, under the influence of the earth's magnetic field which makes the medium doubly refracting, and the development of this analysis is the chief object of the present paper. Hartree's form [†] of the matrix defining the electric displacement vector $\hat{D}$ in terms of the electric field $\hat{E}$ is assumed, for a field possessing a time-factor e^{ikct}; the further assumption $\hat{E} = \hat{F}e^{-ikS}$ gives a partial differential equation of the first order and fourth degree for S, which can be considered the equation of Jacobi for a system of differential equations in the Hamiltonian form. If S is known, an assumed asymptotic expansion $\hat{F} = \hat{F}^0(G_0 + k^{-1}G_1 + k^{-2}G_2 + \cdot\cdot\cdot)$ leads to a differential equation for each G_m whose solution can be written down. The construction of E is similar to the method of Birkhoff [‡] for the asymptotic expansion of a solution of Schrodinger's equation; but the necessity of constructing a vector function requires an analysis similar to that of Hadamard for the discussion of the characteristic surfaces of a system of partial differential equations.[§] A first result is that the trajectories of the Hamiltonian equations are tangent to the Poynting vector of energy flow in the first approximation, if the absorption is negligible.

The trajectories of interest are those for which the Hamiltonian function H vanishes; in the present problem H can be written $H = H_1H_2H_3H_4$, and each factor may be considered a new Hamiltonian function. Hence the method of successive approximations can be applied to each wave form independently of the others; and no splitting up of a plane wave occurs on entering the ionized region if this wave is suitably polarized at the transmitter close to the earth's surface. Hence to the extent that the present approximate analysis is valid; it would be possible, with a sufficiently concentrated short wave beam, to transmit just one wave form to a distant receiver; possible advantages are indicated by formulae for transmission along the magnetic field, in which case the absorption for one wave form may be much larger than that for the other, if both paths lie in regions of high electron density over a considerable distance.

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[*] A. Sommerfeld and J. Runge, *Annalen der Physik*, ser. 4, vol. 35 (1911), p. 277; R. Gans, *Annalen der Physik*, ser. 4, vol. 47 (1915), p. 709; K. Försterling, *Physikalische Zeitschrift*, vol. 14 (1913), p. 265.

[†] D. R. Hartree, *Proceedings of the Cambridge Philosophical Society*, vol. 27 (1931), p. 143; also E. V. Appleton, *Proceedings of the Physical Society* (London), vol. 37 (1925), 16D, and S. Goldstein, *Proceedings of the Royal Society*, A, vol. 121 (1928), p. 260.

[‡] *Proceedings of the National Academy of Sciences*, March (1933).

[§] *Leçons sur la propagation des ondes* (1903), pp. 276-280.

1. In the field of a plane wave of frequency ν, each ion in the ionized region describes a trajectory, calculated from the acting field [$\hat{E}' = (U - \beta\sigma)^{-1}L$ in Hartree's notaation] and these trajectories produce a dipole moment

$$d\hat{M} = (1/4\pi)\sigma\hat{E}dx_0dy_0dz_0, \qquad d\hat{\Pi}_0 = d\hat{M}e^{-ikr}/r, \qquad k = 2\pi\nu/c.$$

If the ionic density is constant, and the impressed magnetic field $\hat{H}^0$ is uniform, the field equations are,

$$(1.1) \qquad D = \epsilon\hat{E}, \quad ik\hat{D} = \operatorname{curl}\hat{H}, \quad -ik\hat{H} = \operatorname{curl}\hat{E}$$

where ϵ is a three-rowed matrix defined as follows. Let

$$(1.2) \qquad k_0^2 = 4\pi Ne^2/mc^2, \qquad k_h = -e\,|\,\hat{H}^0\,|/mc^2, \qquad Z = 1 - ik_c/k,$$

N being the number of free ions per cubic centimeter, e the charge on the ion in e. s. u., m its mass, and ck_c the number of collisions of the ion per second. Then if

$$u = k_0^2/k^2(Z + \beta k_0^2/k^2), \qquad \hat{v} = k_h\hat{H}^0/[k\,|\,\hat{H}^0\,|\,(Z + \beta k_0^2/k^2)],$$
$$w = -u/(1 - v^2)$$

Hartree's matrix is:

$$(1.3) \qquad \epsilon = \begin{Vmatrix} 1 - w(1 - v_x^2) & iwv_z - wv_xv_y & -iwv_y - wv_xv_z \\ -iwv_z - wv_xv_y & 1 + w(1 - v_y^2) & iwv_x - wv_yv_z \\ iwv_y - wv_xv_z & -iwv_x - wv_yv_z & 1 + w(1 - v_z^2) \end{Vmatrix}.$$

The equations (1.1) will be assumed valid in the ionized region where the parameters u, v, w change only by very small fractions of their values within a wave-length. An approximate solution of the field equations (1.1) may be constructed with the aid of a function S determined as follows. The equation

$$(1.4) \qquad k^2\hat{D} = \operatorname{grad}\operatorname{div}\hat{E} - \Delta E$$

is a consequence of (1. 1); if $\hat{E} = \hat{F}e^{-ikS}$ three equations result from (1. 4) of which the first is,

$$k^2(\epsilon\hat{F})_x = k^2L_{2x}(p, q, r) - ikL_{1x}(p, q, r) + L_{0x}, \tag{1. 5}$$

if $p = \partial S/\partial x, \quad q = \partial S/\partial y, \quad r = \partial S/\partial z, \quad \hat{L}_0 = \text{grad div}\,\hat{F} - \Delta\hat{F},$
$\tilde{n} = (p, q, r),\ n^2 = p^2 + q^2 + r^2,$

L_{1x} and L_{2x} being defined by

$$L_{2x} = (L_2\hat{F})_x, \qquad L_2 = \left\| \begin{matrix} n^2 - p^2 & -pq & -pr \\ -qp & n^2 - q^2 & -qr \\ -rp & -rq & n^2 - r^2 \end{matrix} \right\|$$

$$\begin{aligned} L_{1x} = {} & F_x\partial p/\partial x + F_y\partial p/\partial y + F_z\partial p/\partial z + p\,\text{div}\,\hat{F} + p\partial F_x/\partial x + q\partial F_y/\partial x + r\partial F_z/\partial x \\ & - F_x(\partial p/\partial x + \partial q/\partial y + \partial r/\partial z) - 2(p\partial F_x/\partial x + q\partial F_x/\partial y + r\partial F_x/\partial z). \end{aligned}$$

If $\tilde{K}$ is the matrix $\epsilon - L_2$, (1. 5) can be written

$$\tilde{K}\hat{F} = -ik^{-1}\hat{L}_1 + k^{-2}\hat{L}_0 \tag{1. 6}$$

and a first approximate solution for large k results from $\tilde{K}\hat{F} = 0$. This equation is simplified if the $X - Z$ plane is chosen to be that defined by the vectors $\hat{n}$ and $\hat{H}^0$; then if

$$\begin{gathered} p = n\sin\theta,\ r = n\cos\theta,\ a^2 = 1 - u,\ b^2 = [(1 + w)^2 - w^2v^2]/(1 + w), \\ v_t = v_x\cos\theta - v_z\sin\theta,\ v_n = v_x\sin\theta + v_z\cos\theta, \end{gathered} \tag{1. 7}$$

and the determinant K of the linear equation's $\tilde{K}\hat{F} = 0$ is equated to zero, the index of refraction n satisfies

$$n^4[1 - wv_n^2/(1 + w)] - n^2[a^2 + b^2 - wv_n^2/(1 + w)] + a^2b^2 = 0.$$

The solutions, after some reductions, can be written

$$n^2 - 1 = -u/\{1 - v_t^2/2a^2 \pm [v_n^2 + v_t^4/4a^4]^{1/2}\}. \tag{1. 8}$$

For an arbitrary coördinate system,

$$\begin{gathered} v_n = (pv_x + qv_y + rv_z)/n, \\ v^2 = v_x^2 + v_y^2 + v_z^2, \qquad v_t^2 = v^2 - v_n^2. \end{gathered}$$

The vector $\hat{F}$ is defined, except for a constant factor, by

$$\frac{F_x}{a^2v_z + n^2\sin\theta v_t} = \frac{iwF_y}{(1 + w)(n^2 - a^2) - n^2wv_n^2} = \frac{F_z}{-a^2v_x + n^2\cos\theta v_t}.$$

2. If $K(p, q, r, x, y, z)$ is the determinant of the matrix $\tilde{K}$, as above, this determinant is zero everywhere if the function S satisfies the partial differential equation $K = 0$, with

$$p = \partial S/\partial x, \qquad q = \partial S/\partial y, \qquad r = \partial S/\partial z. \tag{2.1}$$

The characteristics of this partial differential equation are in the Hamiltonian form

$$\begin{aligned} dx/dt &= \partial K/\partial p, & dy/dt &= \partial K/\partial q, & dz/dt &= \partial K/\partial r, \\ dp/dt &= -\partial K/\partial x, & dq/dt &= -\partial K/\partial y, & dr/dt &= -\partial K/\partial z. \end{aligned} \tag{2.2}$$

The classical construction of a solution S of Jacobi's equation $K = 0$ in the form

$$S = \int_{(x^0,y^0,z^0)}^{(x,y,z)} p\,dx + q\,dy + r\,dz$$

by means of integrals of (2.2) will be assumed, with the condition $K = 0$. A solution of (1.6) will be assumed to have the form

$$\hat{F} = \hat{F}_0 + k^{-1}\hat{F}_1 + k^{-2}\hat{F}_2 + \cdots$$

from which there results the set of equations

$$\begin{aligned} &\tilde{K}\hat{F}_m = -iL_1(\hat{F}_{m-1}) + L_0(\hat{F}_{m-2}), \qquad m \geq 2, \\ &\tilde{K}\hat{F}_0 = 0, \qquad \tilde{K}\hat{F}_1 = -iL_1(\hat{F}_0). \end{aligned} \tag{2.3}$$

Since the determinant of $\tilde{K}$ vanishes, the right-hand members must satisfy a condition

$$\begin{aligned} &c_1[-iL_1(\hat{F}_{m-1}) + L_0(\hat{F}_{m-2})]_x \\ &+ c_2[-iL_1(\hat{F}_{m-1}) + L_0(\hat{F}_{m-2})]_y + c_3[-iL_1(\hat{F}_{m-1}) + L_0(\hat{F}_{m-2})]_z = 0 \end{aligned} \tag{2.4}$$

in which (c_1, c_2, c_3) is a solution of the equations adjoint to $\tilde{K}\hat{F} = 0$. Also,

$$c_1L_{1x}(\hat{F}^0G) + c_2L_{1y}(\hat{F}^0G) + c_3L_{1z}(\hat{F}^0G) = 0$$

if $\hat{F}^0$ is a solution of $\tilde{K}\hat{F} = 0$ arbitrarily chosen, G a scalar function to be determined. This is a linear partial differential equation for G which can be written in either of two ways if $Z = 1$ in (1.2) corresponding to negligible absorption; in this case the matrix $\tilde{K}$ is Hermitian, and a possible choice for the c's is: $c_1 = \bar{F}_x{}^0$, $c_2 = \bar{F}_y{}^0$, $c_3 = \bar{F}_z{}^0$. Let

$$dx/dn = p/n, \qquad dy/dn = q/n, \qquad dz/dn = r/n.$$

Then from the definition of L_1, the equation for G becomes,

$$(2.5)\quad [F_x(\hat{\bar{F}}\hat{n}) + \bar{F}_x(\hat{F}\hat{n}) - 2p\,|\,\hat{F}\,|^2]\partial G/\partial x$$
$$+ [F_y(\hat{\bar{F}}\hat{n}) + \bar{F}_y(\hat{F}\hat{n}) - 2q\,|\,\hat{F}\,|^2]\partial G/\partial y$$
$$+ [F_z(\hat{\bar{F}}\hat{n}) + \bar{F}_z(\hat{F}\hat{n}) - 2r\,|\,\hat{F}\,|^2]\partial G/\partial z$$
$$+ G[\hat{\bar{F}} \operatorname{grad}(\hat{F}\hat{n}) + (\hat{\bar{F}}\hat{n}) \operatorname{div}\hat{F} - |\,\hat{F}\,|^2 \Delta S - 2\hat{\bar{F}}n d\hat{F}/dn] = 0.$$

From $\hat{E} = \hat{F}e^{-ikS}$ and the field equations, in the first approximation the conjugate imaginary of $\hat{H}$ is $\hat{\bar{H}} = [\tilde{n}, \hat{\bar{F}}]\, e^{ikS}$; hence

$$-2R[\hat{E}, \hat{\bar{H}}] = \hat{\bar{F}}(\hat{n}\hat{F}) + \hat{F}(\hat{n}\hat{\bar{F}}) - 2\hat{n}\,|\,\hat{F}\,|^2$$

a numerical constant times the Poynting energy vector $(c/4\pi)R[\hat{E}, \hat{\bar{H}}]$. Let

$$dx/ds = R[\hat{E}, \hat{\bar{H}}]_x, \qquad dy/ds = R[\hat{E}, \hat{\bar{H}}]_y, \qquad dz/ds = R[\hat{E}, \hat{\bar{H}}]_z.$$

Then

$$-2dG/ds + mG = 0, \qquad \log G = (1/2)\int_{s_0}^{s} m ds.$$

Another form for the differential equation for G will be derived, from which it follows that the integral for G is along a trajectory of (2.2). From the definition of L_2,

$$\begin{aligned} L_{2x} &= (q^2 + r^2)F_x - p(qF_y + rF_z) \\ L_{2y} &= (r^2 + p^2)F_y - q(pF_x + rF_z) \\ L_{2z} &= (p^2 + q^2)F_z - r(pF_x + qF_y). \end{aligned}$$

The substitution $\hat{F} = G\hat{F}^0$ in L_1, with the assumption that $\hat{F}^0$ is a function of (x, y, z) alone, gives the equations

$$\begin{aligned} L_{1x} &= GL_{1x}(\hat{F}^0) - [(\partial L_{2x}/\partial p)\partial G/\partial x + (\partial L_{2x}/\partial q)\partial G/\partial y + (\partial L_{2x}/\partial r)\partial G/\partial z] \\ L_{1y} &= GL_{1y}(\hat{F}^0) - [(\partial L_{2y}/\partial p)\partial G/\partial x + (\partial L_{2y}/\partial q)\partial G/\partial y + (\partial L_{2y}/\partial r)\partial G/\partial z] \\ L_{1z} &= GL_{1z}(\hat{F}^0) - [(\partial L_{2z}/\partial p)\partial G/\partial x + (\partial L_{2z}/\partial q)\partial G/\partial y + (\partial L_{2z}/\partial r)\partial G/\partial z]. \end{aligned}$$

If $\bar{K} = \|\, k_{ij} \,\|$, and K_{ij} is the co-factor of k_{ij} in the determinant K, any two rows of the matrix

$$\begin{Vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{Vmatrix}$$

are proportional, from which

$$K_{11}K_{22} = K_{12}K_{21}, \qquad K_{11}K_{23} = K_{13}K_{21}, \text{ etc.}$$

Now

$$\begin{aligned} -\partial L_{2x}/\partial p &= (\partial/\partial p)\{[\epsilon_{11} - (q^2 + r^2)]F_x{}^0 + (\epsilon_{12} + pq)F_y{}^0 + (\epsilon_{13} + pr)F_z{}^0\} \\ &= (\partial/\partial p)[k_{11}F_x{}^0 + k_{12}F_y{}^0 + k_{13}F_z{}^0] \\ &= C[K_{21}\partial k_{11}/\partial p + K_{22}\partial k_{12}/\partial p + K_{23}\partial k_{13}/\partial p] \end{aligned}$$

if

$$F_x{}^0 = CK_{21}, \quad F_y{}^0 = CK_{22}, \quad F_z{}^0 = CK_{23} \qquad C = C(x, y, z).$$

Let

$$c_1 = \bar{C}K_{12}, \quad c_2 = \bar{C}K_{22}, \quad c_3 = \bar{C}K_{32}.$$

The equation for G becomes

$$C\bar{C}K_{22}\left\{\begin{array}{l} \quad K_{11}\partial k_{11}/\partial p + K_{12}\partial k_{12}/\partial p + K_{13}\partial k_{13}/\partial p \\ + K_{21}\partial k_{21}/\partial p + K_{22}\partial k_{22}/\partial p + K_{23}\partial k_{23}/\partial p \\ + K_{31}\partial k_{31}/\partial p + K_{32}\partial k_{32}/\partial p + K_{33}\partial k_{33}/\partial p \end{array}\right\}\partial G/\partial x + \cdots$$

$$= C\bar{C}K_{22}(\partial K/\partial p)\partial G/\partial x + \cdots.$$

Hence the equation

$$\begin{aligned} 0 = G[c_1L_{1x}(\hat{F}^0) + c_2L_{1y}(F^0) + c_3L_{1z}(\hat{F}^0)] \\ + C\bar{C}K_{22}[(\partial K/\partial p)\partial G/\partial x + (\partial K/\partial q)\partial G/\partial y + (\partial K/\partial r)\partial G/\partial z] \end{aligned}$$

which is identical with (2.5) if the absorption is zero; from a comparison with (2.2) it is seen that in any case this differential equation can be integrated along the trajectories of these equations.

The functions $F_x{}^0 = CK_{21}$, etc., contain some of the variables p, q, r explicitly; these must be expressed as functions of x, y, z with the aid of Jacobi's function S before G can be constructed by the preceding method.

If the function G is known, let $\hat{F}_0 = G\hat{F}^0$; then $\hat{F}_1$ can be determined from (2.3) except for an additive term $G_1\hat{F}^0$ where $\hat{F}^0$ is any solution of $\bar{K}\hat{F} = 0$. With the substitution $\hat{F}_1 = \hat{F}_1{}^0 + G_1\hat{F}^0$ the equations (2.3) can be solved for $\hat{F}_2$ only if the right-hand member satisfies (2.4), which gives a differential equation for G_1 similar to that for G; G_1 being determined, $\hat{F}_2$ is known except for an additive term $G_2\hat{F}^0$, and the equation for $\hat{F}_3$ can be solved if G_2 is determined from (2.4), and so on. The resulting expansion is of the form

$$\hat{F} = \hat{F}^0(G_0 + k^{-1}G_1 + k^{-2}G_2 + \cdots).$$

3. If the determinant K can be factored, let $K = K_1K_2\cdots K_m$; the condition $K = 0$ requires that some factor vanish; if this is K_1, equations (2.2) become

$$(3.1) \qquad dx/dt = (K/K_1)\partial K_1/\partial p, \cdots \; dp/dt = -(K/K_1)\partial K_1/\partial x, \; K_1 = 0,$$

or replacing dt by $(K/K_1)\,dt$,

$$dx/dt = \partial K_1/\partial p, \; dp/dt = -\partial K_1/\partial x, \cdots \; K_1 = 0,$$

and a corresponding function S_1 is determined by the preceding analysis applied to this special system.

From the first section, except for a neglible factor,

$$K = n^4(1-\gamma v_n^2) - n^2(a^2 + b^2 - \gamma v_n^2) + a^2b^2,\ \gamma = w(1+w)^{-1}.$$
$$v_n = n^{-1}(pv_x + qv_y + rv_z),\ n = (p^2+q^2+r^2)^{1/2}.$$

If the parameters u, v, w are functions of z alone, $\partial K/\partial x = \partial K/\partial y = 0$, and from (2.2) p and q are constants. Let

$$A = p^2 + q^2,\ B = pv_x + qv_y.$$

The equation in r becomes,

$$r^4(1-\gamma v_z^2) - 2\gamma B v_z r^3 + r^2\,[2A - a^2 - b^2 - \gamma B^2 + \gamma v_z^2(1-A)]$$
$$+ 2B\gamma v_z(1-A)r + (A-a^2)(A-b^2) + \gamma B^2(1-A) = 0.$$

If $r = R_i$ are the roots,

$$K = (1-\gamma v_z^2)(r-R_1)(r-R_2)(r-R_3)(r-R_4)$$

and the 4 systems of trajectories result:

$$dx/dt = -\,\partial R_i/\partial p,\ dy/dt = -\,\partial R_i/\partial q,\ dz/dt = 1,$$
$$dr/dt = \partial R_i/\partial z;\quad p = \text{const.},\quad q = \text{const.},\quad r = R_i.$$

from which

$$x - x_0 = -\int_{z_0}^{z}(\partial R_i/\partial p)\,dz,\ y - y_0 = -\int_{z_0}^{z}(\partial R_i/\partial q)\,dz.$$

The wave normal is vertical if $p = q = A = B = 0$; but the trajectory is vertical only if $\partial R_i/\partial p = \partial R_i/\partial q = 0$ when p, q are set equal to zero, a condition satisfied over the magnetic poles where $v_x = v_y = 0$, also if $v_z = 0$ since R_i becomes a function of A, B^2.

A trajectory is parallel to the XY plane if $dz = 0$, hence $\partial K/\partial r = 0$, and 2 roots R_i are equal; if this condition is imposed for $z = z_0$, the condition of equal roots gives a single equation between p and q.

4. To determine the components of $\hat{E}$ below the ionized region, for either of the two wave forms being propagated into this region, it is cnly necessary to determine the limits of the components defined at the end of section 1 as the electron density approaches zero. From (1.8)

$$\lim_{w\to 0}(1+w)(n^2-a^2)/w = v_n^2 + v_t^2/2 \mp A',\ A' = (v_n^2 + v_t^4/4)^{1/2};$$

hence if

$$F_x = a^2v_z + n^2\sin\theta v_t,\ F_y = -\,i\,[(1+w)(n^2-a^2)/w - n^2v_n^2],$$
$$F_z = -\,a^2v_x + n^2\cos\theta v_t$$

these limits are

$$(4.1) \qquad F_x = v_n \cos\theta,\ F_y = -i(v_t^2/2 \mp A'),\ F_z = -\sin\theta v_n.$$

If F_L is the component of $\hat{F}$ in the plane of $\hat{H}^0$ and the normal, F_y the component perpendicular to this plane, $F_L = v_n$. Let $\hat{E}'$ be any given electric field below the ionized region, and let

$$E'_L = E'_x \cos\theta - E'_z \sin\theta.$$

$\hat{E}'$ can be expressed in terms of the two special wave forms (4.1) by a solution of the equations for α, β : $\hat{E}' = \hat{F}' e^{-ikS}$,

$$F'_L = \alpha v_n + \beta v_n,\ F'_y = -i\,[\alpha(v_t^2/2 - A') + \beta(v_t^2/2 + A')].$$

If the angle ϕ is defined by

$$v_n = A' \sin\phi,\ v_t^2/2 = A' \cos\phi$$

the solution becomes

$$(4.2) \qquad \alpha = (A'/2)\,[F'_L \cot(\phi/2) - iF'_y] \quad \beta = (A'/2)\,[F'_L \tan(\phi/2) + iF'_y].$$

A solution corresponding to an index of refraction given by (1.8) with the plus sign is obtained if $\beta = 0$, the minus sign corresponding to $\alpha = 0$. For vertical waves directed upwards in the northern hemisphere $v_n < 0$, hence $\beta = 0$ corresponds to a right-hand polarized wave, $\alpha = 0$ to a left-hand polarization.

5. To set up the Hamiltonian equations in spherical coördinates, let

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta\,.$$

$$(5.1) \qquad p_\phi = \partial S/\partial\phi,\ p_\theta = \partial S/\partial\theta,\ P_r = \partial S/\partial r,\ P_\theta = p_\theta/r,\ P_\phi = p_\phi/r\sin\theta$$

and the quantities p, q, r are transformed by the equations

$$(5.2) \qquad \begin{aligned} p &= \sin\theta\cos\phi\, P_r - \sin\phi\, P_\phi + \cos\theta\cos\phi\, P_\theta,\\ q &= \sin\theta\sin\phi\, P_r + \cos\phi\, P_\phi + \cos\theta\sin\phi\, P_\theta,\\ r &= \cos\theta\, P_r - \sin\theta\, P_\theta\\ n^2 &= p^2 + q^2 + r^2 = P_r^2 + P_\phi^2 + P_\theta^2. \end{aligned}$$

The equations of the characteristics become

$$(5.3) \qquad \begin{aligned} &dr/dt = \partial K/\partial P_r,\ d\phi/dt = (1/r\sin\theta)\partial K/\partial P_\phi,\ d\theta/dt = r^{-1}\partial K/\partial P_\theta,\\ &dP_r/dt = -\partial K/\partial r,\ dp_\phi/dt = -\partial K/\partial\phi,\ dp_\theta/dt = -\partial K/\partial\theta. \end{aligned}$$

The magnetic field of the earth can be approximated by a magnetic doublet at the center; if $\hat{m}$ is the magnetic moment, the magnetic potential is equal to

$$V = (\hat{m}\hat{r})/r^3 = r^{-2}\,[m_x \sin\theta\cos\phi + m_y \sin\theta\sin\phi + m_z\cos\theta]$$

from which

$$H_r = -\partial V/\partial r,\ H_\theta = -r^{-1}\partial V/\partial\theta,\ H_\phi = -(1/r\sin\theta)\partial V/\partial\phi$$

and

$$v_n = n^{-1}\,[P_r v_r + P_\theta v_\theta + P_\phi v_\phi],\ v_r = vH_r/\,|\,\hat{H}\,|, \text{ etc.}$$

These relations can be used in the approximate calculation of the horizontal angle between the wave normal and the trajectory if the latter may be assumed approximately parallel the earth's surface; then P_r is small, and if the coördinate system is so chosen that the normal is tangent to a meridian at the initial point of the small arc considered, P_ϕ is also a small quantity. From (5.3) there result the approximate equations, with (1.8) written

$$n^2 - f = 0,\ v_t = (v^2 - v_n^2)^{1/2},\ d\theta/dt = 2P_\theta/r,$$
$$d\phi/dt = \partial(n^2 - f)/\partial p_\phi = -\ (\partial f/\partial v_n - (v_n/v_t)\partial f/\partial v_t)\partial v_n/\partial p_\phi\ |\ p_\phi = 0.$$

From (1.8)

$$f = 1 - u/D,\ D = 1 - v_t^2/2a^2 + (v_n^2 + v_t^4/4a^4)^{1/2}.$$

Substituting $P_\theta = n$ after the differentiation,

$$d\theta/dt = 2n/r,\ d\phi/dt = -\ [f(1-f)v_n/a^2A']\partial v_n/\partial p_\phi\ |\ p_\phi = 0$$
$$= -\ n(1-n^2)v_n v_\phi/a^2A'r\sin\theta,$$

hence

$$\Delta S_\phi^{+}/\Delta S_\theta = r\sin\theta\, d\phi/rd\theta = (n_+^2 - 1)v_n v_\phi/2a^2A'. \tag{5.4}$$

The corresponding ratio with $-A'$ in the denominator D is obtained by substitution:

$$\Delta S_\phi^{-}/\Delta S_\theta = -(n_-^2 - 1)v_n v_\phi/2a^2A'. \tag{5.41}$$

Both deviations vanish if either of the impressed field components H_n^0, H_ϕ^0 is zero, and (5.4) and (5.41) give opposite signs for the deviations if $(n_+^2 - 1)$ and $(n_-^2 - 1)$ have the same sign.

If the direction of propagation is across the magnetic field, $(\hat{n}\hat{H}^0) = 0$, and the two indices of refraction are $n_1 = a$, $n_2 = b$. For very short waves the ratio k_c/k may be considered a small quantity; if n_1 and n_2 are expanded in powers of this ratio and only the first power retained,

$$n_1 = n_1{}^0 + n'_1 k_c/k, \quad n_2 = n_2{}^0 + n'_2 k_c/k.$$

The ratio n'_2/n'_1 is a function of the remaining parameters from which the relative magnitudes of the absorption losses along the same path may be estimated. From the definitions,

$$[du/dZ]_{Z=1} = [dv/dZ]_{Z=1} = (1 + \beta k_0{}^2/k^2)^{-2},$$

and

$$\begin{aligned} n'_2/n'_1 = {} & (n_1/n_2)(dn_2{}^2/dZ)/(dn_1{}^2/dZ)\,|_{Z=1} \\ & = (n_1/n_2)(1 - u + v)(1 + 2uv - u - v)/(1 - u - v^2)^2. \end{aligned}$$

If the propagation is along the magnetic lines of force, and

$$n_1{}^2 = 1 - u/(1 + v), \; n_2{}^2 = 1 - u/(1 - v)$$

the limiting ratio becomes,

$$\begin{aligned} n'_1/n'_2 = {} & (1 - v)(1 + v)^{-1}(1 + u - v)^{-1}\,[(1 - v)(1 + v)^{-1}(1 + v - u) \\ & (1 - v - u)]^{1/2}. \end{aligned}$$

The relatively smaller absorption for the refraction index n_1 can be seen from numerical examples, n_1 corresponding to right-hand polarization. For example, if the ions are free electrons, and $|\hat{H}^0| = 0.5$, $N = 3x10^5$, $k_h \sim 212$ meters, $k \sim 20$ meters, $u = 0.10$, $v = 0.10$, and $n'_1/n'_2 = 0.65$; if $k \sim 40$ meters, $u = 0.38$, $v = 0.185$, and $n'_1/n'_2 = 0.28$.

6. The analogy between (2. 2) and the equations of motion of a dynamical system is complete if no magnetic field is impressed, corresponding to the simpler Larmor theory; in this case $n^2 = 1 - u(xyz)$, and $K = 2H$,

$$H = (1/2)[p^2 + q^2 + r^2] + u/2 - 1/2.$$

(2. 2) are equivalent to

$$dx/dt = \partial H/\partial p, \; dp/dt = -(\partial/\partial x)(u/2) = \text{etc.}$$

where the function $(u - 1)/2$ takes the place of the potential function; but only those trajectories are permissible for which $H = 0$.

A PROBLEM IN DIOPHANTINE ANALYSIS.

By Hansraj Gupta.

1. Bell and Ward have considered the solutions * of

$$\text{(A)} \qquad x_1x_2x_3x_4\cdots x_n = u_1u_2u_3\cdots u_m,$$

(and other allied problems), in integers. Bell has dealt with the problem by the multiplicative method, and Morgan Ward by the additive method. Bell has taken for the fundamental form the equation

$$\text{(B)} \qquad x_1x_2x_3\cdots x_n = u_1u_2u_3\cdots u_n,$$

and has shown that all solutions of (B) in integers can be expressed in terms of n^2 parameters. He makes use of 'Reciprocal Arrays' in the presentation of his solutions.

In what follows, I have followed in general the method of Bell. I have, however, dealt directly with (A). The proof of the general theorem given by Bell is inductive. I have given a deductive proof for the general case. The G. C. D. conditions stated by Bell have been improved upon in so far as the parameters employed are definite at each stage. No need for reciprocal arrays arises in the solution as presented here.

A special case considered by both Bell and Ward has been considered about the end of this paper, and their result is obtained by a different method.

2. If x, y be two non-zero positive integers, their G. C. D. will be denoted by $[x, y]$ as usual. If the integers x, y are prime to each other, we shall have $[x, y] = 1$.

3. Theorem. *Any solution of* (A) *can be expressed in terms of mn parameters.*

Let $[x_1, u_1] = \phi_1, \quad [x_1/\phi_1, u_2] = \phi_2, \quad [x_1/\phi_1\phi_2, u_3] = \phi_3, \cdots,$

$$[x_1/\prod_1^r \phi_r, u_{r+1}] = \phi_{r+1}, \cdots, \quad [x_1/\prod_1^{m-1} \phi_r, u_m] = \phi_m.$$

Then, $\quad \because \quad u_1u_2u_3\cdots u_m \equiv 0 \pmod{x_1}.$

$$\therefore \quad x_1 = \phi_1\phi_2\phi_3\cdots\phi_m.$$

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* E. T. Bell, "Reciprocal arrays and diophantine analysis"; Morgan Ward, "A type of multiplicative diophantine system," *American Journal of Mathematics*, vol. 55 (1933), pp. 50-67.

Now let

$$[x_2, u_1/\phi_1] = \phi_{m+1}, \quad [x_2/\phi_{m+1}, u_2/\phi_2] = \phi_{m+2}, \quad [x_2/\phi_{m+1}\phi_{m+2}, u_3/\phi_3] = \phi_{m+3}, \cdots,$$

$$[x_2/\prod_{r=1}^{m-1} \phi_{m+r}, u_m/\phi_m] = \phi_{2m}.$$

Then,

$$\because \prod_{r=1}^{m} u_r \equiv 0, \ (\text{mod } x_2).$$

$$\therefore \quad x_2 = \phi_{m+1}\phi_{m+2}\phi_{m+3} \cdots \phi_{2m}.$$

In general, let

$$[x_{k+1}/\prod_{r=1}^{l-1} \phi_{km+r}, \ u_l/\prod_{r=1}^{k} \phi_{(r-1)m+l}] = \phi_{km+l}, \qquad (k+1 \leq n, \ l \leqq m).$$

Then we obtain

$$(S_1) \qquad x_{k+1} = \prod_{l=1}^{m} \phi_{km+l}. \qquad (k \leq n-1).$$

Moreover, we get

$$\begin{array}{ll} & u_1 = \phi_1 \cdot \phi_{m+1} \cdot \phi_{2m+1} \cdot \phi_{3m+1} \cdots \phi_{(n-1)m+1}, \\ & u_2 = \phi_2 \cdot \phi_{m+2} \cdot \phi_{2m+2} \cdot \phi_{3m+2} \cdots \phi_{(n-1)m+2}, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (S_2) & u_l = \phi_l \cdot \phi_{m+l} \cdot \phi_{2m+l} \cdot \phi_{3m+l} \cdots \phi_{(n-1)m+l}, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & u_m = \phi_m \cdot \phi_{2m} \cdot \phi_{3m} \cdot \phi_{4m} \cdots \phi_{nm}. \end{array}$$

Placing the solutions $[S_1]$ and $[S_2]$ for x's and u's side by side we notice a row-column correspondence which would be more convenient in practice than the row-diagonal correspondence set up by Bell.

3. 1. The following example will show how ϕ's can be determined in practice. Consider the equation

$$x_1x_2x_3x_4 = u_1u_2u_3.$$

One of its solutions is

$$\begin{array}{ll} x_1 = 16, & u_1 = 98 \\ x_2 = 189, & u_2 = 108, \\ x_3 = 56, & u_3 = 576. \\ x_4 = 36. & \end{array}$$

$$\begin{array}{lll} \text{Now,} & [x_1, u_1] = 2 = \phi_1; & [x_2, u_1/2] = 7 = \phi_4; \\ & [x_1/2, u_2] = 4 = \phi_2; & [x_2/7, u_2/4] = 27 = \phi_5; \\ & [x_1/8, u_3] = 2 = \phi_3; & [x_2/189, u_3/2] = 1 = \phi_6; \\ \text{and} & [x_3, u_1/14] = 7 = \phi_7; & [x_4, u_1/98] = 1 = \phi_{10}; \\ & [x_3/7, u_2/108] = 1 = \phi_8; & [x_4, u_2/108] = 1 = \phi_{11}; \\ & [x_3/7, u_3/2] = 8 = \phi_9; & [x_4, u_3/16] = 36 = \phi_{12}. \end{array}$$

We thus obtain

$$\begin{array}{ll}
x_1 = \phi_1\phi_2\phi_3, & u_1 = \phi_1\phi_4\phi_7\phi_{10}, \\
x_2 = \phi_4\phi_5\phi_6, & u_2 = \phi_2\phi_5\phi_8\phi_{11}, \\
x_3 = \phi_7\phi_8\phi_9, & u_3 = \phi_3\phi_6\phi_9\phi_{12}. \\
x_4 = \phi_{10}\phi_{11}\phi_{12}. &
\end{array}$$

3. 2. Denoting the parameter in the i-th column and the j-th row in the solutions for x's, by $\phi_{i,j}$, we can write the solutions to the equations of the form:

$$\prod_1^n x_j = \prod_1^m u_i$$

in the symbolic form:

$$\begin{array}{r|l} x_j = & \phi_{i,j}, \\ \hline & u_i \end{array} \qquad 1 \leq j \leq n, \quad 1 \leq i \leq m.$$

The row-column correspondence noted above at once suggests this symbol. We thus have

$$x_j = \phi_{1,j} \cdot \phi_{2,j} \cdot \phi_{3,j} \cdot \cdot \cdot \phi_{m,j}, \text{ and } u_i = \phi_{i,1} \cdot \phi_{i,2} \cdot \phi_{i,3} \cdot \cdot \cdot \phi_{i,n}.$$

3. 3. From the G. C. D. conditions stated in § 3, it follows that

$$\left[\frac{x_{k+1}}{\prod_{r=1}^{l} \phi_{km+r}}, \frac{u_l}{\prod_{r=1}^{k+1} \phi_{(r-1)m+l}}\right] = 1, \qquad 0 \leq k \leq n-1,\ 1 \leq l \leq m$$

and, using the symbolism of § 3. 2, we have

$$[\phi_{i,j}, \phi_{i-\alpha,j+\beta}] = 1, \text{ also } [\phi_{i,j}, \phi_{i+\alpha,j-\beta}] = 1,$$

where α, β are non-zero positive integers.

In practice the following device for finding the parameters prime to a given parameter shall be very useful. Write down the parameters in rows and columns as they occur in the solutions for x's or u's. Separate the given parameter from the rest by lines parallel to the rows and columns, thus

$$\begin{array}{cccccc|c||cccc}
 & \phi_{1,1} & \phi_{2,1} & \phi_{3,1} & \cdot & \phi_{i-1,1} & \phi_{i,1} & \phi_{i+1,1} & \cdot & \phi_{m,1} & \\
 & \phi_{1,2} & \phi_{2,2} & \phi_{3,2} & \cdot & \phi_{i-1,2} & \phi_{i,2} & \phi_{i+1,2} & \cdot & \phi_{m,2} & \\
B_2 & \phi_{1,3} & \phi_{2,3} & \phi_{3,3} & \cdot & \phi_{i-1,3} & \phi_{i,3} & \phi_{i+1,3} & \cdot & \phi_{m,3} & B_1 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \phi_{1,j-1} & \phi_{2,j-1} & \phi_{3,j-1} & \cdot & \phi_{i-1,j-1} & \phi_{i,j-1} & \phi_{i+1,j-1} & \cdot & \phi_{m,j-1} & \\
\hline
 & \phi_{1,j} & \phi_{2,j} & \phi_{3,j} & \cdot & \phi_{i-1,j} & \phi_{i,j} & \phi_{i+1,j} & \cdot & \phi_{m,j} & \\
\hline\hline
 & \phi_{1,j+1} & \phi_{2,j+1} & \phi_{3,j+1} & \cdot & \phi_{i-1,j+1} & \phi_{i,j+1} & \phi_{i+1,j+1} & \cdot & \phi_{m,j+1} & \\
B_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B_4 \\
 & \phi_{1,n} & \phi_{2,n} & \phi_{3,n} & \cdot & \phi_{i-1,n} & \phi_{i,n} & \phi_{i+1,n} & \cdot & \phi_{m,n} &
\end{array}$$

Then $\phi_{i,j}$ is prime to each one of the parameters in the two blocks marked B_1, B_3. The number of G. C. D. conditions is $m(m-1)\cdot n(n-1)/4$.

4. 1. We now proceed to consider the equations

$$x_1x_2^2 = y_1y_2 = z_1z_2.$$

Solving first the equation $y_1y_2 = z_1z_2$ we have

$$\begin{aligned} y_1 &= \phi_1\phi_2, & z_1 &= \phi_1\phi_3, \\ y_2 &= \phi_3\phi_4, & z_2 &= \phi_2\phi_4, \text{ where } [\phi_2, \phi_3] = 1, \end{aligned}$$

so that $$x_1x_2^2 = \phi_1\phi_2\phi_3\phi_4.$$
We now get

$$\begin{aligned} x_1 &= \theta_1\theta_2\theta_3\theta_4, & \phi_1 &= \theta_1\theta_5, & \phi_3 &= \theta_3\theta_7, \\ x_2^2 &= \theta_5\theta_6\theta_7\theta_8, & \phi_2 &= \theta_2\theta_6, & \phi_4 &= \theta_4\theta_8, \end{aligned}$$

with the G. C. D. conditions stated in § 3. 3.

Let $\theta_5 = \lambda_1^2 \cdot \theta'_5, \quad \theta_6 = \lambda_2^2 \cdot \theta'_6, \quad \theta_7 = \lambda_3^2 \cdot \theta'_7, \quad \theta_8 = \lambda_4^2 \cdot \theta'_8,$

where θ''s have no square factors, and $[\theta'_6, \theta'_7] = 1$, $\because$ $[\phi_2, \phi_3] = 1$. Now, $\because$ $\theta'_5 \cdot \theta'_6 \cdot \theta'_7 \cdot \theta'_8$ is necessarily a perfect square, any prime factor of θ'_5 must be present in one and only one of the three parameters $\theta'_6, \theta'_7, \theta'_8$; for, if it were present in two of them, $\theta'_5 \cdot \theta'_6 \cdot \theta'_7 \cdot \theta'_8$ would not be a perfect square; and if in all three, $[\theta'_6, \theta'_7]$ would not be $= 1$.

Similarly θ'_6, θ'_7 have all their factors present in θ'_5 and θ'_8.

$\therefore$ Putting $[\theta'_5, \theta'_6] = \lambda_5, \quad [\theta'_5, \theta'_7] = \lambda_6, \quad [\theta'_5, \theta'_8] = \lambda_7,$

$$[\theta'_6, \theta'_8] = \lambda_8 \text{ and } [\theta'_7, \theta'_8] = \lambda_9,$$

we get $\theta'_5 = \lambda_5\lambda_6\lambda_7, \quad \theta'_6 = \lambda_5\lambda_8, \quad \theta'_7 = \lambda_6\lambda_9, \quad \text{and} \quad \theta'_8 = \lambda_7\lambda_8\lambda_9.$

Hence we obtain

$$\begin{aligned} x_1 &= \theta_1\theta_2\theta_3\theta_4, & x_2 &= \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8\lambda_9. \\ y_1 &= \theta_1\theta_2\lambda_1^2\lambda_2^2\lambda_5^2\lambda_6\lambda_7\lambda_8, & y_2 &= \theta_3\theta_4\lambda_3^2\lambda_4^2\lambda_9^2\lambda_6\lambda_7\lambda_8. \\ z_1 &= \theta_1\theta_3\lambda_1^2\lambda_3^2\lambda_6^2\lambda_5\lambda_7\lambda_9, & z_2 &= \theta_2\theta_4\lambda_2^2\lambda_4^2\lambda_8^2\lambda_5\lambda_7\lambda_9. \end{aligned}$$

The solution thus employs 13 parameters, a result which agrees with that of Bell and Ward.

4. 2. Let us now consider the equation

$$x^2 = u_1u_2u_3u_4,$$

without the G. C. D. conditions, viz. $[u_2, u_3] = 1$, as in 4. 1.

Replacing x^2 by x_1x_2, where $x_1 = x_2$,

we get $x_1x_2 \doteq u_1u_2u_3u_4$.

Solving we obtain
$$\left.\begin{aligned} x_1 &= \theta_1\theta_2\theta_3\theta_4, \\ x_2 &= \theta_5\theta_6\theta_7\theta_8, \end{aligned}\right\} \quad \begin{aligned} u_1 &= \theta_1\theta_5, & u_2 &= \theta_2\theta_6, \\ u_3 &= \theta_3\theta_7, & u_4 &= \theta_4\theta_8, \end{aligned}$$

with the obvious G. C. D. conditions.

Moreover, $$\theta_1\theta_2\theta_3\theta_4 = \theta_5\theta_6\theta_7\theta_8,$$

whence
$$\begin{aligned} \theta_1 &= \lambda_1\lambda_2\lambda_3\lambda_4, & \theta_5 &= \lambda_1\lambda_5\lambda_9\lambda_{13}, \\ \theta_2 &= \lambda_5\lambda_6\lambda_7\lambda_8, & \theta_6 &= \lambda_2\lambda_6\lambda_{10}\lambda_{14}, \\ \theta_3 &= \lambda_9\lambda_{10}\lambda_{11}\lambda_{12}, & \theta_7 &= \lambda_3\lambda_7\lambda_{11}\lambda_{15}, \\ \theta_4 &= \lambda_{13}\lambda_{14}\lambda_{15}\lambda_{16}. & \theta_8 &= \lambda_4\lambda_8\lambda_{12}\lambda_{16}. \end{aligned}$$

The solution thus seems to involve 16 parameters.

However, making use of the G. C. D. conditions, we get
$$\lambda_5 = \lambda_9 = \lambda_{13} = \lambda_{10} = \lambda_{14} = \lambda_{15} = 1;$$

so that $$x = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_6\lambda_7\lambda_8\lambda_{11}\lambda_{12}\lambda_{16},$$

and $u_1 = \lambda_1{}^2\lambda_2\lambda_3\lambda_4$, $u_2 = \lambda_6{}^2\lambda_2\lambda_7\lambda_8$, $u_3 = \lambda_{11}{}^2\lambda_3\lambda_7\lambda_{12}$, $u_4 = \lambda_{16}{}^2\lambda_4\lambda_8\lambda_{12}$.

The G. C. D. conditions are thus very useful in the reduction of the number of parameters.

5. THEOREM. *The solutions of the equation*
$$x^n = u_1u_2u_3 \cdot \cdot \cdot u_m,$$
can be expressed in terms of ${}^{m+n-1}C_n$ *parameters.*

The following case sets down the method of proof of this theorem. Consider the equation $x^4 = u_1u_2u_3$.

Replacing x_4 by $x_1x_2x_3x_4$, we get
$$\begin{aligned} x_1 &= \phi_{1,1} \cdot \phi_{2,1} \cdot \phi_{3,1}, & u_1 &= \phi_{1,1} \cdot \phi_{1,2} \cdot \phi_{1,3} \cdot \phi_{1,4}, \\ x_2 &= \phi_{1,2} \cdot \phi_{2,2} \cdot \phi_{3,2}, & u_2 &= \phi_{2,1} \cdot \phi_{2,2} \cdot \phi_{2,3} \cdot \phi_{2,4}, \\ x_3 &= \phi_{1,3} \cdot \phi_{2,3} \cdot \phi_{3,3}, & u_3 &= \phi_{3,1} \cdot \phi_{3,2} \cdot \phi_{3,3} \cdot \phi_{3,4}, \\ x_4 &= \phi_{1,4} \cdot \phi_{2,4} \cdot \phi_{3,4}, \end{aligned}$$

with the necessary G. C. D. conditions.

Putting $x_1 = x_2 = x_3 = x_4$, we finally obtain

$$\begin{array}{lll} \phi_{1,1} = \Pi(1, j, k, l), & \phi_{2,1} = \Pi(2, j, k, l), & \phi_{3,1} = \Pi(3, j, k, l), \\ \phi_{1,2} = \Pi(i, 1, k, l), & \phi_{2,2} = \Pi(i, 2, k, l), & \phi_{3,2} = \Pi(i, 3, k, l), \\ \phi_{1,3} = \Pi(i, j, 1, l), & \phi_{2,3} = \Pi(i, j, 2, l), & \phi_{3,3} = \Pi(i, j, 3, l), \\ \phi_{1,4} = \Pi(i, j, k, 1), & \phi_{2,4} = \Pi(i, j, k, 2), & \phi_{3,4} = \Pi(i, j, k, 3), \end{array}$$

where $\Pi(i, j, k, l)$ denotes the product of all the parameters (i, j, k, l) obtained by giving to i, j, k, l the values 1, 2, 3 (i. e., 1 to m) in all possible manners. The solution thus seems to employ 3^4 (i. e., m^n) parameters, if the G. C. D. conditions be neglected.

Making use of the G. C. D. conditions, however, we are able to remove totally all parameters that have deformities, i. e., those of the parameters in which i, j, k, l are not in ascending order of magnitude (i. e., those in which a smaller number follows a greater one). Parameters such as $(2, 3, 3, 1)$ are thus removed, and we are left with only those of the parameters in which

$$l - k, \quad k - j, \quad j - i$$

are either positive or zero; i. e., $i \leq j \leq k \leq l$.

The number of these parameters is ${}^6C_4$ (i. e., ${}^{m+n-1}C_n$). In the particular case here considered, we are left with the following 15 parameters:

$$\begin{array}{lllll} (1,1,1,1), & (1,1,1,2), & (1,1,1,3), & (1,1,2,2), & (1,1,2,3), \\ (1,1,3,3), & (1,2,2,2), & (1,2,2,3), & (1,2,3,3), & (1,3,3,3), \\ (2,2,2,2), & (2,2,2,3), & (2,2,3,3), & (2,3,3,3), & (3,3,3,3). \end{array}$$

Moreover,

$$u_1 = (1,1,1,1)^4 \cdot (1,1,1,2)^3 \cdot (1,1,1,3)^3 \cdot (1,1,2,2)^2 \cdot (1\,1,2,3)^2 \\ \cdot (1,1,3,3)^2 \cdot (1,2,2,2) \cdot (1,2,2,3) \cdot (1,2,3,3) \cdot (1,3,3,3); \text{ etc.}$$

Notice that the power of a parameter in the solution for u_r is the same as the number of times that r occurs in the said parameter.

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A CLASS OF REPRESENTATIONS OF MANIFOLDS. PART II.*

By Charles B. Morrey, Jr.†

In part I of this paper, which appeared in the October 1933 issue of this Journal, surfaces (and manifolds) of "class L" were defined and it was shown that the Lebesgue and Geöcze areas of such surfaces were identical, and both given by the usual integral formula. The present part (sections 7 and 8) extends Green's (in space) and Stokes' formulas to situations where the surfaces involved are of class L.

6. (*continued*). *Generalized conformal representations of surfaces.* As Theorems 2 and 3 of this section were omitted in part I, we shall present them here. We shall first recall the definition of a generalized conformal representation and state Theorem 1 before proceeding to the proofs of Theorems 2 and 3.

Definition. A surface S, S: $x^i = x^i(u, v)$, $i = 1, \cdots, N$, $(u, v) \epsilon R$, is said to be represented generalized conformally on a Jordan region R (in the plane) if

(i) the $x^i(u, v)$ are all A. C. T. in R, with $(x_u{}^i)^2$ and $(x_v{}^i)^2$ summable over the interior of R, $i = 1, \cdots, N$, and

(ii) $E = G$, $F = 0$ almost everywhere interior to R.

Clearly such a representation is of class L.

Theorem 1. *If S is represented generalized conformally on a Jordan region R, then*

$$L(S_h) \leq L(S)$$

where S_h is the surface $x^i = x_h{}^i(u, v)$, $(u, v) \epsilon R_h$,

$$x_h{}^i = (1/h^2) \int_u^{u+h} \int_v^{v+h} x^i(\xi, \eta)\, d\xi d\eta, \qquad (i = 1, \cdots, N),$$

and R_h is that Jordan region consisting of those points joinable to a point P_0 interior to R (independent of h) by an arc all of whose points are at a distance $\geqq 2^{1/2}h$ from the boundary of R.

Theorem 2. *A necessary and sufficient condition that the surface S (of finite area) be represented generalized conformally on R is that*

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* Presented, together with Part I, to the American Mathematical Society, October 29, 1932.

† National Research Fellow.

(i) $\lim_{h\to 0} L(S_h) = L(S)$,

(ii) $\lim_{h\to 0} \iint_{R_h} [\frac{1}{2}(E_h + G_h) - (E_hG_h - F_h^2)^{1/2}]\,dudv = 0$.

Proof. Combining conditions (i) and (ii), we see that

$$L(S) = \lim_{h\to 0} \tfrac{1}{2} \iint_{R_h} (E_h + G_h)\,dudv.$$

Since it is known that $S_h \to S$ and $x_h^i(u, v) \to x^i(u, v)$ uniformly, we see by § 1, Theorem 7, that the $x^i(u, v)$ are A. C. T. with $(x_u^i)^2$ and $(x_v^i)^2$ summable over R. Then, since S is thus of class L, we have, by § 1, Theorem 3, and § 4, Theorem 1, that

$$L(S) = \iint_R [\tfrac{1}{2}(E + G)]\,dudv = \iint_R (EG - F^2)^{1/2} dudv.$$

Since $(EG - F^2)^{1/2} \leq \frac{1}{2}(E + G)$ where each exists, we must have $(EG - F^2)^{1/2} = \frac{1}{2}(E + G)$ almost everywhere, which in turn implies that $E = G$, $F = 0$ almost everywhere. This proves the sufficiency. The necessity is evident.

THEOREM 3. *Let* S, S: $x^i = x^i(u, v)$, *and* S_n, S_n: $x^i = x_n^i(u, v)$, $i = 1, \cdots, N$, $n = 1, 2, \cdots$, *be continuous surfaces represented on* R. *Suppose:* (i) *the representations of the* S_n *are generalized conformal,* (ii) $\lim_{n\to\infty} L(S_n) = L(S)$, *and* (iii) *there exists a sequence* $\{\epsilon_n\}$ *of positive numbers converging to zero such that*

$$\int_a^b \int_c^d [x_n^i(u, v) - x^i(u, v)]\,dudv < \epsilon_n, \qquad (i = 1, \cdots, N),$$

for every rectangle $(a, b; c, d)$ *in* R. *Then the given representation of* S *on* R *is generalized conformal.*

Proof. Since $L(S)$ is finite and $\lim_{n\to\infty} L(S_n) = L(S)$, we see, using Theorem 7, § 1, that the $x^i(u, v)$ are A. C. T. with $(x_u^i)^2$ and $(x_v^i)^2$ summable. Thus S is of class L and hence (using Theorem 7, § 1),

$$L(S) = \lim_{n\to\infty} L(S_n) = \lim_{n\to\infty} \iint_R [\tfrac{1}{2}(E_n + G_n)]\,dudv$$
$$\geq \iint_R [\tfrac{1}{2}(E + G)]\,dudv \geq L(S).$$

Hence $(EG - F^2)^{1/2} = \frac{1}{2}(E + G)$ and thus $E = G$, $F = 0$ almost everywhere.

PART II.

7. *Some topological considerations.* In this section, we shall give a very general definition of a "simply covered" closed surface. It is to be observed that this definition allows closed "surfaces" to be of infinite connectivity, something not allowed by the usual definition of a closed surface. We shall prove an important theorem on the approximation to such surfaces by means of closed polyhedra.

Definition I. We say that a surface, S, is *closed* if it possesses a representation, $x^i = x^i(u, v)$, $(i = 1, \cdots, N)$, on Q such that there exist two (finite or infinite) sequences, $\{I_n^1\}$ and $\{I_n^2\}$, of intervals on the boundary, $B(Q)$, of Q satisfying the following conditions:

(a) Any point of $B(Q)$ is either (i) a point of one of the $\{I_n^i\}$ [class I], (ii) a common end point of two abutting intervals [class II], (iii) a limit point from both sides of points of classes (I) and (II) [class III].

(b) The surface is to be "connected up" along the curve corresponding to $B(Q)$, i. e., certain distinct points of $B(Q)$ are to be made to correspond by the condition that they yield the same point on the surface, both logically and geometrically. We do not exclude the possibility of $x^i(P_1) = x^i(P_2)$, $(i = 1, \cdots, N)$, for points P_2 (of $B(Q)$) other than those made to correspond to P_1, but such extra points yield logically distinct (i. e. multiple) points on the surface. The above correspondence must be such that there is a one to one, (uniformly) continuous, sense reversing [on $B(Q)$] correspondence between the points of I_n^1 and those of I_n^2; each interval I_n^1 *corresponds* to the unique interval I_n^2, which neither coincides with nor is adjacent to I_n^1 (unless $n = 1$).

(c) Let I_1^i $(i = 1$ or $2)$ be any one of these intervals having end points P_1 and Q_1. Suppose I_2^j abuts on I_1^i at P_1 and I_3^k abuts on I_1^i at Q_1 $(j, k = 1$ or $2)$. Let P_2 be the end point of I_2^{3-j} corresponding (by extension) to P_1 and Q_2 that of I_3^{3-k} corresponding to Q_1. We suppose that one of the intervals (P_2, Q_2) is one of the $\{I_n^l\}$ and we call this interval the interval *associated* with I_1^i, provided that $I_2^j \neq I_3^k$. If $I_2^j = I_3^k$, it is clear that there are only two intervals on $B(Q)$. In this case the associated intervals are not defined.

(d) There exist two sequences $\{C_m^1\}$ and $\{C_m^2\}$ of simple Jordan arcs (which may reduce to points) in Q such that (i) no two curves of $\{C_m^i\}$ intersect $(i = 1, 2)$, (ii) a curve of $\{C_m^1\}$ intersects one of $\{C_m^2\}$ in at most one point, (iii) every interval of $\{I_n^1\} + \{I_n^2\}$ is joined to its corresponding interval by a curve of $\{C_m^i\}$ and to its associated interval by a curve of

$\{C_m^{3-i}\}$; (iv) each curve of $\{C_m^1\} + \{C_m^2\}$ joins a point of an interval of $\{I_n^1\} + \{I_n^2\}$ to either one in its associated interval or to one in its corresponding interval; (v) if a curve, C_λ^i, of $\{C_m^i\}$ joins an interval, I, to its corresponding (associated) interval, there are adjacent curves C_μ^i and C_ν^i of $\{C_m^i\}$ and they join the intervals I_1 and I_2, adjacent to I, to their associated (corresponding) intervals.

(e) Let I_1^i be an interval, I_2^j its associate, and I_3^k the associate of its correspondent, I_1^{3-i}; then $I_3^k = I_2^{3-j}$, the correspondent of I_2^j.

The following lemma is an immediate consequence of the above conditions and gives a more complete picture of the way the surface is connected up along the curve corresponding to $B(Q)$.

LEMMA 1. (a) *If I_1^i and I_2^j are associated, then I_1^{3-i} and I_2^{3-j} are also;*

(b) *there are at most two pairs of corresponding intervals which are their own associates;*

(c) *to each point, P_1, which belongs to class II or class III, there correspond exactly three other similar points, P_2, P_3, and P_4.*

We now introduce the idea of a simply covered, closed surface and of a closed surface of class L by the following definitions:

Definition 2. The surface, S, is a *simply covered, closed* surface if it possesses a representation, $x^i = x^i(u, v)$, on Q satisfying the requirements of definition 1 and such that if $(u_1, v_1) \neq (u_2, v_2)$ and (u_1, v_1) and (u_2, v_2) are not corresponding points on $B(Q)$, we have

$$\sum_{i=1}^{N} | x^i(u_2, v_2) - x^i(u_1, v_1) | > 0.$$

Definition 3. A *closed surface, S,* is of *class L* if it is of class L considered as an open surface and (i) $x^i(u, v)$, $(i = 1, \cdots, N)$ is absolutely continuous along $B(Q)$, (ii) $V_0^{(u)1}[x^i(u, V)]$ is metrically continuous in V for $V = 0$ and $V = 1$ $(i = 1, \cdots, N)$, and (iii) $V_0^{(v)1}[x^i(U, v)]$ is metrically continuous in U for $U = 0$ and $U = 1$ $(i = 1, \cdots, N)$.

LEMMA 2. *Let S, $S : x^i = x^i(u, v)$, $(u, v) \epsilon Q$, be a closed surface of class L. Then we can find a sequence, $\{\Pi_n\}$, $\Pi_n : x^i = x_n^i(u, v)$, of closed polyhedra so that the $x_n^i(u, v)$ converge uniformly to the $x^i(u, v)$ and*

$$(7.1) \quad \lim_{n \to \infty} \iint_Q \left\{ \sum_{i=2}^{N} \sum_{j=1}^{i-1} \left| \frac{\partial(x^i, x^j)}{\partial(u, v)} - \frac{\partial(x_n^i, x_n^j)}{\partial(u, v)} \right| \right\} du dv = 0.$$

Proof. The proof consists of the following three parts: (A) in which we form, for each m, certain finite sets, $\{I_{m,n}\} = \{I^1_{m,n}\} + \{I^2_{m,n}\}$, of intervals on $B(Q)$, each set satisfying the postulates, in definition 1, on the sets $\{I_n{}^1\}$ and $\{I_n{}^2\}$; (B) in which we inscribe polygons, C_m, in C so that (i) the representations of C_m on $B(Q)$ satisfy the same conditions with respect to the corresponding sets, $\{I_{m,n}\}$, that the representation of C does with respect to the set $\{I_n\}$, and (ii) the representations of C_m approach that of C uniformly; (C) in which we span certain of the curves, C_{m_n}, of $\{C_m\}$, by means of polyhedra whose representations satisfy the conditions of the theorem and are connected up along C_{m_n}. If $\{I_n\}$ is a finite set of intervals, we define $\{I_{m,n}\} = \{I_n\}$ for each m. Otherwise, we proceed as in (A), below.

(A) Choose all of the intervals, $\bar{\Sigma}_m$, of $\{I_n\} = \{I_n{}^1\} + \{I_n{}^2\}$, which are of length $\leq 1/m$; to these add all intervals (i) (≤ 4 in number) which are their own associates, (ii) which correspond or are associated with an interval of $\bar{\Sigma}_m$, and (iii) which correspond to an interval associated with an interval of $\bar{\Sigma}_m$. Call this set of intervals $\bar{\Sigma}^1_{m,n}$ and the complementary intervals on $B(Q)$, $\bar{\Sigma}^2_{m,n}$. Let $\bar{I}$ be any interval of $\bar{\Sigma}^2_{m,n}$. It abuts on two intervals, $\bar{I}_1$ and $\bar{I}_2$, of $\bar{\Sigma}^1_{m,n}$. Now there is a curve of $\{C_m{}^1\}$ through each of these two intervals. Either these curves join (i) each to its correspondent (in the set $\{I_n\}$), (ii) each to its associate, or (iii) one to its associate and one to its correspondent. Given an interval $\hat{I}_i$ ($\wedge$ being any kind of a mark and i any integer) we shall denote its correspondent by $\hat{I}'_i$, its associate by $\hat{I}_i{}''$, and that interval which is at the same time the correspondent of its associate and the associate of its correspondent by $\hat{I}_i{}'''$.

In cases (i) and (ii) it is clear that there is a single interval $\bar{I}^{(1)}$ (of $\bar{\Sigma}^2_{m,n}$) between $\bar{I}'_1$ and $\bar{I}'_2$, a single interval $\bar{I}^{(2)}$ between $\bar{I}_1{}''$ and $\bar{I}_2{}''$, and a single interval $\bar{I}^{(3)}$ between $\bar{I}_1{}'''$ and $\bar{I}_2{}'''$. In these cases, then, define $I = \bar{I}$, $I' = \bar{I}^{(2)}$, $I'' = \bar{I}^{(1)}$, and $I''' = \bar{I}^{(3)}$.

In case (iii) it is clear that there are single intervals $\bar{I}^{(1)}$ (of $\bar{\Sigma}^2_{m,n}$) between $\bar{I}_1{}''$ and $\bar{I}'_2$, $\bar{I}^{(2)}$ between $\bar{I}'_1$ and $\bar{I}_2{}''$, and $\bar{I}^{(3)}$ between $\bar{I}_1{}'''$ and $\bar{I}_2{}'''$. Now let P be the common end point of $\bar{I}$ and $\bar{I}_1$. There exists an interval $\bar{I}_3$ of $\{I_n\}$ abutting on $\bar{I}_1$ at P. It is clear that $\bar{I}_3$ does *not* abut on $\bar{I}_2$. It also is clear that $\bar{I}'_3$ abuts on $\bar{I}_1{}''$, and occupies part of $\bar{I}^{(1)}$, $\bar{I}_3{}''$ abuts on $\bar{I}'_1$ and occupies part of $\bar{I}^{(2)}$, and $\bar{I}_3{}'''$ abuts on $\bar{I}_1{}'''$ and occupies part of $\bar{I}^{(3)}$. Now define $I = \bar{I} - \bar{I}_3$, $I' = \bar{I}^{(2)} - \bar{I}_3{}''$, $I'' = \bar{I}^{(1)} - \bar{I}'_3$, and $I''' = \bar{I}^{(3)} - \bar{I}_3{}'''$.

We repeat the above process for each interval, $\bar{I}$, of $\bar{\Sigma}^2_{m,n}$ and thus obtain a new set $\Sigma^2_{m,n}$ of intervals in which we have defined the relations of correspondence and association. We obtain the set, $\Sigma^1_{m,n}$ of intervals all belonging to $\{I_n\}$ from $\bar{\Sigma}^1_{m,n}$ by adding to this latter set all the above intervals $\bar{I}_3$, $\bar{I}'_3$, $\bar{I}_3{}''$, and $\bar{I}_3{}'''$. We define the relations of association and correspondence for intervals of this set to be the same as they were as intervals in $\{I_n\}$. We thus obtain the set $\{I_{m,n}\} = \{\Sigma^1_{m,n}\} + \{\Sigma^2_{m,n}\}$ and we see that it can be split up into two sets $\{I^1_{m,n}\}$ and $\{I^2_{m,n}\}$ satisfying the desired conditions.

(B) Merely replace the part of C corresponding to an interval of $\Sigma^2_{m,n}$ by the line segment joining the end points of the arc corresponding to that interval, the representation on $B(Q)$ being linear with respect to the arc length. Divide up each interval of $\Sigma^1_{m,n}$ into a finite number of intervals each of length $\leq 1/m$, in each of which we replace the arc of C corresponding to that interval by the corresponding straight line segment; care must be taken to inscribe the same polygon (in the reverse order of course) in the arc of C corresponding to the corresponding interval, the points of C_m corresponding to corresponding points on $B(Q)$ being geometrically identical. It is clear that $l(C_m) \leq l(C)$ and that the representations of C_m on $B(Q)$ approach that of C.

(C) Let $\tilde{S}_h$ be defined by

$$\tilde{S}_h : x^i = \tilde{x}_h{}^i(u, v), \quad \tilde{x}_h{}^i(u, v) = (1/4h^2) \int_{u-h}^{u+h} \int_{v-h}^{v+h} x^i(\xi, \eta)\, d\xi d\eta,$$

$$(h \leq u, v \leq 1 - h,\ i = 1, \cdots, N).$$

We know that

$$\lim_{h \to 0} \int_\alpha^{1-\alpha} \int_\alpha^{1-\alpha} \left| \frac{\partial(x^i, x^j)}{\partial(u, v)} - \frac{\partial(\tilde{x}_h{}^i, \tilde{x}_h{}^j)}{\partial(u, v)} \right| dudv = 0,$$

$$(0 < \alpha < \tfrac{1}{2},\ i, j = 1, \cdots, N);$$

$$l(\tilde{C}_h) = \int_h^{1-h} \left\{ \sqrt{\sum_{i=1}^N [\tilde{x}^i{}_{h,t}(h, t)]^2} + \sqrt{\sum_{i=1}^N [\tilde{x}^i{}_{h,t}(1 - h, t)]^2} + \sqrt{\sum_{i=1}^N [\tilde{x}^i{}_{h,t}(t, h)]^2} + \sqrt{\sum_{i=1}^N [\tilde{x}^i{}_{h,t}(t, 1 - h)]^2} \right\} dt,$$

where $\tilde{x}^i{}_{h,t}(h, t)$ stands for $\partial\tilde{x}^i(h, t)/\partial t$, etc. Letting $\tilde{x}_h{}^i(u, v) = \tilde{x}_h(u, v)$, we see that

$$\int_h^{1-h} | \tilde{x}_v{}^h(h, v) |\, dv = (1/4h^2) \int_h^{1-h} \left| \int_0^{2h} \int_{-h}^{h} x_\eta(\xi, v + \eta)\, d\xi d\eta \right| dv$$

$$\leq (1/4h^2) \int_{-h}^{h} \left[\int_0^{2h} \left\{ \int_{h+\eta}^{1-h+\eta} | x_v(\xi, v) |\, dv \right\} d\xi \right] d\eta$$

$$\leq (1/2h) \int_0^{2h} V_0{}^{(v)1}[x(\xi, v)]\, d\xi < 2V_0{}^{(v)1}[x(0, v)],$$

for h sufficiently small. Thus it follows that $l(\tilde{C}_h) < M$ (independent of h). Furthermore, we know that the distance between corresponding points of $\tilde{C}_h$ and C approaches zero with h, the correspondence being determined by the radial correspondence, with $(\frac{1}{2}, \frac{1}{2})$ as center, between $B(Q_h)$ and $B(Q)$, $Q_h : h \leq u, v \leq 1 - h$.

Now define $x^i_{m,h}(u, v)$ on the whole of Q by

$$x^i_{m,h}(u, v) = \tilde{x}_h{}^i(u, v),\ (u, v) \epsilon Q_h, \qquad (i = 1, \cdots, N);$$
$$x^i_{m,h}(u, v) = \lambda x_m{}^i(U, V) + (1 - \lambda)\tilde{x}_h{}^i[(1 - 2h)U + h, (1 - 2h)V + h],$$
$$\begin{cases} u = \lambda U + (1 - \lambda)[U(1 - 2h) + h] \\ v = \lambda V + (1 - \lambda)[V(1 - 2h) + h] \\ (U, V) \epsilon B(Q), \\ 0 \leq \lambda \leq 1,\ (i = 1, \cdots, N); \end{cases}$$

where $x^i = x_m{}^i(U, V)$ is the above representation of C_m on $B(Q)$. For each m and each $h > 0$, it is clear that $x^i_{m,h}(u, v)$ is continuous together with its first partial derivatives except along a finite number of segments of Q, all corresponding to rectifiable curves. It is also clear that

$$\lim_{\substack{h \to 0 \\ m \to \infty}} \iint_Q \left\{ \sum_{i=2}^{N} \sum_{j=1}^{i-1} \left| \frac{\partial(x^i_{m,h}, x^j_{m,h})}{\partial(u, v)} - \frac{\partial(x^i, x^j)}{\partial(u, v)} \right| \right\} du dv = 0.$$

Hence it is clear that we can replace the surfaces, $S_{m,h}$, by a sequence of polyhedra spanning certain of the C_m and having the desired properties.

The proof of the above theorem clearly gives a method of constructing sequences of closed polyhedra which approach any given closed surface, S. Let P be a point not on a given closed surface, S, in ordinary 3-space and let its distance from S be ρ. Let Π_1 and Π_2 be closed polyhedra represented on Q so that the distance between corresponding points of Π_i and S is $< \rho/2$, $(i = 1, 2)$. Then it is clear * that $O(P, \Pi_1) = O(P, \Pi_2) =$ an integer. Thus it is clear that we can define $O(P, S) = \lim_{n \to \infty} O(P, \Pi_n)$ where $\{\Pi_n\}$ is any sequence of closed polyhedra approaching S and P is not on S; as before, we define $O(P, S) = 0$, for P on S. The following four lemmas are well known for surfaces closed in the ordinary sense and may easily be extended as below to the more general closed surfaces treated here (S is assumed to be in ordinary 3-space for the remainder of this section).

LEMMA 3. *The function* $O(P, S)$ *is continuous in* P *and* S *if* P *is not on* S.

LEMMA 4. *If two points* P_1 *and* P_2 *can be joined by an arc which crosses the closed surface* S *and intersects it in exactly one point in the neighborhood of which* S *is a* $1 - 1$ *image of a Jordan region, then* $| O(P_1, S) - O(P_2, S) | = 1$.

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* See Tannery, *loc. cit.*

Proof. Let S be represented on Q so that the given point p_0 corresponds to a point interior to Q and the representation is $1-1$ in a Jordan open neighborhood r of p_0 in Q. This can clearly always be done for if the point p_0 is on the boundary it necessarily is of class I or II and we may merely move the cut slightly.

Then let $\{\Pi_n\}$ be a sequence of closed polyhedra approaching S, the representation of each of which is $1-1$ and continuous on r.

Now the arc joining P_1 and P_2 has no points in common with $S-R$, R being the portion of S corresponding to r, and hence is at a distance $d>0$ from $S-R$. Now for n sufficiently large, the arc is at a distance $> d/2$ from $\Pi_n - R_n$, R_n being the part of Π_n corresponding to r; hence any points the arc has in common with Π_n are in R_n. Let P'_n be the first point (for n large, such points will obviously exist) of $P_1P_2 \cdot R_n$ and P''_n the last such point. Replace arc $P'_nP''_n$ of P_1P_2 by an arc of R_n joining them; it is clear that we can modify the new arc P_1P_2 slightly so that it has just one point in common with R_n. Thus we have reduced our theorem to the case where S is a polyhedron, for which case it is already known.

LEMMA 5. *A necessary and sufficient condition that a closed set in three-space be completely disconnected* (i. e. contain no connected subset containing two points) *is that there exist a sequence of simply covered closed surfaces of finite genus containing no points of the set, and closing down on any given point of the set.*

Proof. This lemma is obvious; merely consider the boundaries of the set of points at a distance $\leq \rho$ from the given set for each $\rho > 0$.

LEMMA 6. *If S is a simply covered closed surface, it divides space into two regions, an "interior" in which $O(P,S) = \pm 1$, and an "exterior" in which $O(P,S) = 0$.*

Proof. Let $x^i = x^i(u,v)$ be a representation of S satisfying the conditions of definition 3. Let P_0 be the point of S corresponding to p_0, an interior point of Q, say. We shall show below that every point of space not on S can be joined to P_0 by an arc containing no point of S other than P_0. When this is shown, we choose a simple closed surface Σ including P_0 and which has in common with S, a simple closed curve which divides Σ into just two parts Σ_1 and Σ_2 on opposite sides of S. Now ∞ can be joined to P_0 by an arc as above; let P_2 be the first point of this arc on Σ; call the part of Σ on which it lies Σ_2; let P_1 be a point of Σ_1. Now P_1 and P_2 can be joined by an arc which cuts S in exactly one point so $|O(P_1,S) - O(P_2,S)| = 1$ and hence $O(P_1,S) = \pm 1$, since $O(P_2,S) = 0$, by lemma 3. Now let P be any point of space and PP_0 the above arc joining it to P_0. Let P' be the first point of PP_0 on Σ; if $P' \epsilon \Sigma_1$, P can clearly be joined to P_1 by an arc having no points in common with S, $O(P, S) = O(P_1, S) = \pm 1$, and we say $P \epsilon I_1$. Otherwise $O(P, S) = O(P_2, S) = 0$ and $P \epsilon I_2$. Clearly I_1 and I_2 are regions and, by lemma 3, mutually exclusive. Obviously S is their common boundary, since p_0 was an arbitrary point not on the cut.

Now, let P be any point not on S. Let R be a point of S nearest to P. If R corresponds to a point interior to Q, well known theorems of analysis situs show that P can be joined as stated to P_0. If R corresponds to a point on $B(Q)$ of classes I or II, the situation is precisely the same as if S were an ordinary simply covered closed surface of finite connectivity and R was on a cut. In this case, also, P can be joined as stated to P_0. Now the set of points on $B(Q)$ of class III is completely disconnected and closed so that it is easy to see that the corresponding set of points on S also has this property. Hence if R is one of these, let σ be a simply covered closed surface of finite connectivity containing R in its interior, of diameter $<$ one-half that of S, and containing no points corresponding to points of class III on $B(Q)$. Since S is connected $\sigma \cdot S \neq 0$ and hence P can be joined as stated to a point of S not corresponding to a point of class III on $B(Q)$ and hence also to P_0.

The following lemma also is known and follows readily from the meaning of the expressions involved.

LEMMA 7. *If* Π *is a closed polyhedron and* $F(x, y, z)$ *is a function continuous with its first partial derivative with respect to* z, *and defined throughout space,*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\partial F/\partial z)\, O(x, y, z; \Pi)\, dx dy dz$$
$$= \int_{-R}^{R} \int_{-R}^{R} \int_{-R}^{R} (\partial F/\partial z)\, O(x, y, z; \Pi)\, dx dy dz = \iint_{\Pi} F(x, y, z)\, dx dy,$$

for R *sufficiently large.*

The following two lemmas, together with lemma 2, from a connecting link between the above general geometric concepts and closed surfaces of class L.

LEMMA 8. *If* S, $S : x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, *is a closed surface of class* L, *and the point* (r, s, t) *is not on* S, *then*

$$O(r, s, t; S) = \frac{1}{4\pi} \iint_{Q} \begin{vmatrix} x(u, v) - r & y(u, v) - s & z(u, v) - t \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \frac{du dv}{\rho^3},$$
$$\rho = [(x - r)^2 + (y - s)^2 + (z - t)^2]^{1/2}.$$

Proof. Let $\{\Pi_n\}$ be any sequence of closed polyhedra approaching S such that

$$\lim_{n\to\infty}\iint_Q \left\{ \left| \frac{\partial(x_n, y_n)}{\partial(u,v)} - \frac{\partial(x,y)}{\partial(u,v)} \right| + \left| \frac{\partial(y_n, z_n)}{\partial(u,v)} - \frac{\partial(y,z)}{\partial(u,v)} \right| + \left| \frac{\partial(z_n, x_n)}{\partial(u,v)} - \frac{\partial(z,x)}{\partial(u,v)} \right| \right\} dudv = 0,$$

where $\Pi_n : x = x_n(u,v),\ y = y_n(u,v),\ z = z_n(u,v)$, and x_n, y_n, z_n approach x, y, and z uniformly. Then we know that $O(r, s, t, \Pi_n)$ approaches $O(r, s, t, S)$ and the integral expression for $O(r, s, t; \Pi_n)$ approaches the corresponding one with Π_n replaced by S. This proves the lemma.

LEMMA 9. *Let S, $S : x = x(u,v), y = y(u,v), z = z(u,v)$, be a closed surface of class L, and having zero three dimensional measure. Then $O(x, y, z, S)$ is integrable all over space and we can find a sequence of polyhedra $\{\Pi_n\}$, $\Pi_n : x = x_n(u,v)$, etc., such that*

(i) *$x_n(u,v)$, $y_n(u,v)$, and $z_n(u,v)$ approach $x(u,v)$, $y(u,v)$, and $z(u,v)$ uniformly;*

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} | O(x, y, z; \Pi_n) |\, dxdydz < M,$$

M independent of n;

(iii) $$\lim_{n\to\infty}\iint_Q \left\{ \left| \frac{\partial(x_n, y_n)}{\partial(u,v)} - \frac{\partial(x,y)}{\partial(u,v)} \right| + \left| \frac{\partial(y_n, z_n)}{\partial(u,v)} - \frac{\partial(y,z)}{\partial(u,v)} \right| + \left| \frac{\partial(z_n, x_n)}{\partial(u,v)} - \frac{\partial(z,x)}{\partial(u,v)} \right| \right\} dudv = 0;$$

(iv) $$\lim_{n\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} O(x, y, z; \Pi_n)\, dxdydz = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} O(x, y, z; S)\, dxdydz$$

for each set of finite constants (a_i, b_i).

Proof. In the first place, it is clear that

$$F^{(x)}(x, y, z) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \frac{| x - r |\, drdsdt}{[(x-r)^2 + (y-s)^2 + (z-t)^2]^{3/2}} = \int_{a_1-x}^{b_1-x}\int_{a_2-y}^{b_2-y}\int_{a_3-z}^{b_3-z} \frac{| r |\, drdsdt}{(r^2+s^2+t^2)^{3/2}},$$

the function $F_2^{(x)}(x, y, z)$ obtained from $F_1^{(x)}(x, y, z)$ by replacing $| x - r |$

by $(x - r)$, and the corresponding $F_i^{(y)}$, $F_i^{(z)}$, $(i = 1, 2)$, are defined, bounded, and uniformly continuous over all of space.

By lemma 2, we may find a sequence, $\{\Pi_n\}$, of closed polyhedra satisfying (i) and (iii). Using the results of the above paragraph, we see that

$$(7.2)\quad \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |\, O(x, y, z; \Pi_n) \,|\, dxdydz$$
$$\leq \int_0^1\int_0^1 \Big\{ F_1^{(x)}(x_n, y_n, z_n) \left| \frac{\partial(y_n, z_n)}{\partial(u, v)} \right| + F_1^{(y)}(x_n, y_n, z_n) \left| \frac{\partial(z_n, x_n)}{\partial(u, v)} \right|$$
$$+ F_1^{(z)}(x_n, y_n, z_n) \left| \frac{\partial(y_n, z_n)}{\partial u \partial v} \right| \Big\} dudv < M, \qquad (n = 1, 2, \cdots).$$

Thus, $O(x, y, z; S)$ is integrable over all of space, being zero outside a large sphere; this demonstrates (ii). Furthermore, by lemma 8, since S is of measure zero, we can write

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} O(x, y, z; S)\, dxdydz$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} (1/4\pi) \iint_Q \begin{vmatrix} x(u,v) - r & y(u,v) - s & z(u,v) - t \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \frac{dudv}{\rho^3},$$
$$\rho = [(x - r)^2 + (y - s)^2 + (z - t)^2]^{1/2}.$$

Using the inequality (7.2), we see that we may interchange the order of integration and get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} [O(x, y, z; \Pi_n) - O(x, y, z; S)]\, dxdydz$$
$$= \iint_Q \Big\{ \left[\frac{\partial(y_n, z_n)}{\partial(u, v)} - \frac{\partial(y, z)}{\partial(u, v)} \right] F_2^{(x)}(x_n, y_n, z_n)$$
$$+ \frac{\partial(y, z)}{\partial(u, v)} [F_2^{(x)}(x_n, y_n, z_n) - F_2^{(x)}(x, y, z)] \Big\} dudv + \bigstar,$$

where the ★ indicates the other terms obtained from these two by permuting the letters x, y, z cyclically. It is clear that the expression on the right approaches zero with $1/n$.

8. *Green's and Stokes' formulas.*

Definition 1.* A function $f(x)$ is said to be *approximately continuous at* x_0, if for every $\epsilon > 0$, the set of values of x for which $|f(x) - f(x_0)| < \epsilon$ is of metric density unity at x_0 (x may stand for n-variables).

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* Hobson, *Functions of a real variable*, vol. I.

Remark 1. A measurable function is approximately continuous almost everywhere.

Remark 2. A bounded measurable function is metrically continuous at every point of approximate continuity.

LEMMA 1. *Suppose* (i) $F(x,y,z)$ *is defined and measurable in an open region* D, *and* (ii) *for almost all* (x,y), $F(x,y,z)$ *is continuous in* z *on every open interval of the line* $x=x$, $y=y$ *which lies in* D. *Then* $F(x,y,z)$ *is approximately continuous at every point* (x,y,z) *in* D *for which* (x,y) *does not belong to a certain set of measure zero.*

Proof. We can find a sequence $\{D_i\}$ of closed cubes, each lying entirely interior to D and such that each point of D is *interior* to one of the D_i. Thus, if we prove the lemma for the case that D is the unit cube and with the continuity in hypothesis (ii) uniform, the lemma will clearly hold for the general case.

Thus, let D be the unit cube and define

$$\phi(x,y,z,h)=\max_{z\leq\zeta\leq z+h}|F(x,y,\zeta)-F(x,y,z)|=\overline{\lim_{n\to\infty}}|F(x,y,r_n)-F(x,y,z)|$$

where the sequence, $\{r_n\}$, consists of all rational numbers, $z\leq r_n\leq z+h$, (x,y) being a point for which F is continuous in z. It is easy to see that, for h fixed, $\phi(x,y,z,h)$ is continuous in z and measurable in (x,y,z). Hence, for each h, $\eta(x,y,h)$ is measurable in (x,y) where

$$\eta(x,y;h)=\max_{0\leq z\leq 1-h}\phi(x,y,z,h),$$

(x,y) again being a point such that F is continuous in z, being defined arbitrarily for all other (x,y). Also, if (x,y) belongs to S_1, the set of points (x,y) for which $F(x,y,z)$ is uniformly continuous in z, then $\eta(x,y;h)$ approaches zero with h.

Now, since $F(x,y,z)$ is measurable, it is measurable in (x,y) for almost all z, and, since it is continuous in z, therefore measurable in (x,y) for *each* fixed z. Let S_2 be the set, of measure unity, of points (x_0,y_0) such that $F(x,y,z)$ is metrically continuous in (x,y) at (x_0,y_0) for each rational z. Let S_3 be the set where $\eta(x,y,h)$ is approximately continuous for each rational h. Then $S=S_1\cdot S_2\cdot S_3$ is of measure unity.

Let (x_0,y_0,z_0) be a point in the interior of D such that (x_0,y_0) is in S. Choose an $\epsilon>0$. Choose a rational h_0 so small that $\eta(x_0,y_0,h_0)<\epsilon/4$, and (x_0,y_0,z_0+h) is in D for $|h|<h_0$. Choose a rational $\bar{z}$ so that $|\bar{z}-z_0|<h_0/2$ and hence

$$| F(x_0, y_0, z_0) - F(x_0, y_0, \bar{z}) | < \epsilon/4.$$

Now we know: (1) that $\eta(x, y, h_0)$ is approximately continuous in (x, y) at (x_0, y_0) so that we can find a set, E_1, of metric density unity at (x_0, y_0) in which $\eta(x, y, h_0) < \epsilon/2$, (2) that $F(x, y, \bar{z})$ is approximately continuous in (x, y) at (x_0, y_0) so that we can find a set, E_2, of metric density unity at (x_0, y_0) so that $| F(x, y, \bar{z}) - F(x_0, y_0, \bar{z}) | < \epsilon/4$ for (x, y) in E_2. Then let (x, y, z) be any point of D such that $(x, y) \epsilon E_1 \cdot E_2$ and $| z - z_0 | < h_0/2$. Then, since $| z - \bar{z} | < h_0$, we have

$$| F(x, y, z) - F(x_0, y_0, z_0) | < \epsilon.$$

Clearly the above determined set of points (x, y, z) is of metric density unity at (x_0, y_0, z_0).

LEMMA 2. *Suppose that S, S : $x = x(u, v)$, $y = y(u, v)$, is a surface of class L. Let $E_{x,y}$ be a subset, of measure zero, of the set, $Q_{x,y}$, of points into which Q is carried by S. Let D be the set of points of Q where $\partial(x, y)/\partial(u, v) \neq 0$, and E all the points of Q which are carried by S into points of $E_{x,y}$. Then*

$$\text{meas } (E \cdot D) = 0.$$

Proof. Suppose $m_e(E \cdot D) > 0$. Let $\{O_p\}$ be a sequence of open sets each including the next and covering $E_{x,y}$ such that $\lim\limits_{p \to \infty} m(O_p) = 0$. Then the sets, E_p, of points of Q which are carried into points of $O_p \cdot Q_{x,y}$ are all open, each includes the next, and all include E. Then, if $\bar{E} = \prod\limits_{p=1}^{\infty} E_p$, $m(\bar{E} \cdot D) > 0$ and $m(\bar{E}_{x,y}) = 0$. Let $\bar{E}_l$ be the subset of $\bar{E}$ in Q_l, Q_l : $l \leq u$, $v \leq 1 - l$. Then for l_0 sufficiently small, $m(\bar{E}_{l_0} \cdot D) > 0$. Let $\bar{Q}$ be a rectangle of Q including Q_{l_0} in its interior, along the boundary of which $x(u, v)$ and $y(u, v)$ satisfy the hypotheses of lemma 4, § 4. Now let $\bar{F} = \bar{E} \cdot \bar{Q} \cdot D$ and $\bar{F}_r^+$ be the subset of F in which $\partial(x, y)/\partial(u, v) \geq r$ and $\bar{F}_r^-$ that where $\partial(x, y)/\partial(u, v) \leq -r$. Then for r small enough $m(\bar{F}_r^+) > 0$ or $m(\bar{F}_r^-) > 0$. Choose one of these of positive measure and let Σ be a closed subset of that one for which $m(\Sigma) > 0$. We shall show that $m(\Sigma_{x,y}) > 0$ which will contradict the fact that $m(\bar{E}_{x,y}) = 0$, since $\Sigma_{x,y} \subset \bar{E}_{x,y}$.

Let $\{T_n\}$ be a sequence of subdivisions of $\bar{Q}$ into rectangles, $R_i^{(n)}$, $(i = 1, \cdots, N_n)$ of diameter $< \epsilon_n$ (where $\lim\limits_{n \to \infty} \epsilon_n = 0$) by means of lines parallel to the axes, such that $x(u, v)$ and $y(u, v)$ satisfy the hypotheses of lemma 4, § 4 on the boundary of each rectangle of each subdivision. Suppose also that the division lines of T_n are included among those of T_{n+1}. For each point (s, t) not on any of the curves (the totality of which form a set of measure zero) corresponding to these division lines, define

$$M(s, t; \bar{S}) = \lim_{n\to\infty} M_n(s, t; \bar{S}); \ M_n(s, t; \bar{S}) = \sum_{i=1}^{N_n} | O_{x,y}(s, t; C_i^{(n)}) |,$$

where, as usual, $C_i^{(n)}$ is the curve of $\bar{Q}_{x,y}$ corresponding to $B(R_i^{(n)})$. This limit clearly exists since the sum on the right cannot decrease with n. From the definition of $Y(S)$ and Theorem 1, § 4, it follows immediately that

$$Y(S) = L(S) \geq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M_n(s, t; \bar{S})\, ds dt.$$

$$\therefore \quad Y(S) = L(S) \geq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M(s, t; \bar{S})\, ds dt,$$

and $M(s, t; \bar{S})$ is summable and finite except on a set of measure zero.

Let (s_0, t_0) be a point where $M(s_0, t_0; \bar{S})$ is defined, finite, and > 0. Then there is at least one point, P, of Q corresponding to (s_0, t_0). Let P be such a point; it does not lie on any division line of one of the T_n. Let $R_1^{(n)}$ be the unique rectangle of T_n which contains P in its interior. For each $j \geq 2$

$$O_{x,y}(s_0, t_0; C_1^{(j)}) = O_{(x,y)}(s_0, t_0; C_1^{(j-1)}) - \sum_{i=2}^{p_j} O_{x,y}(s_0, t_0; C_i^{(j)})$$

where $C_1^j, \cdots, C_{p_j}^j$ are the rectangles of T_j which add up to C_1^{j-1}. Thus

$$O_{x,y}(s_0, t_0; C_1^{(j)}) = O_{x,y}(s_0, t_0; C_1^{(1)}) - \sum_{k=2}^{j} \sum_{i=2}^{p_k} O_{x,y}(s_0, t_0; C_i^{(k)}).$$

But now

$$M(s_0, t_0; \bar{S}) \geq \sum_{k=2}^{j} \sum_{i=2}^{p_k} | O_{x,y}(s_0, t_0; C_i^{(k)}) |$$

for every j. Since M is finite, it follows that there exists a j_p such that

$$O_{x,y}(s_0, t_0; C_1^{j}) = O_{x,y}(s_0, t_0; C_1^{j_p}), \qquad j \geq j_p.$$

Hence for each point P corresponding to (s_0, t_0), $\lim_{j\to\infty} O_{x,y}(s_0, t_0; C_1^{j})$ exists and can be different from zero at at most $M(s_0, t_0; \bar{S})$ distinct such points.

Now, let (s, t) be any point not on the above division curves. Let H_n be the set of rectangles of T_n which contain points of Σ and define

$$L_{\bar{S}}^{(n)}(s, t; \Sigma) = \sum_{R_i^{(n)} \epsilon H_n} O(s, t; C_i^{(n)}).$$

Now $\Sigma = \prod_{n=1}^{\infty} \bar{H}_n$ ($\bar{H}_n = H_n$ plus its boundary), so that (1) if (s, t) is not in $\Sigma_{x,y}$, any point corresponding to it will be outside $\bar{H}_n$ for n large enough and so $L_{\bar{S}}^{n}(s, t; \Sigma) = 0$ for such values of n. (2) if (s, t) is in $\Sigma_{x,y}$ and $M(s, t; \bar{S})$ is finite, then it is clear (from the discussion in the previous paragraph and from the fact that Σ is closed), that there is a function $L_{\bar{S}}(s, t; \Sigma)$ [defined almost everywhere and zero outside $\Sigma_{x,y}$] such that, for almost every (s, t),

$$\lim_{n\to\infty} L_{\bar{S}}^{(n)}(s, t; \Sigma) = L_{\bar{S}}(s, t; \Sigma).$$

From lemma 4, § 4, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L_{\bar{S}}^{(n)}(s, t; \Sigma)\, ds dt = \iint_{H_n} \frac{\partial(x, y)}{\partial(u, v)}\, du dv.$$

Now, since $| L_{\bar{S}}^{(n)}(s, t; \Sigma) | \leq M(s, t; \bar{S})$ and $\lim_{n\to\infty} L_{\bar{S}}^{(n)}(s, t; \Sigma) = L_{\bar{S}}(s, t; \Sigma)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L_{\bar{S}}(s, t; \Sigma)\, ds dt = \iint_{\Sigma} \frac{\partial(x, y)}{\partial(u, v)}\, du dv = \iint_{\Sigma_{x,y}} L_{\bar{S}}(s, t; \Sigma)\, ds dt \neq 0.$$

Hence $m(\Sigma_{x,y}) > 0$.

THEOREM 1. *Let*

(A) *D be an open region in space and R another one which lies, together with its boundary, entirely interior to D;*

(B) *the boundary of R consist of the oriented simply covered closed surfaces S_i,*

$$S_i : x = x_i(u, v),\ y = y_i(u, v),\ z = z_i(u, v), \quad (i = 1, 2, \cdots, (u, v) \epsilon Q),$$

of class L such that (1) $\sum_{i=1}^{\infty} \left| \frac{\partial(x_i, y_i)}{\partial(u, v)} \right|$ *is integrable over Q and* (2) *the space measure of these surfaces together with their limit points is zero;*

(C) *$F(x, y, z)$ be a function* (1) *defined and measurable in D,* (2) *absolutely continuous in z for almost all (x, y), and* (3) *such that $D_z(F)$ is summable in D, $D_z(F)$ being one of the four Dini partial derivatives of F.*

Then:

(i) $F_{m,n}[x_i(u, v),\ y_i(u, v),\ z_i(u, v)] \dfrac{\partial(x_i, y_i)}{\partial(u, v)}$ *is measurable* $(i = 1, 2, \cdots)$;

(ii) $\dfrac{\partial F}{\partial z}$ *exists almost everywhere in D and is summable; and*

$$\text{(iii)}\quad \int\!\!\int_R\!\!\int \frac{\partial F}{\partial z}\,dxdydz = \lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=1}^{\infty}\int\!\!\int_Q F_{m,n}(x_i, y_i, z_i)\,\frac{\partial(x_i, y_i)}{\partial(u, v)}\,dudv,$$

$$F_{m,n}(x, y, z) = \begin{cases} F(x, y, z), & -m \leq F(x, y, z) \leq n \\ -m, & F(x, y, z) < -m \\ n, & F(x, y, z) > n, \text{ or } F(x, y, z) \text{ undefined.} \end{cases}$$

Proof. In the first place, it is easy to show that $\partial F/\partial z$ exists almost everywhere and is summable (Cf. § 1, Theorem 1). We also remark here that $\partial F_{m,n}/\partial z$ exists almost everywhere, has the same sign as $\partial F/\partial z$ and is summable, satisfying

$$| \partial F_{m,n}/\partial z | \leq | \partial F/\partial z |,$$

wherever both exist.

For m_0 sufficiently large, we see that the set, G_m, of points (not in R) bounded by S_m, $m > m_0$, is a subset of D. Hence define R_m and Σ_m by

$$R_m = R + \sum_{i=m+1}^{\infty} (S_i + G_i), \quad m > m_0, \qquad \Sigma_m = \sum_{i=1}^{m} S_i.$$

Now let $\{\Pi_{p,q}\}$ be a sequence of closed polyhedra fulfilling the conditions (i), (ii), (iii) and (iv) of the conclusion of lemma 7, § 7, where the S of that lemma is the surface S_p. Call $R_{p,q}$ the region bounded by $Z + \Sigma_{p,q}$ where Z is the set of limit points of the S_i which do not belong to any S_i and $\Sigma_{p,q} = \sum_{i=1}^{p} \Pi_{i,q}$.

Now, it is clear that $F^{(h)}_{m,n}(x, y, z)$ and $\partial F^{(h)}_{m,n}/\partial z$ are uniformly continuous in D_h, their region of definition and therefore that

$$\text{(8.1)}\quad \lim_{q\to\infty}\int\!\!\int_Q F^{h}_{m,n}[x_{p,q}(u, v), y_{p,q}(u, v), z_{p,q}(u, v)]\,\frac{\partial(x_{p,q}, y_{p,q})}{\partial(u, v)}\,dudv$$

$$= \int\!\!\int_Q F^{(h)}_{m,n}(x_p, y_p, z_p)\,\frac{\partial(x_p, y_p)}{\partial(u, v)}\,dudv,$$

$$F^{(h)}_{m,n} = (1/h^3) \int_x^{x+h}\int_y^{y+h}\int_z^{z+h} F_{m,n}(\xi, \eta, \zeta)\,d\xi d\eta d\zeta.$$

Furthermore, using a well known theorem on Stieltjes integrals, defining the set functions $\phi_q(R)$ and $\phi(R)$ by

$$\phi_q(R) = \int\!\!\int_R\!\!\int O(x, y, z; \Sigma_{p,q})\,dxdydz, \quad \phi(R) = \int\!\!\int_R\!\!\int O(x, y, z; \Sigma_p)\,dxdydz,$$

and observing that, for q sufficiently large, all the points where $O(x, y, z; \Sigma_{p,q}) \neq 0$ lie in D_h, and that $O(x, y, z; \Sigma_p)$ is unity inside R_p and zero elsewhere (by lemma 6, § 7), we see that

$$\lim_{q\to\infty}\iiint_{D_h} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, O(x, y, z; \Sigma_{p,q})\, dxdydz = \lim_{q\to\infty}\iiint_{D_h} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, d\phi_q(e) = \iiint_{D_h} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, d\phi(e) = \iiint_{R_p} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, dxdydz. \tag{8.2}$$

By § 7, lemma 7, we have, for each sufficiently large q,

$$\iiint_{D_h} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, O(x, y, z; \Sigma_{p,q})\, dxdydz = \sum_{i=1}^{p} \iint_{Q} F^{(h)}_{m,n}(x_{i,q}, y_{i,q}, z_{i,q}) \frac{\partial(x_{i,q}, y_{i,q})}{\partial(u, v)}\, dudv. \tag{8.3}$$

From (8. 1) and (8. 2), we may allow q to become infinite, and we see that

$$\iiint_{R_p} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, dxdydz = \sum_{i=1}^{p} \iint_{Q} F^{(h)}_{m,n}(x_i, y_i, z_i) \frac{\partial(x_i, y_i)}{\partial(u, v)}\, dudv. \tag{8.4}$$

Now, since $\partial F^{(h)}_{m,n}/\partial z$ is summable, $F^{(h)}_{m,n}$ is bounded, and $\sum_{i=1}^{\infty} \left| \frac{\partial(x_i, y_i)}{\partial(u, v)} \right|$ is integrable over Q, we may let p become infinite in (8. 4) getting

$$\iiint_{R} \frac{\partial F^{(h)}_{m,n}}{\partial z}\, dxdydz = \sum_{i=1}^{\infty} \iint_{Q} F^{(h)}_{m,n}(x_i, y_i, z_i) \frac{\partial(x_i, y_i)}{\partial(u, v)}\, dudv. \tag{8.5}$$

Now, for each $h > 0$, each integrand on the right is measurable and is zero when $\partial(x_i, y_i)/\partial(u, v) = 0$. By lemmas 1 and 2, $F^{(h)}_{m,n}(x_i, y_i, z_i)$ converges to $F_{m,n}(x_i, y_i, z_i)$ at almost all points of Q where $\partial(x_i, y_i)/\partial(u, v) \neq 0$. Thus condition (i) of the conclusion is demonstrated.

Since $F_{m,n}(x, y, z)$ is bounded and $\partial F_{m,n}/\partial z$ is summable, we see that

$$\iiint_{R} \frac{\partial F_{m,n}}{\partial z}\, dxdydz = \sum_{i=1}^{\infty} \iint_{Q} F_{m,n}(x_i, y_i, z_i) \frac{\partial(x_i, y_i)}{\partial(u, v)}\, dudv.$$

It is now clear that we may allow m and n to become infinite independently, thus obtaining the formula (iii).

Remark. If we restrict F and $\partial F/\partial z$ to be defined and bounded in a bounded region D, which contains the closed surfaces S_i, of class L, and their limit points, the whole being of space measure zero, and also the set R of all points for which

$$\sum_{i=1}^{\infty} O(x, y, z; S_i) = O(x, y, z; \Sigma) \neq 0, \qquad \Sigma = \sum_{i=1}^{\infty} S_i(+z),$$

this function being integrable, the above proof clearly establishes the formula

$$\int\int_D\int \frac{\partial F}{\partial z} O(x, y, z; \Sigma)\, dxdydz = \sum_{i=1}^{\infty} \int_Q\int F(x_i, y_i, z_i) \frac{\partial(x_i, y_i)}{\partial(u, v)}\, dudv.$$

A method of proof very similar to the above serves to establish the theorem below. It is clear that this theorem may be proved with the hypotheses altered in many different ways.

THEOREM 2. *Let*

(A) *S, $S: x = x(u, v), y = y(u, v), z = z(u, v)$, be a surface of class L such that x, y, and z are absolutely continuous on $B(Q)$ with $V_0^{(u)1}[f(u, V)]$ metrically continuous at $V = 0$ and $V = 1$, and $V_0^{(v)1}(f)$ metrically continuous at $U = 0$ and $U = 1$, $f = x, y, z$ in turn.*

(B) *R be a region including S and*

(C) *$P(x, y, z)$ be a function (1) defined and measurable in R, (2) absolutely continuous in y for almost all (x, z) and in z for almost all (x, y), (3) summable in R together with $\partial P/\partial y$ and $\partial P/\partial z$ which are assumed to exist almost everywhere in R, (4) metrically continuous in (x, y, z) at every point on C, x not belonging to Z_x of measure zero, and dominated on C by a function summable on C. (5) with $\partial P/\partial y$ metrically continuous at every point, (x, y, z), of S, (x, y) not in $Z_{x,y}$ of measure zero, and $\partial P/\partial z$ metrically continuous at every point (x, y, z) of S, (x, z) not in $Z_{x,z}$ of measure zero, where there exists a summable function, $G(u, v)$, such that*

$$\left| \frac{\partial P}{\partial y} \frac{\partial(x, y)}{\partial(u, v)} \right| + \left| \frac{\partial P}{\partial z} \frac{\partial(x, z)}{\partial(u, v)} \right| < G(u, v).$$

Then:

(i) *P is measurable and thus summable on C, and $\dfrac{\partial P}{\partial y} \dfrac{\partial(y, x)}{\partial(u, v)}$ and $\dfrac{\partial P}{\partial z} \dfrac{\partial(z, x)}{\partial(u, v)}$ are measurable and thus summable on Q, and*

(ii) $$\int_Q\int \left[\frac{\partial P}{\partial y} \frac{\partial(y, x)}{\partial(u, v)} + \frac{\partial P}{\partial z} \frac{\partial(z, x)}{\partial(u, v)} \right] dudv = \int_{B(Q)} P(x, y, z) \frac{\partial x}{\partial s}\, ds,$$

s = arc length on $B(Q)$.

COROLLARY. *We see that hypothesis* (C) *is fulfilled for the following classes of functions for every surface* S *in* R, *which satisfies hypothesis* (A):

(1) $P(x, y, z)$, $\partial P/\partial y$, *and* $\partial P/\partial z$ *are continuous in* R,

(2) $P(x, y, z)$, $\partial P/\partial y$, *and* $\partial P/\partial z$ *are all bounded and measurable in a cube containing* R, *in which* $P(x, y, z)$ *has the form*

$$P(x, y, z) = P(x, a, b) + \int_a^y P_1(x, \eta, b)\, d\eta + \int_b^z P_2(x, a, \zeta)\, d\zeta + \int_a^y \int_b^z P_3(x, \eta, \zeta)\, d\eta d\zeta,$$

where a *and* b *are independent of* x *and the functions* $P_1(x, \eta, b)$, $P_2(x, a, \zeta)$, *and* $P_3(x, \eta, \zeta)$ *are summable in the Greek letters for almost all values of* x.

NON-ALTERNATING TRANSFORMATIONS.

By G. T. Whyburn.

If A and B are compact metric spaces and $T(A) = B$ is a single valued continuous transformation which, as indicated, sends A into B, then T is said to be *non-alternating* provided that for no two distinct points x and y of B does the set $T^{-1}(x)$ separate the set $T^{-1}(y)$ in A, i. e., there exists no separation $A - T^{-1}(x) = A_1 + A_2$ such that $A_1 \cdot T^{-1}(y) \neq 0 \neq A_2 \cdot T^{-1}(y)$. It will be noted that in case A is a circle, this is equivalent to saying that for any two points x and y of B no pair of points of $T^{-1}(x)$ alternates with any pair of points of $T^{-1}(y)$ on A.* Whence the term *non-alternating.* Such a transformation T will be called *monotone* † provided that for each $x \epsilon B$, the set $T^{-1}(x)$ is connected. Obviously any monotone transformation is also non-alternating.

Since any continuous transformation of A into B is equivalent ‡ to an upper semi-continuous decomposition § of A into disjoint closed sets with hyperspace homeomorphic with B, it follows that any non-alternating or monotone transformation of A into B is equivalent to such a decomposition into sets which do not separate each other in A or into continua in A respectively.

1. Characteristic properties. Throughout this section it will be supposed that A and B are compact and metric and that $T(A) = B$ is continuous.

(1.1) *If b is any limit point of a subset B_1 of B, then* $T^{-1}(b) \cdot \overline{T^{-1}(B_1)} \neq 0$.

For let points $b_i \epsilon B_1$ be selected so that b_i converges to b. For each i select a point a_i from $T^{-1}(b_i)$. We can select a sequence of integers n_i such that a_{n_i} converges to some point a of A. Now by the continuity of T we have $T(a) = b$, and this gives $a \epsilon T^{-1}(b) \cdot \overline{T^{-1}(B_1)}$.

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* In connection with this condition see A. Denjoy, *Comptes Rendus*, vol. 197 (1933), p. 572.

† This term has been suggested by C. B. Morrey. See his paper "The topology of path surfaces."

‡ See Kuratowski, *Fundamenta Mathematicae*, vol. 11 (1928), p. 172.

§ See R. L. Moore, *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 416-428; also Alexandroff, *Mathematische Annalen*, vol. 96 (1926), pp. 551-571.

(1.2) *If E is any connected subset of B, then for any possible separation $T^{-1}(E) = F_1 + F_2$, there exists a $p \in E$ such that $F_1 \cdot T^{-1}(p) \neq 0 \neq F_2 \cdot T^{-1}(p)$.*

For if there were no such point in E, then $T(F_1)$ and $T(F_2)$ would be disjoint; and since E is connected and $E = T(F_1) + T(F_2)$, one of these sets, say $T(F_1)$, would have to contain a limit point p of the other. But then (1.1) would give $T^{-1}(p) \cdot \bar{F}_2 \neq 0$, which is impossible since $T^{-1}(p) \subset F_1$.

(1.3) *In order that T be monotone it is necessary and sufficient that connectedness be invariant under T^{-1}.*

The sufficiency is immediate, since for each $b \in B$, $T^{-1}(b)$ must be connected because b is connected. The necessity results from (1.2), because if E is any connected subset of B, then since there can exist no $p \in E$ with $F_1 \cdot T^{-1}(p) \neq 0 \neq F_2 \cdot T^{-1}(p)$, there can exist no separation of $T^{-1}(E)$ into sets F_1 and F_2.

(1.4) *In order that T be non-alternating it is necessary and sufficient that for any $x \in B$, $T^{-1}(x)$ should separate two points a_1 and a_2 in A if and only if x separates $T(a_1)$ and $T(a_2)$ in B.*

Proof. Sufficiency: if $x, y \in B$, then since x cannot separate y in B, $T^{-1}(x)$ cannot separate any two points of $T^{-1}(y)$ in A. Necessity: If x separates $T(a_1)$ and $T(a_2)$ in B, then $T^{-1}(x)$ separates a_1 and a_2 in A by virtue of the continuity of T. On the other hand if $T^{-1}(x)$ separates a_1 and a_2, we have a separation $A - T^{-1}(x) = A_1 + A_2$, where $a_i \in A_i$. Now since T is non-alternating, no set $T^{-1}(z)$, $z \in B$, can intersect both A_1 and A_2. Thus if $B_1 = T(A_1)$ and $B_2 = T(A_2)$, we have $B_1 \cdot B_2 = 0$. But also $B_1 \cdot \bar{B}_2 = 0 = \bar{B}_1 \cdot B_2$; for if say $b \in B_1 \cdot \bar{B}_2$, then by (1.2), $T^{-1}(b) \cdot \overline{T^{-1}(B_2)} \subset A_1 \cdot \bar{A}_2 \neq 0$; and this is contrary to the fact that A_1 and A_2 are mutually separated. Thus $B - x = B_1 + B_2$ is a separation between $T(a_1)$ and $T(a_2)$.

(1.41) *If B is connected and T is non-alternating, then a point x of B is a cut point of B if and only if $T^{-1}(x)$ separates A.*

In particular, if B has no cut point, or is cyclicly connected, no one of the sets $T^{-1}(x)$ separates A.

(1.5) *If A is locally connected, then in order that T be non-alternating it is necessary and sufficient that for each $x \in B$ and each component K of $B - x$, the set $T^{-1}(K)$ be connected.*

Proof. Necessity: Since A is locally connected and T is non-alternating, it follows that some single component C of $A - T^{-1}(x)$ contains $T^{-1}(K)$.

Since $T(C)$ is connected and contains K but does not contain x, we have $T(C) = K$. Whence $T^{-1}(K) = C$. Sufficiency: By virtue of (1.4) we have only to show that if a point x does not separate $T(a_1)$ and $T(a_2)$ in B, then $T^{-1}(x)$ cannot separate a_1 and a_2 in A. Now under these conditions, since B also is locally connected, $T(a_1)$ and $T(a_2)$ lie together in a component K of $B - x$; and since by hypothesis $T^{-1}(K)$ is connected and does not intersect $T^{-1}(x)$, our conclusion follows.

2. Product and factor theorems. In this section the spaces A, B, and C referred to are compact and metric and all transformations used are assumed to be continuous.

Let $T_1(A) = B$ and $T_2(B) = C$, and let $T = T_2T_1$. In other words, T is the result of first applying T_1 to A and then applying T_2 to $T_1(A)$, so that we have $T(A) = T_2[T_1(A)] = T_2(B) = C$.

(2.1) *If T_1 is monotone and T_2 is non-alternating, then T is non-alternating. If T is non-alternating, T_2 must be non-alternating regardless of T_1.*

To prove the first statement, let $x, y \in C$, $T^{-1}(x) = X$, $T^{-1}(y) = Y$ and suppose, contrary to the theorem, that we do have a separation $A - X = A_1 + A_2$, where $Y \cdot A_1 \supset a_1$ and $Y \cdot A_2 \supset a_2$. Now since

$$\text{(i)} \qquad T_1(X) = T_2^{-1}(x), \qquad T_1(Y) = T_2^{-1}(y),$$

we have

$$\text{(ii)} \qquad T_1(A_1) \cdot T_2^{-1}(y) \neq 0 \neq T_1(A_2) \cdot T_2^{-1}(y).$$

Furthermore $T_1(A_1) \cdot T_1(A_2) = 0$; for if this set contained a point p, then $T_1^{-1}(p)$, being connected, would have to intersect X, which is impossible since $T_1(X) \cdot T_1(A_1) = T_1(X) \cdot T_1(A_2) = 0$. Whence, by (1.1), we have that $T_1(A_1)$ and $T_1(A_2)$ are separated. But $B - T_1(X) = T_1(A_1) + T_1(A_2)$, and this, by virtue of (ii), gives that $T_2^{-1}(x)$ separates $T_2^{-1}(y)$ in B contrary to the fact that T_2 is non-alternating.

To establish the second conclusion, let T be non-alternating, let $x, y \in C$, $T_2^{-1}(x) = X$, and $T_2^{-1}(y) = Y$. Then if we had a separation $B - X = B_1 + B_2$, where $B_1 \cdot Y \neq 0 \neq B_2 \cdot Y$, we would likewise have the separation $A - T_1^{-1}(X) = T_1^{-1}(B_1) + T_1^{-1}(B_2)$, where $T_1^{-1}(B_1) \cdot T_1^{-1}(Y) \neq 0 \neq T_1^{-1}(B_2) \cdot T_1^{-1}(Y)$; and this is impossible since $T_1^{-1}(X) = T^{-1}(x)$, $T_1^{-1}(Y) = T^{-1}(y)$ and T is non-alternating. Thus we have no such separation of $B - X$, and accordingly T_2 is non-alternating.

It is easily seen by simple examples that T_1 can be non-alternating and

T_2 monotone and yet T be alternating. Also both T and T_2 may be monotone and yet T_1 be alternating.

Concerning monotone transformations we have the following corresponding theorem:

(2. 2) *If T_1 and T_2 are monotone, so also is T, $(T = T_2T_1)$. If T is monotone, T_2 must be monotone regardless of T_1.*

We shall now prove a result which may be regarded as a factor theorem for arbitrary continuous transformations.

(2. 3) *If $T(A) = B$ is continuous, then there exist continuous transformations T_1 and T_2 such that $T_2T_1(x) \equiv T(x)$ on A, T_1 is monotone and, for each $b \in B$, dim $T_2^{-1}(b) = 0$.*

Proof. Let A' denote the hyperspace of the upper semi-continuous decomposition G of A into the components of the sets $T^{-1}(b)$, $b \in B$. Then there exists * a continuous transformation $T_1(A) = A'$ such that for each $a' \in A'$, $T_1^{-1}(a') \in G$. Thus T_1 is monotone.

For each $a' \in A'$, define $T_2(a') = T[T_1^{-1}(a')]$. Then clearly T_2 is continuous, and for each $x \in A$, we have $T_2T_1(x) = T(x)$, so that $T_2(A') = B$ and $T_2T_1 \equiv T$.

Now for each $b \in B$, we have $T_2^{-1}(b) = T_1[T^{-1}(b)]$, and since T_1 is monotone so that, by (1. 3), connectedness is invariant under T_1^{-1}, it follows that dim $[T_2^{-1}(b)] = 0$, (where dim X means the Menger-Urysohn dimensionality of the set X). This completes the proof.

(2. 31) *If T is monotone, T_2 is a homeomorphism, so that T is equivalent to T_1; on the other hand, if dim $T^{-1}(b) = 0$ for each $b \in B$, then T_1 is a homeomorphism so that T is equivalent to T_2.*

Still using the notation of (2. 3) and its proof, by virtue of a result of Menger's † we have

(2. 32) *dim $B \geq$ dim A'.*

Thus if the dimension of A is lowered under the transformation T, it must be lowered under T_1 and it cannot be lowered under T_2.

It is interesting to note, on the other hand, that since the property of unicoherence is invariant under monotone transformations,‡ it follows that if this property is altered by the transformation T, it must be altered under T_2.

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* See Kuratowski, *loc. cit.*

† See *Dimensionstheorie*, Teubner, 1928, p. 235.

‡ See Kuratowski, *loc. cit.*, p. 182.

3. Applications to locally connected continua.

(3.1) *Let A and B be compact locally connected continua and suppose $T(A) = B$ is non-alternating. Then for each true cyclic element E_b of B there exists a true cyclic element E_a of A such that $T(E_a) \supset E_b$.*

To prove this, let $x, y \in E_b$ and let uv be an arc in A such that $uv \cdot T^{-1}(x) = u$ and $uv \cdot T^{-1}(y) = v$. Then u and v must be conjugate in A, i. e., no point separates them in A. For if some point z separates A between u and v, we would have $z \subset uv - (u + v)$ so that $x \neq T(z) \neq y$. But then since $T^{-1}[T(z)]$ separates u and v in A, it follows by (1.4) that $T(z)$ must separate x and y in B; and this is impossible, because x and y are conjugate.

Let E_a denote the cyclic element of A containing $u + v$. Then E_a is the desired element. To prove this we have only to show that for each $p \in E_b$, $T^{-1}(p) \cdot E_a \neq 0$. Suppose, on the contrary, that for some $p \in E_b$, $T^{-1}(p) \cdot E_a = 0$. Let $q \in E_a$ be the boundary point of a component Q of $A - E_a$ which intersects $T^{-1}(p)$. Either q non $\in T^{-1}(x)$ or q non $\in T^{-1}(y)$, say q non $\in T^{-1}(x)$. Then since $T^{-1}[T(q)]$ separates u and any point of $T^{-1}(p) \cdot Q$ in A, it follows by (1.4) that $T(q)$ must separate x and p in B; but this is impossible, because x and p are conjugate.

(3.11) *If A is a dendrite, so also is B.*

While this result is a corollary to (3.1), it is interesting to note that it is a consequence also of the following two facts:

(i) *Any non-alternating transformation defined on a dendrite is necessarily monotone,* and (ii) the image under any monotone transformation of a dendrite is itself a dendrite.*

(3.2) *With conditions as in (3.1), there exists a continuous non-alternating transformation $Z(E_a) = E_b$ which is monotone provided that T is monotone.*

Proof. It is understood, of course, that E_a is to be found as in (3.1). For each $x \in E_a$, let us define $Z(x)$ as follows: (i) if $x \in T^{-1}(b)$ for some $b \in B$, let $Z(x) = b$; (ii) if not, then let A_x be the component of $E_a - E_a \cdot T^{-1}(E_b)$ containing x; then there must exist a $p \in E_b$ such that $T^{-1}(p) \supset \bar{A}_x - A_x$, because $T(A_x)$ is contained in some component of $B - E_b$ and hence has only one limit point in E_b; in this case we let $Z(x) = p$.

Now clearly $Z(E_a) = E_b$. Also Z is continuous. This is obvious in case (ii), since Z is constant on A_x; and in case (i) it follows from the continuity of T. Finally, since for each $p \epsilon E_b$, $Z^{-1}(p) = T^{-1}(p) \cdot E_a +$ all components of $E_a - E_a \cdot T^{-1}(p)$ except the one containing $E_a[T^{-1}(E_b) - T^{-1}(p)]$, it follows that $E_a - Z^{-1}(p)$ is connected and that $Z^{-1}(p)$ is connected provided $T^{-1}(p)$ is connected. Thus Z is non-alternating and is monotone provided T is monotone.

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* See Kuratowski, *loc. cit.*, p. 182; R. L. Moore, *Foundations of Point Set Theory*, p. 344.

(3.3) *If M is a locally connected continuum and A is any A-set* in M, there exists a monotone retracting*† *transformation $T(M) = A$.*

To see this we have only to define T as follows. If $x \epsilon A$, let $T(x) = x$. If $x \epsilon (M - A)$, let $T(x)$ be that point of A which is the boundary of the component of $M - A$ containing x. Obviously T is monotone and retracting and $T(M) = A$.

(3.4) *If A is a compact locally connected continuum and $T(A) = B$ is non-alternating, then for each true cyclic element E_b of B there exists a non-alternating transformation $W(A) = E_b$ such that W is monotone if T is monotone. Furthermore none of the sets $W^{-1}(x)$ can separate A.*

By virtue of (3.2) there exists a true cyclic element E_a of A and a non-alternating transformation $T_2(E_a) = E_b$ which is monotone in case T is monotone. Also by (3.3) there exists a monotone retracting transformation $T_1(A) = E_a$. For each $x \epsilon A$, let $W(x) = T_2 T_1(x)$. Then $W(A) = E_b$. By (2.1), W is non-alternating, and by (2.2) W is monotone provided T is monotone. Finally, since E_b is cyclicly connected, by (1.41) no set $W^{-1}(x)$ can separate A.

(3.5) *If A is a compact locally connected continuum; if $T(A) = B$ is non-alternating, and if B is cyclicly connected, then there exists a true cyclic element E_a of A such that $T(E_a) = B$. Furthermore if $T_1(A) = E_a$ is a monotone retracting transformation, we have $T(x) \equiv TT_1(x)$ on A.*

The first part of this theorem is a direct consequence of (3.1). To prove the second part it is apparent that we have only to show that if x is any point of $A - E_a$ and p is the boundary point of the component Q of $A - E_a$ containing x, then $T(x) = T(p)$. But this follows immediately from the fact that since B is cyclicly connected, the set $T^{-1}[T(p)]$ cannot separate A.

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* That is, A is closed and contains every arc xy in M where $x + y \subset A$. See Kuratowski and Whyburn, *Fundamenta Mathematicae*, vol. 16, p. 309.

† This means that for each $a \epsilon A$, $T(a) = a$. For this notion and for the result (3.3) in case A is a cyclic element of M, see K. Borsuk, *Fundamenta Mathematicae*, vol. 18, p. 204.

If we agree to call a given upper-semi-continuous decomposition *monotone* or *non-alternating* provided the corresponding continuous transformation is monotone or non-alternating respectively, then (3. 4) gives the following result.

(3. 6) *Any* $\left\{\begin{matrix} \textit{monotone} \\ \textit{non-alternating} \end{matrix}\right\}$ *upper semi-continuous decomposition of a compact locally connected continuum A is equivalent to a* $\left\{\begin{matrix} \textit{monotone} \\ \textit{non-alternating} \end{matrix}\right\}$ *upper semi-continuous decomposition of A into sets not separating A in the sense that each cyclic element of the hyperspace is itself the hyperspace of some* $\left\{\begin{matrix} \textit{monotone} \\ \textit{non-alternating} \end{matrix}\right\}$ *upper semi-continuous decomposition of A into sets not separating A.*

(3. 7) *If A is compact, locally connected, and unicoherent and $T(A) = B$ is non-alternating, the true cyclic elements of B are cantorian manifolds of dimension ≥ 2.*

Let us break T up into the two factor transformations T_1 and T_2 as in (2. 3). Let $A' = T_1(A)$. Then since unicoherence is invariant under monotone transformations, A' is unicoherent. Now let E be any true cyclic element of B and suppose contrary to our theorem that some closed, 0-dimensional subset D of E disconnects E. Then $T_2^{-1}(D)$ disconnects A' and hence must reduce to a single point, because the dimensionality of no such set is lowered under T_2 and A' is unicoherent. But this means that D consists of a single point, which is impossible since E is cyclicly connected.

4. Applications to special curves and surfaces.

In this section it will be shown how our results apply in a number of interesting particular cases.

(4. 1) Suppose A is a topological sphere (simple closed surface). Then, using R. L. Moore's theorem * that the hyperspace of any upper semi-continuous decomposition of A into continua not separating A is itself a topological sphere, our theorem (3. 2) gives that each true cyclic element of the image B of A under any monotone transformation is itself a topological sphere so that B is a cactoid. Stated in terms of upper semi-continuous decompositions this will be recognized as a more recent result of Moore's.†

(4. 2) Now suppose A is a cactoid. Then by (3. 2) it follows that if B is the image of A under a monotone transformation, each true cyclic element of B is the image under some monotone transformation of some true cyclic

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* See *Transactions of the American Mathematical Society, loc. cit.*

† *Monatshefte für Mathematik und Physik*, vol. 36 (1929), pp. 81-88.

element in A, and hence of a sphere. Thus by the theorem o
above, each true cyclic element of B is a topological sphere ar
Whence, *the image under any monotone transformation of an*
a cactoid. In other words, monotone transformations car
cactoids.

(4. 3) Let A be a simple closed curve. Then if B is the ir
any non-alternating transformation, by (3. 2) each true cyclic
is the image of A under some non-alternating transformati
any non-alternating transformation throwing A into E mu
since the sets $T^{-1}(x)$ cannot separate A and any closed set in
A must be connected. It follows from this that each such E
closed curve.

We shall call such a continuum as B, i. e., a compact
continuum each true cyclic element of which is a simple
boundary curve. This term seems justified in view of the fac
curve is homeomorphic with the boundary of some plane bou
any locally connected continuum which is the boundary of su
boundary curve in this sense.

(4. 4) Now let A be any boundary curve. Then by (3. 2
have immediately that *the image under any non-alternating t*
a boundary curve is itself a boundary curve. Thus non-al
formations carry boundary curves into boundary curves.

In view of R. L. Moore's theorem (*loc. cit.*) that any cac
under some monotone transformation of the sphere, the qu
arises as to whether every boundary curve is the image u
alternating transformation of the circle. That this is inde
now be shown.

Let C denote the unit circle, let H be a boundary curve
be continuous and, for each $p \in H$, let $m(p)$ denote the multip
T and let $a(p)$ be the number of components of $H - p$. Th

(4. 5) *If* $m(p) = a(p)$ *for each* $p \in H$ *for which* $a(p) = 1$
non-alternating.

Let $x, y \in H$. Since H is a boundary curve, it follows t
be separated in H by a set X consisting either of two non-
or of a single point p of H which cuts H into just two compo
case it follows from our hypothesis that $T^{-1}(X)$ consists of

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* See W. L. Ayres, *Fundamenta Mathematicae,* vol. 14 (1929),

u and v on C. And since $T^{-1}(X)$ necessarily separates $T^{-1}(x)$ and $T^{-1}(y)$ in C, it follows that one of these sets lies on one of the arcs of C from u to v and the other one on the other arc, so that these sets cannot separate each other in C.

(4.6) *For any boundary curve H there exists a non-alternating transformation $T(C) = H$, where C is a circle, which is not constant on any arc of C.*

Proof. Let T be the transformation W described in § 5 of the author's paper in the *American Journal of Mathematics*, vol. 54 (1932), p. 372 ff., so set up that $W(C) = H$ (see p. 376). Then by (II) and (III), pp. 374-375 of that paper and by (4.5) above, it follows that this transformation has all the desired properties.

Thus it is seen that the class of all non-alternating transformations definable on the circle is equivalent to the class of all boundary curves, just as the class of all monotone transformations definable on the sphere is equivalent to the class of all cactoids.

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THE REPRESENTATION OF INTEGERS AS SUMS OF VALUES OF CUBIC POLYNOMIALS.*

By R. D. James.†

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1. *Introduction.* As we shall prove in Lemma 7, a cubic polynomial in x is an integer for all integers $x \geqq 0$ if and only if it is of the form $a(x^3 - x)/6 + b(x^2 - x)/2 + cx + d$, where a, b, c, and d are integers. In considering the representation of integers as sums of values of this polynomial we may evidently assume $d = 0$. Let

$$P(x) = a(x^3 - x)/6 + b(x^2 - x)/2 + cx. \tag{1.1}$$

We also assume that $a > 0$ and that a, b, and c have no common factor. For, if $p \mid a$, $p \mid b$, $p \mid c$ then $p \mid P(x)$ and $\sum_{\nu=1}^{s} P(x_\nu)$ would represent only multiples of p.

When $a = 1, b = c = 0$ the polynomial $P(x)$ becomes a pyramidal number $(x^3 - x)/6$. It is known that every sufficiently large integer is a sum of eight pyramidal numbers,[3] ‡ and that every integer is a sum of nine pyramidal numbers.[8]

When $(a, 3) = 1, b = 0, c = 1$, L. E. Dickson [2] has found explicit values of C and ν such that every integer $\geqq C \cdot 3^{3\nu}$ is a sum of nine values of $P(x)$. In some cases he showed by a short table that this is also true for every integer $< C \cdot 3^{3\nu}$ and hence obtained universal theorems.

When $a = 3a_1, b = 0, c = 1$, Frances E. Baker [1] and G. C. Webber [7] have applied this method to show that nine, nine, or ten values suffice according as $a_1 \equiv 0, 1$, or $2 \pmod 3$. They also obtained universal theorems in some cases. Similar results were proved by Webber without the restriction $b = 0, c = 1$, but with $a \mid b$.

In the first part of this paper we prove the result:

THEOREM 1. *Let $s \geqq 9$ be an integer and $P(x)$ any polynomial of the form* (1.1) *with $a \not\equiv 4c \pmod 8$. Then there exists a number C_1 depending only on s, a, b, and c such that every integer $n > C_1$ is a sum of s values of $P(x)$. That is, every sufficiently large integer is a sum of nine values of $P(x)$.*

We may state this theorem in another form:

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[*] Presented to the American Mathematical Society, April 6, 1934.

[†] National Research Fellow.

[‡] See the list of references at the end of the paper.

THEOREM 2. *Let $G(P)$ denote the least value of s such that the equation*

$$n = \sum_{\nu=1}^{s} P(x_\nu), \qquad x_\nu \geqq 0 \tag{1.2}$$

is solvable for all sufficiently large integers n. Then for every polynomial of the form (1.1) with $a \not\equiv 4c \pmod 8$ *we have*

$$G(P) \leqq 9.$$

The method of proof is as follows. Let

$$Q(x) = Q(x; v, t) = P(vx + t) = Ax^3 + Bx^2 + Cx + D, \tag{1.3}$$

where $t \geqq 0$ is a definite integer depending on a, b, c; and

(1.41) $v = 6$ when $3 \nmid a$, $2 \nmid (a, b)$;

(1.42) $v = 3$ when $3 \nmid a$, $2 \mid (a, b)$;

(1.43) $v = 2$ when $3 \mid a$, $2 \nmid (a, b)$;

(1.44) $v = 1$ when $3 \mid a$, $2 \mid (a, b)$.

In all cases the coefficients in $Q(x)$ are integers and $A = av^3/6 > 0$. Let $r_s(n)$ denote the number of solutions of

$$n = \sum_{\nu=1}^{s} Q(x_\nu), \qquad x_\nu \geqq 0.$$

Then E. Landau[5] has proved a result which for the special case of cubic polynomials becomes

$$\left| r_s(n) - \frac{\Gamma^s(4/3)}{A^{s/3}\Gamma(s/3)} \mathfrak{S} n^{(s-3)/3} \right| < C_2 n^{(s-3-\delta)/3} \tag{1.5}$$

where C_2 and δ are positive constants depending only on s, A, B, C, D; that is, only on s, a, b, c. The function $\mathfrak{S} = \mathfrak{S}(n)$ is real and is called the Singular Series. It is defined as follows: Let ρ be a primitive q-th root of unity,

$$\rho = e^{2\pi i r/q}, \qquad (r, q) = 1;$$

$$S_\rho = \sum_{h=0}^{q-1} \rho^{Q(h)};$$

$$A(q) = \sum_{\substack{r=0 \\ (r,q)=1}}^{q-1} q^{-s} S_\rho^{\,s}\, e^{-2\pi i r n/q};$$

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q).$$

If $\mathfrak{S} \geqq \eta > 0$, where η is independent of n, then it follows from (1. 5) ~~that~~ $r_s(n) > 0$ when

$$n > C_1 = \left(\frac{C_2 A^{s/3}\Gamma(s/3)}{\eta\Gamma^s(4/3)}\right)^{3/\delta}.$$

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The proof of Theorem 1 is thus reduced to the proof of

THEOREM 3. *If* $s \geqq 9$ *then*

$$\mathfrak{S} \geqq \eta > 0,$$

where η *depends only on* s, a, b, *and* c.

In the proof of this result we follow the method of Landau.[6]

In the second part of this paper we are concerned with polynomials $\psi(x)$ for which $\sum_{\nu=1}^{s} \psi(x_\nu)$ represents all integers. Such polynomials must represent 1. Hence let

$$(1.6) \qquad \psi(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k, \qquad \alpha_0 > 0$$

be an integer $\geqq 0$ for all integers $x \geqq 0$; and let $\psi(x_1) = 1$. If we define $g(\psi)$ to be the least value of s such that

$$\sum_{\nu=1}^{s} \psi(x_\nu) = n, \qquad x_\nu \geqq 0$$

is solvable for every integer n, then we have

THEOREM 4.* *Let* $\psi(x_2)$ *be the least value of* $\psi(x) > \psi(x_1) = 1$, *and let* $\psi(x_3)$ *be the least value of* $\psi(x) > \psi(x_2)$. *Then if* $l = [\psi(x_3)/\psi(x_2)]$ *we have*

$$g(\psi) \geqq l + \psi(x_2) - 2$$

for every polynomial of the form (1. 6).

In the particular case in which $\psi(x) = P(x)$, $a \geqq 3$, $b = c = 1$ we have $x_2 = 2$, $P(x_2) = a + 3$; $x_3 = 3$, $P(x_3) = 4a + 6$; $l = [(4a + 6)/(a + 3)] = [3 + (a - 3)/(a + 3)] = 3$. Then $g(P) \geqq a + 4$. It is highly probable that $g(P) = a + 4$ in this case. This is an analogue of the theorem that every integer is a sum of $a + 2$ polygonal numbers $a(x^2 - x)/2 + x$. The proof that $g(P) = a + 4$ depends on the evaluation of the constant C_1.

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* This is a generalization of the result $g(k) \geqq [(3/2)^k] + 2^k - 2$ in Waring's Problem.

I.

2. *Preliminary considerations.* We begin by proving some results for a polynomial of degree k. Let

$$\phi(x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_k;$$

where $a_0 > 0, a_1, \cdots, a_k$ are integers. For every prime p let $\theta(w) = \theta(w, p)$ denote the highest power of p which divides $k - w$ $(0 \leqq w \leqq k-1)$. That is,

$$p^{\theta(w)} \mid (k-w), \qquad p^{\theta(w)+1} \nmid (k-w).$$

Similarly, let

$$p^{\alpha(w)} \mid a_w, \qquad p^{\alpha(w)+1} \nmid a_w.$$

Let

$$\gamma(w) = \begin{cases} \theta(w) + 2, & p = 2, \\ \theta(w) + 1, & p > 2; \end{cases}$$

$$\theta = \theta(p) = \operatorname*{Min}_{0 \leqq w \leqq k-1} (\theta(w) + \alpha(w));$$

$$\gamma = \gamma(p) = \operatorname*{Min}_{0 \leqq w \leqq k-1} (\gamma(w) + \alpha(w))$$

$$= \begin{cases} \theta + 2, & p = 2, \\ \theta + 1, & p > 2. \end{cases}$$

Then θ is the highest power of p which divides every coefficient of $\phi'(x)$. Write $\phi_0(x) = p^{-\theta}\phi'(x)$. Let $M(m) = M(m, n)$ denote the number of solutions of

$$(2.1) \qquad \sum_{\nu=1}^{s} \phi(x_\nu) \equiv n \pmod{m}, \qquad 0 \leqq x_\nu < m.$$

For $m = p^l$ let $N(p^l) = N(p^l, n)$ denote the number of solutions of this congruence in which at least one $\phi_0(x_\nu)$ is prime to p. These solutions correspond to the primitive solutions in the case $\phi(x) = x^k$.

LEMMA 1. *If* $\lambda \geqq \gamma(w) + 1$ *and* $x = y + zp^{\lambda-\theta(w)-1}$ *then*

$$x^{k-w} \equiv y^{k-w} + (k-w) y^{k-w-1} z p^{\lambda-\theta(w)-1} \pmod{p^\lambda}.$$

Proof. This is Landau,[4] Theorem 290 with l, k, γ, and θ replaced by λ, $k-w$, $\gamma(w)$, $\theta(w)$, respectively.

LEMMA 2. *If* $l \geqq \gamma + 1$ *and* $x = y + zp^{l-\theta-1}$ *then*

$$\phi(x) \equiv \phi(y) + zp^{l-1}\phi_0(y) \pmod{p^l},$$
$$\phi_0(x) \equiv \phi_0(y) \pmod{p}.$$

Proof. Since $l + \theta(w) - \theta = l + \gamma(w) - \gamma \geqq \gamma(w) + 1$ we may apply Lemma 1 with $\lambda = l + \theta(w) - \theta$. Then

$$x^{k-w} \equiv y^{k-w} + (k-w)y^{k-w-1}zp^{l-\theta-1} \pmod{p^{l+\theta(w)-\theta}}. \tag{2.21}$$

Since $p^{\alpha(w)} \mid a_w$ and $\theta(w) + \alpha(w) \geqq \theta$ we have

$$a_w x^{k-w} \equiv a_w y^{k-w} + (k-w)a_w y^{k-w-1}zp^{l-\theta-1} \pmod{p^l}.$$

If we sum both sides of this congruence for $w = 0, 1, \cdots, k-1$ we obtain

$$\phi(x) - a_k \equiv \phi(y) - a_k + zp^{l-\theta-1}\phi'(y) \pmod{p^l},$$
$$\phi(x) \equiv \phi(y) + zp^{l-1}\phi_0(y) \pmod{p^l}.$$

This proves the first result. Again, since $l + \theta(w+1) - \theta > l - \theta - 1 \geqq 1$ we have from (2.21) with w replaced by $w + 1$

$$x^{k-w-1} \equiv y^{k-w-1} \pmod{p}$$
$$p^{-\theta}(k-w)a_w x^{k-w-1} \equiv p^{-\theta}(k-w)a_w y^{k-w-1} \pmod{p}.$$

Summing both sides we obtain the second result.

LEMMA 3. *If* $l \geqq \gamma + 1$ *then*

$$N(p^l) = p^{s-1}N(p^{l-1}).$$

Proof. In (2.1) with $m = p^l$ write $x_\nu = y_\nu + z_\nu p^{l-\theta-1}$ where $0 \leqq y_\nu < p^{l-\theta-1}$, $0 \leqq z_\nu < p^{\theta+1}$; and p does not divide every $\phi_0(x_\nu)$. Then by Lemma 2, p does not divide every $\phi_0(y_\nu)$ and (2.1) becomes

$$\sum_{\nu=1}^{s} \phi(y_\nu) + p^{l-1}\sum_{\nu=1}^{s} z_\nu\phi_0(y_\nu) \equiv n \pmod{p^l}, \quad \begin{array}{l} 0 \leqq y_\nu < p^{l-\theta-1}, \\ 0 \leqq z_\nu < p^{\theta+1}. \end{array} \tag{2.31}$$

Then to each of the $N(p^l)$ solutions of (2.31) there corresponds a solution of the two congruences

$$\sum_{\nu=1}^{s} \phi(y_\nu) \equiv n \pmod{p^{l-1}}, \ 0 \leqq y_\nu < p^{l-\theta-1}, \ p \nmid \text{every } \phi_0(y_\nu); \tag{2.32}$$

$$\sum_{\nu=1}^{s} z_\nu\phi_0(y_\nu) \equiv p^{-l+1}\Big(\sum_{\nu=1}^{s} \phi(y_\nu) - n\Big) \pmod{p}, \ 0 \leqq z_\nu < p^{\theta+1}; \tag{2.33}$$

and conversely.

From Lemma 2 it follows that if $x_\nu \equiv y_\nu \pmod{p^{l-\theta-1}}$ then $\phi(x_\nu) \equiv \phi(y_\nu) \pmod{p^{l-1}}$. Hence each solution of (2.32) gives $p^{\theta s}$ solutions of

$$\sum_{\nu=1}^{s} \phi(y_\nu) \equiv n \pmod{p^{l-1}}, \quad 0 \leqq y_\nu < p^{l-1}, \quad p \nmid \text{every } \phi_0(y_\nu).$$

That is, the congruence (2.32) has $p^{-\theta s}N(p^{l-1})$ solutions.

Since p does not divide every $\phi_0(y_\nu)$ we may assume $p \nmid \phi_0(y_1)$. Then $z_2, z_3, \cdots, z_s$ may be chosen arbitrarily mod $p^{\theta+1}$ in (2.33) and z_1 is determined uniquely mod p. Thus for any choice of $z_2, z_3, \cdots, z_s$ mod $p^{\theta+1}$ there are p^θ solutions z_1 with $0 \leqq z_1 < p^{\theta+1}$. Hence (2.33) has $p^{(\theta+1)(s-1)+\theta}$ solutions.

Since the number of solutions of (2.31) is equal to the product of the numbers of solutions of (2.32) and (2.33), we have

$$N(p^l) = p^{(\theta+1)(s-1)+\theta-\theta s} N(p^{l-1}) = p^{s-1} N(p^{l-1}).$$

Lemma 4. *If $l \geqq \gamma$ then*

$$N(p^l) = p^{(l-\gamma)(s-1)} N(p^\gamma).$$

Proof. If $l = \gamma$ the result is trivial. If $l > \gamma$ then by Lemma 3

$$N(p^l) = p^{s-1} N(p^{l-1}) = p^{2(s-1)} N(p^{l-2}) = \cdots = p^{(l-\gamma)(s-1)} N(p^\gamma).$$

Lemma 5. *We have*

$$M(m) = m^{s-1} \sum_{q|m} A(q).$$

Lemma 6. *If p_h denotes the h-th prime then*

$$\sum_{q|p_1{}^l \cdots p_t{}^l} A(q) = \prod_{p \leqq p_t} \sum_{q|p^l} A(q).$$

Both these results are proved in Landau[6] for the case $\phi(x) = x^k$. The proofs given there apply without change to the general case.

3. *Cubic polynomials.*

Lemma 7. *A polynomial $P(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ is an integer for all integers $x \geqq 0$ if and only if*

$$\alpha = a/6, \quad \beta = b/2, \quad \gamma = (6c - a - 3b)/6, \quad \delta = d, \tag{3.11}$$

where a, b, c, and d are integers.

Proof. (1) Let (3.11) be satisfied. Then

$$P(x) = a(x^3 - x)/6 + b(x^2 - x)/2 + cx + d.$$

Since $x^3 - x$ and $x^2 - x$ are divisible by 6 and 2, respectively, it is obvious that $P(x)$ is an integer. (2) Let $P(x)$ be an integer for all integers $x \geqq 0$. Then in particular

$$\begin{aligned}\delta &= P(0) &&= \text{an integer } d,\\ \alpha+\beta+\gamma &= P(1)-\delta &&= \text{an integer } i_1,\\ 8\alpha+4\beta+2\gamma &= P(2)-\delta &&= \text{an integer } i_2,\\ 27\alpha+9\beta+3\gamma &= P(3)-\delta &&= \text{an integer } i_3.\end{aligned}$$

If we solve these equations for α, β, and γ we obtain the conditions (3.11).

Now let

$$\phi(x) = Q(x) = P(vx+t) = Ax^3 + Bx^2 + Cx + D,$$

where

$$\begin{aligned}&A = av^3/6; \quad B = (at+b)v^2/2;\\ (3.12) \qquad &C = (3at^2+6bt-a-3b+6c)v/6;\\ &D = a(t^3-t)/6 + b(t^2-t)/2 + ct.\end{aligned}$$

Lemma 8. *If $p > 2$, $\theta = 0$, $\gamma = 1$, $p \nmid B$ then*

$$N(p^\gamma) = N(p) \geqq 1$$

for $s \geqq 6$ and every integer n.

Proof. We determine an integer B' so that $2BB' \equiv 1 \pmod{p}$. By Landau,[4] like Theorem 301 the congruence

$$\sum_{\nu=1}^{3} t_\nu^2 \equiv B'(n-6D) \pmod{p}$$

has a solution for every n with $p \nmid t_1$. Then

$$\sum_{\nu=1}^{3} (Q(t_\nu) + Q(p-t_\nu)) \equiv \sum_{\nu=1}^{3} (2Bt_\nu^2 + 2D) \equiv n \pmod{p}.$$

At least one of $Q_0(t_1)$ and $Q_0(p-t_1)$ is prime to p, since otherwise

$$0 \equiv Q'(t_1) - Q'(p-t_1) \equiv 4Bt_1 \pmod{p}.$$

This contradicts $p > 2$, $p \nmid B$, $p \nmid t_1$.

Lemma 9. *If $p > 2$, $p \nmid A$ then for every n there is a solution of*

$$2A \sum_{\nu=1}^{4} z_\nu^3 \equiv n \pmod{p}$$

with each of z_1, z_2, and z_3 prime to p.

Proof. If $p \mid n$ then obviously

$$2A(1^3 + (p-1)^3 + 1^3 + (p-1)^3) \equiv n \pmod{p}.$$

Hence let $p \nmid n$. We determine A' so that $2AA' \equiv 1 \pmod{p}$. Then by Landau,[4] Theorem 301 there is a solution of

$$\sum_{\nu=1}^{3} z_\nu^3 \equiv A'n \pmod{p}$$

with $p \nmid z_1$. If $p \nmid z_2$, $p \nmid z_3$ the result follows.

The remaining cases are $p \mid z_2$, $p \nmid z_3$; $p \nmid z_2$, $p \mid z_3$; $p \mid z_2$, $p \mid z_3$. In the respective cases we have

$$\begin{aligned} z_1^3 + 1^3 + (p-1)^3 + z_3^3 &\equiv A'n \pmod{p}; \\ z_1^3 + z_2^3 + 1^3 + (p-1)^3 &\equiv A'n \pmod{p}; \\ z_1^3 + 1^3 + (p-1)^3 &\equiv A'n \pmod{p}. \end{aligned}$$

This completes the proof of the lemma.

LEMMA 10. *If h_1, h_2 and h_3 are each prime to $p > 2$ there is a solution of*

$$\sum_{\nu=1}^{3} h_\nu t_\nu^2 \equiv h \pmod{p}, \qquad p \nmid t_1 \tag{3.4}$$

for every integer h.

Proof. Let g be any primitive root mod p. Then $h_\nu \equiv g^{\alpha_\nu} \pmod{p}$, where $\alpha_\nu = 2\beta_\nu + \gamma_\nu$, $\gamma_\nu = 0$ or 1. Hence we may write (3.4) in the form

$$\sum_{\nu=1}^{3} g^{\gamma_\nu} x_\nu^2 \equiv h \pmod{p}, \qquad p \nmid x_1,$$

where $x_\nu = g^{\beta_\nu} t_\nu$. The coefficients are either 1 or g and the congruence is equivalent to one of the congruences

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &\equiv h' \pmod{p}, \qquad p \nmid x_1; \\ x_1^2 + x_2^2 + g x_3^2 &\equiv h' \pmod{p}, \qquad p \nmid x_1. \end{aligned}$$

It is easily seen that both these congruences have a solution. For, if $p \nmid h'$ then by Landau,[4] Theorem 301

$$x_1^2 + x_2^2 \equiv h' \pmod{p}, \qquad p \nmid x_1$$

has a solution and we take $x_3 = 0$. If $p \mid h'$ then $p \nmid (h'-1)$, $p \nmid (h'-g)$. Hence each of the congruences

$$\begin{aligned} x_1^2 + x_2^2 &\equiv h' - 1 \pmod{p}, \qquad p \nmid x_1; \\ x_1^2 + x_2^2 &\equiv h' - g \pmod{p}, \qquad p \nmid x_1 \end{aligned}$$

has a solution and we take $x_3 = 1$.

LEMMA 11. *If* $p > 3$, $\theta = 0$, $\gamma = 1$, $p \mid B$ *then*

$$N^{*}(p^{\gamma}) = N(p) \geqq 1$$

for $s \geqq 8$ *and every integer* n.

Proof. We cannot have $p \mid A$, $p \mid C$ for this would imply $p \mid a$, $p \mid b$, $p \mid c$. If $p \mid A$, $p \nmid C$ then $Q(x) \equiv Cx + D \pmod{p}$ and it is evident that $Cx + D \equiv n \pmod{p}$ has a solution with $Q_0(x) = C$ prime to p.

Hence let $p \nmid A$. By Lemma 9 there is a solution of

$$2A \sum_{\nu=1}^{4} z_\nu^3 \equiv n \pmod{p} \tag{3.51}$$

with each of z_1, z_2, z_3 prime to p. Then by Lemma 10 with

$$h = -\sum_{\nu=1}^{4} (2Cz_\nu + 2D)\,; \quad h_\nu = 6Az_\nu \ (\nu = 1, 2, 3)\,; \quad t_4 = 0$$

there is a solution of

$$\sum_{\nu=1}^{4} 6Az_\nu t_\nu^2 \equiv -\sum_{\nu=1}^{4} (2Cz_\nu + 2D) \pmod{p}, \qquad p \nmid t_1. \tag{3.52}$$

From (3.51) and (3.52) it follows that

$$\sum_{\nu=1}^{4} (Q(z_\nu + t_\nu) + Q(z_\nu - t_\nu)) \\ \equiv \sum_{\nu=1}^{4} (2Az_\nu^3 + 6Az_\nu t_\nu^2 + 2Cz_\nu + 2D) \equiv n \pmod{p}.$$

At least one of $Q_0(z_1 + t_1)$ and $Q_0(z_1 - t_1)$ is prime to p, since otherwise $0 \equiv Q'(z_1 + t_1) - Q'(z_1 - t_1) \equiv 12Az_1t_1 \pmod{p}$. This contradicts $p > 3$, $p \nmid A$, $p \nmid z_1$, $p \nmid t_1$.

LEMMA 12. *If* $p \mid A$, $p \mid B$, $p \nmid C$ *then*

$$N(p^{\gamma}) \geqq p^{\gamma(s-1)}$$

for $s \geqq 1$ *and every integer* n.

Proof. The p^{γ} integers $Q(x)$ $(0 \leqq x \leqq p^{\gamma} - 1)$ are incongruent mod p^{γ}. For suppose $Q(x) \equiv Q(y) \pmod{p^{\gamma}}$, $x \neq y$, $0 \leqq x \leqq p^{\gamma} - 1$, $0 \leqq y \leqq p^{\gamma} - 1$. Then

$$Q(x) - Q(y) = (x - y)(A(x^2 + xy + y^2) + B(x + y) + C) \equiv 0 \pmod{p^{\gamma}}.$$

The second factor is prime to p since $p \mid A$, $p \mid B$, $p \nmid C$. Thus $x \equiv y \pmod{p^{\gamma}}$ and hence $x = y$, a contradiction.

Then in the congruence

$$\sum_{\nu=1}^{s} Q(x_\nu) \equiv n \pmod{p^\gamma}, \qquad 0 \leqq x_\nu \leqq p^\gamma - 1$$

we may choose $x_2, x_3, \cdots, x_s$ arbitrarily mod p^γ and then determine x_1 with $Q'(x_1) \equiv C$ prime to p. This proves the lemma.

Lemma 13. *If for* $x = x_1$ *we have* $p \nmid Q(x_1)$, $p \nmid Q_0(x_1)$ then

$$N(p^\gamma) \geqq 1$$

for $s \geqq p^\gamma$ *and every integer* n.

Proof. The result is trivial since any integer is represented mod p^γ by at most p^γ values $Q(x_1)$ with $p \nmid Q_0(x_1)$.

Lemma 14. *If for* $x = x_1$ *we have* $p \nmid Q(x_1)$ *and for* $x = x_2$, $p \mid Q(x_2)$, $p \nmid Q_0(x_2)$ *then*

$$N(p^\gamma) \geqq 1$$

for $s \geqq (p+1)p^{\gamma-1}$ *and every integer* n.

Proof. Since $p^{\gamma-1}Q(x_2) \equiv 0 \pmod{p^\gamma}$ every integer is represented mod p^γ by at most p^γ values $Q(x_1)$ plus at most $p^{\gamma-1}$ values $Q(x_2)$ with $p \nmid Q_0(x_2)$. Thus $N(p^\gamma) \geqq 1$ for $s \geqq p^\gamma + p^{\gamma-1} = (p+1)p^{\gamma-1}$.

4. *The Case* $p > 3$. In this section we prove

Lemma 15. *For* $Q(x) = P(vx + t)$ *we have*

$$N(p^\gamma) \geqq 1$$

for $p > 3$, $s \geqq 9$ *and every integer* n.

Proof. The coefficients of $Q'(x)$ are $3A$, $2B$ and C. If a prime p divided each of $3A$, $2B$, C it would follow from (3.12) that p divided each of a, b, and c, contrary to our assumption. Hence $\theta = 0$, $\gamma = 1$. The result now follows from Lemmas 8 and 11.

5. *The Case* $p = 3$. We shall prove that Lemma 15 holds also when $p = 3$. We first consider

(5.11) $$a \not\equiv 0 \pmod{3}.$$

Then we have $v = 3$ or 6, $3 \mid A$, $3 \mid B$, $3 \nmid C$. Hence Lemma 12 applies with $p = 3$ and thus $N(3^\gamma) \geqq 3^{\gamma(s-1)} \geqq 1$ for $s \geqq 1$.

Next, let

(5.12) $$a \equiv 0 \pmod{3}, \qquad b \not\equiv 0 \pmod{3}.$$

Then $v=1$ or 2 and $B=(at+b)v^2/2\not\equiv 0 \pmod 3$. Hence $\theta=0$ and Lemma 8 applies.

Now suppose

$$(5.2)\qquad a\equiv b\equiv 0 \pmod 3,\qquad a\not\equiv 6c \pmod 9.$$

Then from (3.12) we have $3\nmid C$ and hence $\theta=0$, $\gamma=1$. By choosing $t\equiv 1 \pmod 3$ we have $3\nmid D$. Thus $3\nmid Q(0)$, $3\nmid Q_0(0)$. By Lemma 13 with $p=3$, $\gamma=1$, $x_1=0$ we have $N(3)\geqq 1$ for $s\geqq 3$.

Finally, let

$$(5.3)\qquad a\equiv b\equiv 0 \pmod 3,\qquad a\equiv 6c \pmod 9.$$

From $a\equiv 6c \pmod 9$, $c\not\equiv 0 \pmod 3$ it follows that $a\not\equiv 0 \pmod 9$. Let $a=3a_1$, $b=3b_1$. Then from (1.43), (1.44), and (3.12)

$$\begin{aligned} A&=a_1v^3/2;\qquad B=3(a_1t+b_1)v^2/2;\\ C&=(3a_1t^2+6b_1t-a_1-3b_1+2c)v/2;\\ D&=a_1(t^3-t)/2+3b_1(t^2-t)/2+ct;\end{aligned}$$

where $v=1$ or 2, and $a_1\not\equiv 0 \pmod 3$. Then $\theta=1$, $\gamma=2$.

We choose t as follows

$$(5.31)\qquad t\equiv 0 \pmod 9 \quad\text{when}\quad a\equiv 24c \pmod{27},\qquad b\equiv 0 \pmod 9;$$

$$(5.32)\qquad t\equiv 1 \pmod 9 \quad\text{when}\quad a\equiv 24c \pmod{27},\qquad b\not\equiv 0 \pmod 9;$$

$$(5.33)\qquad t\equiv 2 \pmod 9 \quad\text{when}\quad a\not\equiv 24c \pmod{27}.$$

In (5.31) we have

$$\begin{aligned} &Q(0)=D\equiv 0 \pmod{3^2},\qquad Q_0(0)=C/3\not\equiv 0 \pmod 3,\\ &Q(1)=A+B+C+D\equiv a_1v^3/2\not\equiv 0 \pmod 3.\end{aligned}$$

Hence any integer not divisible by 9 is represented by at most eight values $Q(1)$ plus one value $Q(0)$ with $3\nmid Q_0(0)$. Any integer divisible by 9 is represented by one value $Q(0)$ with $3\nmid Q_0(0)$. Thus $N(3^2)\geqq 1$ for $s\geqq 9$. In (5.32) and (5.33) we have $Q(0)\not\equiv 0$, $Q_0(0)\not\equiv 0 \pmod 3$, and by Lemma 13 with $p=3$, $\gamma=2$ we have $N(3^2)\geqq 1$ for $s\geqq 9$.

6. *The Case* $p=2$. We shall prove that Lemma 15 holds also when $p=2$. We first consider

$$(6.1)\qquad 2\nmid(a,b).$$

Then we have $v=2$ or 6, $2\mid A$, $2\mid B$ and by choice of t mod 2,

$$C\equiv at^2-a-b\not\equiv 0 \pmod 2.$$

Hence Lemma 12 applies with $p = 2$ and $N(2^{\gamma}) \geqq 2^{\gamma(s-1)}$ for $s \geqq 1$.

Now suppose

$$(6.2) \qquad a \equiv b \equiv 0 \pmod 2, \qquad b \not\equiv 2c \pmod 4.$$

Then from (1.42), (1.44), and (3.12) we have $v = 1$ or 3, $a = 2a_1$, $b = 2b_1$, and

$$\begin{aligned} &A = a_1 v^3/3; \quad B(a_1 t + b_1)v^2; \\ &C = (3a_1 t^2 + 6b_1 t - a_1 - 3b_1 + 3c)v/3; \\ &D = a_1(t^3 - t)/3 + b_1(t^2 - t) + ct. \end{aligned}$$

By choosing $t \equiv 1 \pmod 2$ we have both C and D odd. Hence $\theta = 0$, $\gamma = 2$ and by Lemma 13 with $x_1 = 0$ we have $N(2^2) \geqq 1$ for $s \geqq 4$.

Next, let

$$(6.3) \qquad a \equiv b \equiv 0 \pmod 2; \qquad a \equiv 2,\ b \equiv 2c \pmod 4.$$

In this case $\theta = 0$, $\gamma = 2$. If we choose t even then $Q(0) \equiv 0$, $Q'(0) \equiv Q(1) \equiv 1 \pmod 2$. By Lemma 14 with $p = 2$, $\gamma = 2$ we have $N(2^2) \geqq 1$ for $s \geqq 6$.

Finally, let

$$(6.4) \qquad a \equiv b \equiv 0 \pmod 2; \qquad a \equiv 0,\ b \equiv 2c \pmod 4.$$

In this case $\theta = 1$, $\gamma = 3$. We choose t as follows

$$(6.41) \qquad t \equiv 0 \pmod 4 \quad \text{when} \quad b \equiv 6c \pmod 8;$$

$$(6.42) \qquad t \equiv 1 \pmod 4 \quad \text{when} \quad b \equiv 2c \pmod 8.$$

In (6.41) we have $Q(0) \equiv 0 \pmod 4$, $Q(1) \not\equiv 0 \pmod 2$, and since we are assuming $a \not\equiv 4c \pmod 8$, then $Q_0(0) = C/2 \not\equiv 0 \pmod 2$. Any integer not divisible by 8 can be represented by at most seven values $Q(1)$ plus two values $Q(0)$ with $2 \nmid Q_0(0)$. Any integer divisible by 8 can be represented by two values $Q(0)$ with $2 \nmid Q_0(0)$. Hence $N(2^3) \geqq 1$ for $s \geqq 9$. In (6.42) we have $Q(0) \equiv Q_0(0) \equiv 1 \pmod 2$, and by Lemma 13 with $p = 2$, $\gamma = 3$ we have $N(2^3) \geqq 1$ for $s \geqq 8$.

7. *The Proof of Theorem* 3. Since $M(p^l) \geqq N(p^l)$ we have from Lemmas 5, 4, and 15

$$(7.1) \qquad \begin{aligned} \sum_{q \mid p^l} A(q) &= p^{-l(s-1)} M(p^l) \geqq p^{-l(s-1)} N(p^l) \\ &= p^{-l(s-1)} \cdot p^{(l-\gamma)(s-1)} N(p^{\gamma}) \geqq p^{-\gamma(s-1)} \end{aligned}$$

$$\geqq p^{-3(s-1)} \qquad (l \geqq 3 \geqq \gamma).$$

Also, by Landau,[5] Theorem 8 with $\epsilon = 1/8$

$$\sum_{q|p^l} A(q) = 1 + \sum_{\lambda=1}^{l} A(p^\lambda) > 1 - E_2 \sum_{\lambda=1}^{\infty} p^{-9\lambda/8} \tag{7.2}$$
$$= 1 - E_2(p^{9/8} - 1)^{-1}.$$

Finally, the Singular Series is absolutely convergent for $s \geqq 9$ (Landau,[5] Theorem 9). Hence by (7.1), (7.2), and Lemma 6

$$\begin{aligned} \mathfrak{S} &= \lim_{l\to\infty} \sum_{q|p_1{}^l \cdots p_l{}^l} A(q) = \lim_{l\to\infty} \prod_{p \leqq p_l} \sum_{q|p^l} A(q) \\ &\geqq \prod_p \max\left(p^{-3(s-1)}, 1 - E_2(p^{9/8} - 1)^{-1}\right) \\ &\geqq \eta > 0. \end{aligned}$$

Theorems 1 and 2 now follow as indicated in the Introduction.

II.

8. *The Proof of Theorem* 4. Consider the integer $N = l\psi(x_2) - 1$. We have $N < \psi(x_3)$. Hence the only permissible values of $\psi(x)$ in the representation of N are $\psi(x_1) = 1$ and $\psi(x_2)$. That is,

$$N = (l - h)\psi(x_2) + (h\psi(x_2) - 1)\,\psi(x_1),$$

and $l - h + h\psi(x_2) - 1$ values of $\psi(x)$ are required. Since $\psi(x_2) > 1$ the number required is a minimum when $h = 1$. Then $g(\psi) \geqq l + \psi(x_2) - 2$.

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FUNDAMENTAL REGIONS FOR THE SIMPLE GROUP OF ORDER 168 IN S_4.

By William I. Miller.

Fundamental regions for certain ternary groups have been considered by J. W. Young[*] and Henry F. Price[†]; the former obtained a solution for cyclic groups, the latter for the G_{24} and the G_{60}. The method used here to obtain fundamental regions for the G_{168} is analogous to the method used by Price for the G_{60}.

The G_{168} may be written as a doubly transitive permutation group of degree seven generated by[‡]

$$S = abcdefg, \qquad R = ab \cdot ce, \qquad T = SRS^4RSR = adb \cdot cef.$$

As the corresponding collineations we shall use

$$\begin{array}{rlll} & x_1 = 1/2(-x'_1 - rx'_2 + x'_3), & & \\ S: & x_2 = 1/2(-x'_1 + rx'_2 + x'_3), & & \\ & x_3 = 1/2(-\bar{r}x'_1 - \bar{r}x'_3), & & \\ R: & x_1 = -x'_2, & x_2 = -x'_1, & x_3 = -x'_3, \\ T: & x_1 = x'_2, & x_2 = x'_3, & x_3 = x'_1, \end{array}$$

where $r = 1/2[1 + (-7)^{1/2}], \qquad \bar{r} = 1/2[1 - (-7)^{1/2}].$

The collineation group in this form contains a subgroup of order 24, generated by R, T, and $V = S^4RS^3$, whose coefficients are real; this is the G_{24} studied by Price.

To represent the complex points of the plane as real points in S_4, set $x_1/x_3 = x + iu$, $x_2/x_3 = y + iv$.

Consider the 21 conjugate linear forms b_{ij} $(i = 1, 2, 3;\ j = 0, \cdots, 6)$, in which $b_{10} = \bar{r}x_1 - \bar{r}x_3$ (or the negative of this, as the signs of these forms are of no consequence), and in which the others are so chosen that they are transformed by the generators as follows:

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[*] J. W. Young, "Fundamental regions for cyclic groups of linear fractional transformations on two complex variables," *Bulletin of the American Mathematical Society*, vol. 17, pp. 340-344.

[†] H. F. Price, "Fundamental regions for certain finite groups in S_4," *American Journal of Mathematics*, vol. 40 (1919), pp. 108-114.

[‡] H. F. Blichfeldt, *Finite Collineation Groups*, chap. V, p. 113, Chicago, The University of Chicago Press, 1917.

S: $b_{i,j}b_{i,j+1}$;
R: $b_{10}b_{20}\cdot b_{12}b_{31}\cdot b_{13}b_{14}\cdot b_{15}b_{36}\cdot b_{21}b_{35}\cdot b_{22}b_{34}\cdot b_{25}b_{33}\cdot b_{26}b_{32}$;
T: $b_{1,j}b_{3,4j}b_{2,2j}$.

The second subscript is to be reduced modulo 7 if necessary.

The group on $b_{10},\cdots,b_{36}$ is imprimitive. We may associate with each set of imprimitivity one of the letters a, b, c, d, e, f, g, in such a manner that the group obtained by considering the ways in which these sets of imprimitivity are transformed is the same as the group of degree seven given above. To do this let a represent the set b_{13}, b_{25}, b_{32}; b the set b_{14}, b_{26}, b_{33}; c the set b_{15}, b_{20}, b_{34}; etc. If in each set every second subscript is replaced by its negative modulo 7, another division according to sets of imprimitivity is obtained; this method of division is not used, however, as it leads to an outer isomorphism of the group with itself.

We now form seven positive forms of the type

$$A=b_{13}\bar{b}_{13}b_{25}\bar{b}_{25}+b_{25}\bar{b}_{25}b_{32}\bar{b}_{32}+b_{32}\bar{b}_{32}b_{13}\bar{b}_{13},$$

where $\bar{b}_{13}$ is the conjugate imaginary of b_{13}. The forms $B,\cdots,G$ are formed in the same manner from the sets $b,\cdots,g$. The generators permute the capital letters in the same manner as they permute the corresponding small letters.

Let β_{ij} $(i=1,2,3;\ j=0,\cdots,6)$ represent the 21 differences of $A,\cdots,G$, with $\beta_{10}=G-A$, $\beta_{20}=G-B$, $\beta_{30}=G-D$, and the others so chosen that S permutes them in the order $\beta_{i,j}\beta_{i,j+1}$. It is not necessary to indicate how R and T transform these forms, as it is more convenient to use the permutation group on $A,\cdots,G$.

The forms β_{ij} are used to determine fundamental regions for the G_{168}. If we divide each of these forms by $x_3{}^2\bar{x}_3{}^2$ and equate to zero we obtain 21 hypersurfaces in S_4. Excluding points which lie on these hypersurfaces, every point of S_4 will make each of these forms either positive or negative; and since these are the differences of seven positive forms there are at most 7! possible arrangements of plus and minus signs. Obviously no operator except the identity can transform a point into another point which produces the same arrangement of signs. An arrangement of signs is most conveniently given by writing the order of magnitude of $A,\cdots,G$, and if $A>B>C>D>E>F>G$ we shall indicate this by the *value system* $ABCDEFG$. To obtain a fundamental region we select 30 value systems no two of which are conjugates under the group.

As an example of a fundamental region we take the 30 value systems which have the form $ABCXXXX$ or the form $ABFCXXX$, where X may be any one of the remaining letters.

If we set $u = 0$, $v = 0$ in each of the equations of the 21 hypersurfaces we obtain the curves of intersection with the "real plane." Among these curves will be found nine straight lines which divide the plane into 24 parts which are projectively the same. These are the nine lines used by Price to determine fundamental regions for the G_{24} on the plane *; this is not surprising in view of the fact that this G_{24} is the largest subgroup of the G_{168} that leaves this plane invariant. Similarly the plane $x = 0$, $y = 0$, is invariant under a subgroup of order eight, and among the intersections of the hypersurfaces with this plane will be found five straight lines which divide the plane into eight regions that are projectively the same.

If the curves of intersection of the hypersurfaces with these two planes are drawn it is not difficult to determine all the value systems that appear. It is found that 27 of the 30 conjugate sets are represented. The remaining three sets are represented by the points $x = 1/2$, $u = -1/2$, $y = \epsilon$, $v = 0$; $x = \epsilon$, $u = 1 + \epsilon$, $y = \epsilon$, $v = \epsilon$; $x = -\epsilon$, $u = 1 + \epsilon$, $y = -\epsilon$, $v = \epsilon$, where ϵ is small and positive. Hence all of the 7! value systems occur.

To complete the determination of a fundamental region we start with the 30 value systems of the form $ABCXXXX$ or $ABFCXXX$ and set two or more of the forms $A, \cdots, G$ equal in all possible ways. After discarding duplicates the remaining value systems are arranged into sets of conjugates under the group and from each conjugate set one value system is selected. This gives 327 additional value systems which must be added to the original 30.

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* H. F. Price, *loc. cit.*

A METHOD OF DETERMINING ALL THE SOLVABLE GROUPS OF GIVEN ORDER AND ITS APPLICATION TO THE ORDERS 16*p* AND 32*p*.

By A. C. Lunn and J. K. Senior.

In a recent paper, P. Hall[*] has shown that if G is any solvable group of order yz where y is prime to z, then

(a) G contains at least one subgroup of order y.

(b) All subgroups of order y in G are conjugate.

(c) Every subgroup whose order divides y is contained in a subgroup of order y.

These propositions, for the case where z is a power of a prime, together with the well-known theorems on the representations of abstract groups[†] yield the theorem upon which the following paper is based.

Before proceeding to the statement and proof of that theorem, it will be convenient to establish a corollary of Hall's theorem. The symbol T will be used throughout to denote a transitive solvable substitution group of degree $x = p^a$ and order kx where k is not divisible by p. The symbol $(T_1 : T_2 \cdots T_h)$ indicates a group on h sets of transitivity formed by using isomorphisms between the transitive groups $T_1 \cdots T_h$. The limiting case of direct product formation by any pair of transitive elements is included in the definition. The symbol R will denote an abstract group simply isomorphic with T.

Corollary. *If T_1 and T_2 are distinct substitution groups, they are not simply isomorphic.*

Proof. If T_1 and T_2 are simply isomorphic then $k_1x_1 = k_2x_2$.

Two cases may be distinguished.

(1) $p_1 = p_2$. In this case R_1 (by Hall's theorem) contains only one set of conjugates of index x_1 and hence has only one transitive representation of degree x_1. Therefore T_1 and T_2 are not simply isomorphic.

(2) $p_1 \neq p_2$. In this case R_1 contains a single set of conjugate subgroups of order k_1, no one of which contains any invariant subgroup of R_1 besides identity. But (by Hall's theorem) every subgroup of R_1 whose order divides k_1 is contained in a subgroup of order k_1. Therefore (besides identity) no subgroup of R_1 whose order divides k_1 is invariant, and the order of every proper invariant subgroup of R_1 is divisible by p_1.

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[*] P. Hall, *Journal of the London Mathematical Society*, vol. 3 (1928), p. 98.

[†] Burnside, *Theory of Groups*, second edition (1911), p. 231 *et seq.*

If R_1 and R_2 were simply isomorphic, then the order of every proper invariant subgroup in either of them would be divisible by p_1p_2. But any R is solvable, and hence contains a characteristic Abelian subgroup (of order greater than 1) the Sylow subgroups of which are invariant under R and of prime power order. Therefore R_1 and R_2 (and hence T_1 and T_2) are not simply isomorphic. Thus the corollary is proved for all cases.

The fundamental theorem of the following paper may be stated in terms of the notation already defined. If, for $i = 1 \cdots n$, T_i is of degree $x_i = p_i^{a_i}$ (where the p_i's are distinct primes), and R_i is the abstract group simply isomorphic with T_i, then such a list of n R's will be called a complete array for the order $g = \Pi x_i$.

THEOREM. *Every solvable group G of order $g = \Pi x_i$ determines its complete array of R's and the isomorphism according to which each pair of R's must be combined to give a representation of G. Any representation of G which involves every R of a complete array belonging to the order g is simply isomorphic with G.*

Proof. G contains for each i a single set of conjugate subgroups of index x_i, thus determining T_i and R_i. The intersection of this set is C_i, a characteristic subgroup of G; and T_i (hence R_i) is simply isomorphic with the quotient group G/C_i. Let C_{ij} be the intersection of C_i and C_j. If T_i and T_j are written on distinct sets of letters, the intransitive representation $(T_i : T_j)$ is simply isomorphic with the quotient group G/C_{ij}, and hence $(R_i : R_j)$ is uniquely determined. It is thus shown that G uniquely determines its complete array of R's, and that between each pair of R's in the array there is only one isomorphism which leads to a representation of G.

The order of the intransitive representation $(T_1 : T_2 \cdots T_n)$ must be a factor of g and a multiple of the degree of each T_i. But since the degree of T_i is x_i, the order of $(T_1 : T_2 \cdots T_n)$ must be a multiple of $\Pi x_i = g$. So $(T_1 : T_2 \cdots T_n)$ and hence $(R_1 : R_2 \cdots R_n)$ are of the same order as G, and thus simply isomorphic with G. This completes the proof of the theorem.

The above theorem may be used to obtain all the abstract solvable groups of given order g by proceeding as follows. Pick a complete array of R's belonging to the order g, and combine them by isomorphism so as to give a group of that order. (It is to be understood that the word isomorphism is here used to include the case of direct product formation, or isomorphism according to a quotient group of order 1.) If this process is carried out in all possible ways with all distinct complete arrays belonging to the order g, the list of abstract groups thus obtained includes all the solvable abstract groups of that order. No two of the groups of order g thus obtained are simply isomorphic unless the two arrays are identical, and the respective isomorphisms between the members of all corresponding pairs of R's are also identical.

Certain general conclusions from the above theorem may also be drawn.

(1) There exists a category of abstract groups (for which the name Hall groups is here proposed) which are defined as being simply isomorphic with solvable transitive groups of degree p^a and order kp^a where k is not divisible by p. Every solvable abstract group which is not itself a Hall group may be obtained (as explained above) by a suitable combination of Hall groups.

(2) Every abstract solvable group G of order $g = \Pi x_i$ can be expressed as an intransitive group of degree Σx_i with sets of transitivity of degree x_i. If it is so expressed, each transitive constituent of the intransitive group is uniquely determined as a substitution group.

The theorem previously proven will be used to determine all the groups of orders $16p$ and $32p$. For this purpose the following symbolism will be used. G^x_{xk} indicates a solvable abstract group simply isomorphic with a transitive group of degree x and order xk where $x = p^a$ and k is not divisible by p. The symbol $(G^{x_1}_{x_1k_1} : G^{x_2}_{x_2k_2})_{k_1k_2}$ indicates that two such groups $(p_1 \neq p_2)$ are combined by isomorphism according to a quotient group of order k_1k_2. Reference will be made repeatedly to the number of distinct cases of a given abstract quotient group involved in a list of abstract groups $G_1 \cdots G_n$. Any two such cases (with quotient groups denoted as usual by G_i/H_i and G_j/H_j) are to be understood as non-distinct when and only when $G_i \equiv G_j$, and H_i and H_j are conjugate under the holomorph of G_i.

The Groups of Order $16p$.* Every abstract group of order $16p$ is solvable and thus determines a $(G^{16}_{16k_1} : G^p_{pk_2})_{k_1k_2}$. $(k_1 = 1$ or $p.)$ The only solvable transitive groups of degree p are the metacyclic group of order $p(p-1)$ and its invariant subgroups. Therefore k_2 must be a common divisor of 16 and $(p-1)$, and for any possible pair of values of p and k_2, there will be but one $G^p_{pk_2}$. Moreover k_2 is the order of a cyclic quotient group of order 2^n in $G^p_{pk_2}$, and for any one $G^p_{pk_2}$ there is only one such quotient group of given order.

Every group of order $16p$ occurs therefore in one of the following divisions:

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* The groups of order $16p$ have been treated by Le Vavasseur, *Annales de la Faculté des Sciences de l'Université de Toulouse*, ser. 2, vol. 5 (1903), p. 63. But as the article is not entirely free from error, an independent demonstration is here given.

Division (a) $= (G^{16}_{16} : G^{p}_{p})_1$
" (b) $= (G^{16}_{16} : G^{p}_{2p})_2$
" (c) $= (G^{16}_{16} : G^{p}_{4p})_4$
Division (d) $= (G^{16}_{16} : G^{p}_{8p})_8$
" (e) $= (G^{16}_{16} : G^{p}_{16p})_{16}$
" (f) $= (G^{16}_{16p} : G^{p}_{pk_2})_{pk_2}$

Division (*a*). $(G^{16}_{16} : G^{p}_{p})_1$. A group in this division is the direct product of its Sylow subgroups. Since there are fourteen different groups of order 16 and one group of order p, there are, for every p, fourteen groups of division (a); five of these are abelian.

Division (*b*). $(G^{16}_{16} : G^{p}_{2p})_2$. Since $(p-1)$ is divisible by 2, there are groups of this division for every p. The fourteen groups of order 16 permit in all 28 distinct dimidiations;* hence for every p there are 28 groups of division (b).

Division (*c*). $(G^{16}_{16} : G^{p}_{4p})_4$. Groups of this division exist only when $(p-1)$ is divisible by 4. The fourteen groups of order 16 involve nine distinct cases of cyclic quotient group of order 4, and each case gives rise to a single isomorphism.† Hence there are nine groups of division (c) if 4 divides $(p-1)$.

Division (*d*). $(G^{16}_{16} : G^{p}_{8p})_8$. Groups of this division exist only when $(p-1)$ is divisible by 8. Of the fourteen groups of order 16, only the cyclic group and the abelian group of type (3, 1) involve cyclic quotient groups of order 8. Each of these involves one case of such a quotient group and this case gives rise to a single isomorphism. Hence there are two groups of division (d) when 8 divides $(p-1)$.

Division (*e*). $(G^{16}_{16} : G^{16}_{16p})_{16}$. Groups of this division exist only when $(p-1)$ is divisible by 16. The only group of order 16 which can be used in such an isomorphism is the cyclic group, and it gives rise to a single isomorphism. Hence there is one group of division (e) when 16 divides $(p-1)$.

Division (*f*). $(G^{16}_{16p} : G^{p}_{pk_2})_{pk_2}$. Groups of this division exist only when p divides (2^n-1) and $1 < n < 5$—that is when $p=3$, 5 or 7. These three cases will therefore be considered separately.

When $p=3$, the number of groups of division (f) has been shown to be ten.‡

When $p=5$, the number of groups of division (f) has been shown to be one.§

When $p=7$, the groups in question are of order 112. If a group G of this order contained no subgroup of order 56, it would contain an invariant subgroup K of order 16, but could contain no invariant subgroup of order 8, since it could involve no quotient group of order 14. Hence K would have to contain seven subgroups of order 8 conjugate under G, and could contain no other subgroups of this same order. There is no group of order 16 which meets this restriction, and hence every group of order 112 contains a subgroup H of order 56.

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* G. A. Miller, *Quarterly Journal of Pure and Applied Mathematics*, vol. 30 (1898), p. 243.

† G. A. Miller, unpublished work.

‡ *Loc. cit.*

§ *Loc. cit.*

If G is of division (f), it contains 8 subgroups of order 7 all of which are contained in H. As there is only one group of order 56 containing 8 subgroups of order 7, H is uniquely determined. It contains a characteristic abelian subgroup L of order 8 and type (1, 1, 1) which must be invariant under G. In H, every subgroup of order 7 is its own normalizer, but in G, the normalizer N of a subgroup of order 7 is of order 14. Hence G can be generated by the adjunction to H of an operator $T (T^2 = 1)$ which with a subgroup of order 7 generates N. N is non-invariant under G since its single subgroup of order 7 is non-invariant.

Since all the operators of order 2 in N are conjugate under N, if T were non-invariant under G, then N would contain no subgroup besides identity invariant under G, and G could be expressed transitively on 8 letters. As there is no such transitive group of degree 8,[*] T is invariant under G. G thus contains an invariant subgroup L (of order 8) and an invariant subgroup T (of order 2) which is not contained in L. Together L and T generate an invariant abelian subgroup K of order 16 and type (1, 1, 1, 1), and G corresponds to a set of conjugate subgroups of order 7 in the i-group of K. As the subgroups of order 7 in this i-group are Sylow subgroups, there is only one group of division (f) when $p = 7$.

The groups of order $16p$ can then be tabulated as follows:

p	No. of groups of division						Total
	(a)	(b)	(c)	(d)	(e)	(f)	
3	14	28	0	0	0	10	52
5	14	28	9	0	0	1	52
7	14	28	0	0	0	1	43
> 7							
$(p - 1)$ divisible by 2 but not by 4.	14	28	0	0	0	0	42
$(p - 1)$ divisible by 4 but not by 8.	14	28	9	0	0	0	51
$(p - 1)$ divisible by 8 but not by 16.	14	28	9	2	0	0	53
$(p - 1)$ divisible by 16.	14	28	9	2	1	0	54

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[*] G. A. Miller, *American Journal of Mathematics*, vol. 21 (1899), p. 326.

The Groups of Order 32p. Every group of order $32p$ is solvable and thus determines a $(G^{32}_{32k_1} : G^{p}_{pk_2})_{k_1k_2}$. $(k_1 = 1 \text{ or } p)$. In regard to the group $G^{p}_{pk_2}$, obviously the considerations for $16p$ hold also for $32p$ save that in the latter case k_2 may take the additional value 32. Every group of order $32p$ occurs therefore in one of the following divisions:

Division (a) $(G^{32}_{32} : G^{p}_{p})_1$
" (b) $(G^{32}_{32} : G^{p}_{2p})_2$
" (c) $(G^{32}_{32} : G^{p}_{4p})_4$
Division (d) $(G^{32}_{32} : G^{p}_{8p})_8$
" (e) $(G^{32}_{32} : G^{p}_{16p})_{16}$
" (f) $(G^{32}_{32} : G^{p}_{32p})_{32}$
Division (g) $(G^{32}_{32p} : G^{p}_{pk_2})_{pk_2}$.

Division (*a*). $(G^{32}_{32} : G^{p}_{p})_1$. For every p the number of groups of this division is 51—that being the number of groups of order 32. Seven of these groups are abelian.

Division (*b*). $(G^{32}_{32} : G^{p}_{2p})_2$. For every p, the number of groups of this division is equal to the number of distinct dimidiations of the 51 groups of order 32. This number has been shown to be 144.*

Division (*c*). $(G^{32}_{32} : G^{p}_{4p})_4$. In 24 of the groups of order 32, all the operators which are squares are contained in the commutator subgroup. No such group can give rise to a group of division (c), and each of the remaining 27 groups must give rise to at least one group of this division. In 15 of these groups, all the invariant subgroups complementary to cyclic quotient groups of order four form one set of conjugates under the holomorph of the group; in 11 groups they form two sets; in one group three sets. As each of these cases of cyclic quotient group of order 4 gives rise to a single isomorphism, there are $15 + (2 \times 11) + 3 = 40$ groups of division (c) if $(p - 1)$ is divisible by 4. Otherwise the number is 0. Each such group contains just one invariant subgroup H of order 8, and the groups may be classified as follows:

H	No. of groups
Cyclic	9
Abelian of type (2, 1)	16
Dihedral	4
Dicyclic	4
Abelian of type (1, 1, 1)	7
Total	40

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* G. A. Miller, *Annals of Mathematics*, ser. 2, vol. 31 (1930), p. 163.

Division (*d*). $(G^{32}_{32} : G^{p}_{8p})_8$. For seven of the groups of order 32, the commutator quotient group has an invariant $\geqq 8$. Each such group must give rise to at least one group of division (d), and no other group of order 32 can give rise to a group of this division. In five of the seven groups, the invariant subgroups complementary to cyclic quotient groups of order 8 form one set of conjugates under the holomorph of the group; in each of the other two groups they form two such sets. As each of these cases of cyclic quotient group of order 8 gives rise to a single isomorphism, there are $5 + (2 \times 2) = 9$ groups of division (d) if $(p - 1)$ is divisible by 8. Otherwise the number is 0. Every group of this division contains just one invariant subgroup of order 4. In five of the nine groups, this subgroup is cyclic.

Division (*e*). $(G^{32}_{32} : G^{p}_{16p})_{16}$. Of the groups of order 32, only the cyclic group and the abelian group of type (4, 1) involve cyclic quotient groups of order 16. Each of these involves one case of such a quotient group and each of these cases gives rise to a single isomorphism. The number of groups of division (e) is therefore two if $(p - 1)$ is divisible by 16; otherwise the number is 0.

Division (*f*). $(G^{32}_{32} : G^{p}_{32p})_{32}$. The only group of order 32 which can be used in such an isomorphism is the cyclic group, and it gives rise to a single isomorphism. Hence the number of groups of division (f) is one if $(p - 1)$ is divisible by 32; otherwise the number is 0.

Division (*g*). $(G^{32}_{32p} : G^{p}_{pk_2})_{pk_2}$. Groups of this division exist only when p divides $(2^n - 1)$ and $1 < n < 6$—that is when $p = 3, 5, 7$ or 31. These cases will therefore be considered separately.

When $p = 3$, the number of groups of division (g) has been shown to be 36.*

When $p = 5$, the groups in question are of order 160.

(1) Let G be a group of order 160 and division (g) containing no subgroup of order 80. G must therefore contain an invariant subgroup K of order 32, but can contain no invariant subgroup of order 16 since it can involve no quotient group of order 10. K must thus contain exactly 15 subgroups of order 16 which must be simply isomorphic in sets of $5n$. There is only one group of order 32 which fulfills this condition, so K is uniquely determined. G corresponds to a set of conjugate subgroups of order 5 in the i-group of K. As these subgroups are Sylow subgroups, there is only one such group.

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* *Loc. cit.* A private communication from Dr. G. A. Miller informs the present authors that there are 26 instead of 25 groups of order 96 which contain four subgroups of order 3. Thus there are 36 instead of 35 groups of this order which contain more than one subgroup of order 3, and the total number of groups of order 96 is 231 instead of 230 as originally stated.

(2) Let G be a group of order 160 and division (g) containing a subgroup H of order 80. H contains all the 16 subgroups of order 5 in G, and, since there is only one group of order 80 which meets this condition, H is uniquely determined. It contains a characteristic abelian subgroup L of order 16 and type (1, 1, 1, 1) which must be invariant under G.

In H, each subgroup of order 5 is its own normalizer, but in G the normalizer N of such a subgroup is of order 10. N is non-invariant under G since its single subgroup of order 5 is non-invariant. G can be generated by the adjunction to H of an operator T ($T^2 = 1$) which with a subgroup of order 5 generates N.

If T is invariant under G, it generates with L an invariant abelian subgroup K of order 32 and type (1, 1, 1, 1, 1), and G corresponds to a set of conjugate subgroups of order 5 in the i-group of K. Since this i-group contains Sylow subgroups of order 5, there is one group when T is invariant under G.

If T is non-invariant under G, then N contains no subgroup besides identity which is invariant under G, since all the operators of order 2 in N are conjugate under N. G can therefore be expressed transitively on 16 letters, and when thus expressed, L is regular, so that G corresponds to a set of conjugate subgroups of order 10 in the i-group of L. As this i-group contains only one such set, there is one group when T is non-invariant under G.

It has thus been shown that when $p = 5$ there are three groups of division (g).

When $p = 7$, the groups in question are of order 224. If a group G of this order contained no subgroup of order 112, it would necessarily contain an invariant subgroup K of order 32. But G could not contain an invariant subgroup of order 16, since it could involve no quotient group of order 14. Hence K would necessarily contain 7 subgroups of order 16 conjugate under G, and no other subgroups of this order. But as there is no group of order 32 which meets this condition, the case does not arise, and G must contain a subgroup H of order 112.

If G is of division (g), H can contain no invariant subgroup of order 7 and is hence uniquely determined.* It contains a characteristic abelian subgroup L of order 16 and type (1, 1, 1, 1) which is invariant under G. In H, the normalizer of a subgroup of order 7 is generated by S ($S^{14} = 1$ and S^7 is in L); but in G, the normalizer N of such a subgroup is of order 28. G can therefore be generated by the adjunction to H of an operator T ($T^2 = 1$ or $T^2 = S^7$) which with S generates N. N is non-invariant under G since its single subgroup of order 7 is non-invariant. S^7 is characteristic under H

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* See p. 323 of this paper.

and hence invariant under G. If the subgroup of order 4 generated by S^7 and T were non-invariant under G, then N would contain no proper subgroup invariant under G other than that generated by S^7. There would then exist a transitive representation of G of degree 8 and order 112. As no such transitive group of degree 8 exists,* the group generated by S^7 and T is invariant under G. Hence T and L generate an invariant subgroup K of order 32, and G corresponds to a set of conjugate subgroups of order 7 in the i-group of K.

K must contain L (an abelian group of type $(1, 1, 1, 1)$), and there are five groups of order 32 which meet this condition. Three of these are non-abelian and their i-groups contain no subgroups of order 7. Thus these groups give rise to no groups of order 224 and division (g). The i-groups of the two abelian groups contain Sylow subgroups of order 7, and hence each of these groups gives rise to one group of order 224 and division (g).

It has thus been shown that when $p = 7$, there are two groups of division (g).

When $p = 31$, the groups in question are of order 992. Let G be a group of this order and division (g). G must contain 32 subgroups of order 31 which involve 960 operators of this order. So G must contain an invariant subgroup K of order 32 and must correspond to a set of conjugate subgroups of order 31 in the i-group of K. Hence K must contain 31 operators all of the same order, and can thus be only the abelian group of type $(1, 1, 1, 1, 1)$. Since the i-group of this group contains Sylow subgroups of order 31, there is only one group of division (g) when $p = 31$.

The numbers of groups of order $32p$ can then be tabulated as follows:

p	No. of groups of division							Total
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	
3	51	144	0	0	0	0	36	231
5	51	144	40	0	0	0	3	238
7	51	144	0	0	0	0	2	197
31	51	144	0	0	0	0	1	196
> 7 and $\neq 31$								
$(p-1)$ divisible by 2 but not by 4.	51	144	0	0	0	0	0	195
$(p-1)$ divisible by 4 but not by 8.	51	144	40	0	0	0	0	235
$(p-1)$ divisible by 8 but not by 16.	51	144	40	9	0	0	0	244
$(p-1)$ divisible by 16 but not by 32.	51	144	40	9	2	0	0	246
$(p-1)$ divisible by 32.	51	144	40	9	2	1	0	247

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* *Loc. cit.*

DETERMINATION OF THE GROUPS OF ORDERS 101-161, OMITTING ORDER 128.

By J. K. Senior and A. C. Lunn.

In a recent paper,[*] G. A. Miller has listed the numbers of the groups of every order g where g does not exceed 100. General methods are known for determining the groups of any order g where g is the product of less than five prime factors,[†] and if $100 < g < 162$ there are only six integers where g is the product of more than four such factors. These are

$$g = 108 = 2^2 \cdot 3^3 \qquad g = 128 = 2^7$$
$$g = 112 = 2^4 \cdot 7 \qquad g = 144 = 2^4 \cdot 3^2$$
$$g = 120 = 2^3 \cdot 3 \cdot 5 \qquad g = 160 = 2^5 \cdot 5$$

The groups of orders 112 and 160 have been determined by Lunn and Senior.[‡] To determine the number of groups of order 128 is a very laborious task, and no attempt will here be made to solve the problem. But brief arguments suffice to cover the cases of 108, 120 and 144, and these orders will therefore be treated in some detail. Where g is the product of less than five prime factors, since the general methods are known, only the results will be given. The references to the transitive groups of degrees less than ten are taken from the tables of G. A. Miller [§] and F. N. Cole.[¶]

In treating the solvable groups of orders 108, 120 and 144, the argument will be based [‖] on the

THEOREM. *Every solvable group G of order $g = \Pi x_i$ determines its complete array of R's and the isomorphism according to which each pair of*

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[*] G. A. Miller, *American Journal of Mathematics*, vol. 52 (1930), p. 617.

[†] Hölder, *Mathematische Annalen*, vol. 43 (1893), p. 409, Orders p, p^2, pq, p^3, p^2q, pqr, p^4; "Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen," *Mathematisch-physikalische Klasse* (1895), p. 211, Orders $pqrs$; Le Vavasseur, *Annales scientifiques de l'École Normale Supérieure*, ser. 3, vol. 19 (1902), p. 335, Orders p^2q^2; Western, *Proceedings of the London Mathematical Society*, ser. 1, vol. 30 (1899), p. 209, Orders p^3q; Glenn, *Transactions of the American Mathematical Society*, vol. 7 (1906), p. 137, Orders p^2qr.

[‡] Lunn and Senior, *American Journal of Mathematics*, this vol., pp. 319-327.

[§] G. A. Miller, *American Journal of Mathematics*, vol. 21 (1899), p. 326.

[¶] F. N. Cole, *Quarterly Journal of Pure and Applied Mathematics*, vol. 26 (1893), p. 372.

[‖] Lunn and Senior, *loc. cit.*

R's must be combined to give a representation of G. Any representation of G which involves every R of a complete array belonging to the order g is simply isomorphic with G.

The methods and symbols to be used are those described in the paper just cited. The symbol $G^x{}_{xk}$ indicates a solvable abstract group simply isomorphic with a transitive group of degree x and order xk where $x = p^a$ and k is not divisible by p. The symbol $(G^{x_1}{}_{x_1k_1} : G^{x_2}{}_{x_2k_2})_{k_1k_2}$ indicates that two such groups $(p_1 \neq p_2)$ are combined according to a quotient group of order k_1k_2 to form a group of order x_1x_2. Similarly the symbol $[(G^{x_1}{}_{x_1k_1} : G^{x_2}{}_{x_2k_2})_u : G^{x_3}{}_{x_3k_3}]_v$ indicates that $G^{x_1}{}_{x_1k_1}$ and $G^{x_2}{}_{x_2k_2}$ are combined according to a quotient group of order u, and that the resultant group is then combined with $G^{x_3}{}_{x_3k_3}$ according to a quotient group of order v where $uv = k_1k_2k_3$ and $p_3 \neq p_1$ or p_2. The group thus obtained is uniquely determined by the three groups $G^x{}_{xk}$ and the isomorphisms used to combine them. It is independent of which pair is first combined, so that this choice can be made a matter of convenience.

The Groups of Order $108 = 2^2 \cdot 3^3$. Every group of order 108 is solvable and hence determines a $(G^4{}_{4k_1} : G^{27}_{27k_2})_{k_1k_2}$. $k_1 = 1$ or 3, since 4 and 12 are the only orders which divide 108 for which transitive groups of degree four exist. $k_2 = 1, 2$ or 4. Thus every group of order 108 occurs in one of the following divisions:

Division	(a)	$(G^4{}_4 : G^{27}_{27})_1$	Division	(d)	$(G^4{}_{12} : G^{27}_{27})_3$
"	(b)	$(G^4{}_4 : G^{27}_{54})_2$	"	(e)	$(G^4{}_{12} : G^{27}_{54})_6$
"	(c)	$(G^4{}_4 : G^{27}_{108})_4$	"	(f)	$(G^4{}_{12} : G^{27}_{108})_{12}$

Division (*a*). $(G^4{}_4 : G^{27}_{27})_1$. A group in this division is the direct product of its Sylow subgroups. Since there are five groups of order 27 and two groups of order 4, there are $5 \times 2 = 10$ groups of division (a). Six of these are abelian.

Division (*b*). $(G^4{}_4 : G^{27}_{54})_2$. There are ten distinct groups G^{27}_{54}.* Each of these contains one invariant subgroup of order 27 and can consequently be dimidiated in only one way. Each of the two groups of order four permits but one distinct dimidiation, and so there are $10 \times 2 = 20$ groups in division (b).

Division (*c*). $(G^4{}_4 : G^{27}_{108})_4$. A group in this division corresponds to a set of conjugate subgroups of order 4 in the i-group of a group of order 27. The five groups of this latter order will therefore be considered one at a time.

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* G. A. Miller, *Quarterly Journal of Pure and Applied Mathematics*, vol. 30 (1898), p. 243.

(1) *The cyclic group* has an i-group containing Sylow subgroups of order 2, and hence gives rise to no group of division (c).

(2) *The abelian group of type* (2, 1) has an i-group containing non-cyclic Sylow subgroups of order 4, and hence gives rise to one group of division (c).

(3) *The non-abelian group which contains operators of order* 9 has an i-group containing Sylow subgroups of order 2, and hence gives rise to no groups of division (c).

(4) *The abelian group of type* (1, 1, 1) has an i-group containing Sylow subgroups of order 32. Under the i-group, the subgroups of order 4 are permuted in five sets of conjugates, two of which are cyclic. Thus there arise five groups of division (c).

(5) *The non-abelian group which contains no operators of order* 9 has an i-group containing Sylow subgroups of order 16. Under the i-group, the subgroups of order 4 are permuted in two sets of conjugates, one of which is cyclic. Thus there arise two groups of division (c).

The number of groups of division (c) is thus $1 + 5 + 2 = 8$. In three of these groups, the quotient group of order 4 is cyclic.

Division (*d*). $(G^4{}_{12} : G^{27}_{27})_3$. There is only one group $G^4{}_{12}$. It involves but one case of quotient group of order 3, and this case gives rise to a single isomorphism. The five groups of order 27 involve seven distinct cases of quotient group of order 3, and hence there are seven groups of division (d). Three of these contain operators of order 18.

Division (*e*). $(G^4{}_{12} : G^{27}_{54})_6$. As $G^4{}_{12}$ contains no invariant subgroup of order 2, there are no groups of division (e).

Division (*f*). $(G^4{}_{12} : G^{27}_{108})_{12}$. A group G in this division would contain an invariant subgroup of order 9 complementary to a quotient group simply isomorphic with $G^4{}_{12}$. Hence G would contain an invariant subgroup of order 36 containing three subgroups of order 18 conjugate under G. But every group of order 36 which contains three simply isomorphic subgroups of order 18 contains a characteristic subgroup of order 4, and no such group of order 36 can be an invariant subgroup of G, since a group of division (f) can not contain an invariant subgroup of order 4. Hence there are no groups of division (f).

The number of groups of order 108 is thus

Division	(a)	(b)	(c)	(d)		
Number	10	20	8	7	Total	45

The Groups of Order $120 = 2^3 \cdot 3 \cdot 5$. It is well known that there are just three insolvable groups of order 120.

Every solvable group of order 120 determines a $(G^8_{8k_1} : G^5_{5k_2} : G^3_{3k_3})_{uv}$ $(uv = k_1k_2k_3)$. $k_1 = 1$ or 3, since 8 and 24 are the only orders which divide 120 for which transitive groups of degree 8 exist; $k_2 = 1, 2$ or 4, since 5, 10 and 20 are the only orders for which solvable transitive groups of degree 5 exist; $k_3 = 1$ or 2, since 3 and 6 are the only orders for which transitive groups of degree 3 exist.

Thus every solvable group of order 120 occurs in one of the following divisions:

Division (a) $(G^8_8 : G^5_5 : G^3_3)_1$
" (b) $[(G^8_8 : G^5_{10})_2 : G^3_3]_1$
" (c) $[(G^8_8 : G^3_6)_2 : G^5_5]_1$
" (d′) $[(G^5_{10} : G^3_6)_2 : G^8_8]_2$
" (d″) $[(G^5_{10} : G^3_6)_1 : G^8_8]_4$
" (e) $[(G^8_8 : G^5_{20})_4 : G^3_3]_1$
" (f) $[(G^8_8 : G^5_{20})_4 : G^3_6]_2$
" (g) $[(G^8_{24} : G^3_3)_3 : G^5_5]_1$
" (h) $[(G^8_{24} : G^3_3)_3 : G^5_{10}]_2$
" (i) $[(G^8_{24} : G^3_6)_6 : G^5_5]_1$
" (j) $[(G^8_{24} : G^3_6)_6 : G^5_{10}]_2$
" (k) $[(G^8_{24} : G^3_3)_3 : G^5_{20}]_4$
" (l) $[(G^8_{24} : G^3_6)_6 : G^5_{20}]_4$

Division (a). $(G^8_8 : G^5_5 : G^3_3)_1$. A group in this division is the direct product of its Sylow subgroups. As there are five groups of order 8, one of order 5 and one of order 3, there are five groups of division (a). Three of these are abelian.

Divisions (b) and (c). $[(G^8_8 : G^5_{10})_2 : G^3_3]_1$ and $[(G^8_8 : G^3_6)_2 : G^5_5]_1$. There are seven groups* $(G^8_8 : G^5_{10})_2$ and hence seven groups of division (b). Similarly there are seven groups $(G^8_8 : G^3_6)_2$ and hence seven groups of division (c).

Division (d′). $[(G^5_{10} : G^3_6)_2 : G^8_8]_2$. There is only one group $(G^5_{10} : G^3_6)_2$; it permits but one dimidiation. As the five groups of order 8 permit seven distinct dimidiations, there are seven groups of division (d′).

Division (d″). $[(G^5_{10} : G^3_6)_1 : G^8_8]_4$. There is only one group $(G^5_{10} : G^3_6)_1$. Its only case of quotient group of order 4 is non-cyclic, and, of the three subgroups of index two involved, no two are simply isomorphic. Hence there arise the following groups:

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* G. A. Miller, *Philosophical Magazine and Journal of Science*, ser. 5, vol. 42 (1896), p. 195.

Group of order 8	No. of groups of order 120
Cyclic	0
Abelian of type (2, 1)	3
Dihedral	3
Dicyclic	1
Abelian of type (1, 1, 1)	1
Total	8

There are thus 8 groups of division (d″).

Divisions (*e*) *and* (*f*). $[G^{8}_{8} : G^{5}_{20})_4 : G^{3}_{3}]_1$ and $[(G^{8}_{8} : G^{5}_{20})_4 : G^{3}_{6}]_2$. There are two groups $(G^{8}_{8} : G^{5}_{20})_4$. These two groups permit in all three dimidiations. They may thus be multiplied directly by G^{3}_{3} to give the two groups of division (e) or dimidiated with G^{3}_{6} to give the three groups of division (f).

Divisions (*g*) *and* (*h*). $[(G^{8}_{24} : G^{3}_{3})_3 : G^{5}_{5}]_1$ and $[(G^{8}_{24} : G^{3}_{3})_3 : G^{5}_{10}]_2$. There are two groups $(G^{8}_{24} : G^{3}_{3})_3$. One of these contains a single subgroup of order 12; the other contains no subgroup of this order. Consequently these groups yield the two groups of division (g) by direct multiplication with G^{5}_{5}, and the single group of division (h) by dimidiation with G^{5}_{10}.

Divisions (*i*) *and* (*j*). $[(G^{8}_{24} : G^{3}_{6})_6 : G^{5}_{5}]_1$ and $[(G^{8}_{24} : G^{3}_{6})_6 : G^{5}_{10}]_2$. There is only one group $(G^{8}_{24} : G^{3}_{6})_6$; it contains one subgroup of order 12. Consequently it yields the single group of division (i) by direct multiplication with G^{5}_{5}, and the single group of division (j) by dimidiation with G^{5}_{10}.

Divisions (*k*) *and* (*l*). $[(G^{8}_{24} : G^{3}_{3})_3 : G^{5}_{20}]_4$ and $[(G^{8}_{24} : G^{3}_{6})_6 : G^{5}_{20}]_4$. No group G^{8}_{24} contains an invariant subgroup of order 6, and so there are no groups of divisions (k) and (l).

The number of groups of order 120 is thus

Insolvable	3
Solvable	
Division (a)	5
" (b)	7
" (c)	7
" (d′)	7
" (d″)	8
" (e)	2
" (f)	3
" (g)	2
" (h)	1
" (i)	1
" (j)	1
Total	47

The Groups of Order $144 = 2^4 \cdot 3^2$. Every group of order 144 is solvable and hence determines a $(G^{16}_{16k_1} : G^{9}{}_{9k_2})_{k_1 k_2}$, $k_1 = 1, 3$ or 9. $k_2 = 1, 2, 4, 8$ or 16. Thus every group of order 144 occurs in one of the following divisions:

Division (a)	$(G^{16}_{16} : G^{9}{}_{9})_1$	Division (f)	$(G^{16}_{48} : G^{9}{}_{9})_3$
" (b)	$(G^{16}_{16} : G^{9}{}_{18})_2$	" (g)	$(G^{16}_{48} : G^{9}{}_{18})_6$
" (c)	$(G^{16}_{16} : G^{9}{}_{36})_4$	" (h)	$(G^{16}_{48} : G^{9}{}_{36})_{12}$
" (d)	$(G^{16}_{16} : G^{9}{}_{72})_8$	" (i)	$(G^{16}_{48} : G^{9}{}_{72})_{24}$
" (e)	$(G^{16}_{16k_1} : G^{9}{}_{144})_{16k_1}$	" (j)	$(G^{16}_{144} : G^{9}{}_{9k_2})_{9k_2}$

Division (*a*). $(G^{16}_{16} : G^{9}{}_{9})_1$. A group of this division is the direct product of its Sylow subgroups. As there are fourteen groups of order 16 and two groups of order 9, there are $14 \times 2 = 28$ groups of division (a). Ten of these are abelian.

Division (*b*). $(G^{16}_{16} : G^{9}{}_{18})_2$. There are three groups $G^{9}{}_{18}$, each one of which permits one dimidiation. As the fourteen groups of order 16 permit in all 28 distinct dimidiations,* there are $3 \times 28 = 84$ groups of division (b).

The groups of divisions (c) and (d) depend on certain quotient groups of order 4 and 8 involved in the groups of order 16. It is therefore convenient to list the fourteen abstract groups of this order according to their quotient groups of these types.

G_{16}	No. of distinct cases of quotient group which are				
	Cyclic Order 4	Non-cyclic Order 4	Cyclic Order 8	Dihedral Order 8	Dicyclic Order 8
(1)	1	0	1	0	0
(2)	2	1	1	0	0
(3)	0	1	0	1	0
(4)	0	1	0	1	0
(5)	0	1	0	1	0
(6)	2	1	0	0	0
(7)	1	1	0	0	0
(8)	1	2	0	0	0
(9)	0	3	0	1	0
(10)	0	2	0	0	1
(11)	0	3	0	0	0
(12)	1	1	0	1	1
(13)	1	1	0	1	0
(14)	0	1	0	0	0
	9	19	2	6	2

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* *Loc. cit.*

This listing suffices to identify uniquely each of the groups of order 16 except numbers (3), (4) and (5). These three groups are generated as follows:

$$(3)\quad A^8 = B^2 = 1.\quad B^{-1}AB = A^7.$$
$$(4)\quad A^8 = 1.\quad B^2 = A^4.\quad B^{-1}AB = A^7.$$
$$(5)\quad A^8 = B^2 = 1.\quad B^{-1}AB = A^3.$$

Division (c). $(G^{16}_{16} : G^9{}_{36})_4$. The groups of this division fall into two sub-divisions according to whether the quotient group utilized is cyclic or non-cyclic.

(1) Cyclic quotient group of order four. There is only one case of such a quotient group among the two groups $G^9{}_{36}$ and this case gives rise to a single isomorphism. Hence with the nine cases of cyclic quotient group of order 4 involved in the groups of order 16, there arise 9 groups of order 144.

(2) Non-cyclic quotient group of order 4. There is only one case of such a quotient group among the two groups $G^9{}_{36}$. It involves only one characteristic subgroup of index two. Hence the following groups arise:

G^{16}_{13}	No. of groups of order 144
(1)	0
(2)	2
(3)	2
(4)	2
(5)	3
(6)	2
(7)	1
(8)	2 + 1
(9)	2 + 2 + 2
(10)	2 + 1
(11)	3 + 2 + 1
(12)	2
(13)	2
(14)	1
Total	35

There are thus $9 + 35 = 44$ groups of division (c).

Division (d). $(G^{16}_{16} : G^9{}_{72})_8$. Each one of the three groups $G^9{}_{72}$ contains an invariant Sylow subgroup of order 9. This subgroup is complementary to a quotient group of order 8 which is respectively cyclic, dihedral or dicyclic in the three instances. Therefore the groups of this division fall into three sub-divisions.

(1) Cyclic quotient group of order 8. Each of the two cases of cyclic quotient group of order 8 involved in the groups of order 16 gives rise to a single isomorphism, and hence there are two groups in this sub-division.

(2) Dihedral quotient group of order 8. The group $G^9{}_{72}$ which involves a dihedral quotient group of order 8 contains only one characteristic subgroup of index 2 which corresponds to the cycle of order 4 in the quotient group. Hence this dihedral quotient group gives rise to a single isomorphism. As the groups of order 16 involve in all 6 distinct cases of dihedral quotient group of order 8, there are six groups of this sub-division.

(3) Dicyclic quotient group of order 8. The group $G^9{}_{72}$ which involves a dicyclic quotient group of order 8 contains three subgroups of index 2 which are conjugate under an outer isomorphism. Hence this dicyclic quotient group gives rise to a single isomorphism. As the groups of order 16 involve, in all, two distinct cases of dicyclic quotient group of order 8, there are two groups in this sub-division.

There are thus $2 + 6 + 2 = 10$ groups of division (d).

Division (*e*). $(G^{16}_{16k_1} : G^9{}_{144})_{16k_1}$. Every group in this division is simply isomorphic with its $G^9{}_{144}$. As there is only one transitive group of degree 9 and order 144, there is only one group of division (e).

The groups of divisions (f), (g) and (h) depend on certain of the quotient groups involved in the ten groups G^{16}_{48}. These groups have been described by G. A. Miller.*

Six of them involve one distinct case of quotient group of order 3 each.

Four of them involve one distinct case of cyclic quotient group of order 6 each.

Four of them involve one distinct case of non-cyclic quotient group of order 6 each.

One of them involves one distinct case of dihedral quotient group of order 12.

Division (*f*). $(G^{16}_{48} : G^9{}_9)_3$. Each of the two groups of order 9 involves one distinct case of quotient group of order 3, and this case gives rise to a single isomorphism. By combining these two groups with the six groups G^{16}_{48} which involve quotient groups of order 3, there are obtained the $2 \times 6 = 12$ groups of division (f).

Division (*g*). $(G^{16}_{48} : G^9{}_{18})_6$. The groups of this division fall into two sub-divisions according to whether the quotient group of order 6 utilized is cyclic or non-cyclic.

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* *Loc. cit.*

(1) Cyclic quotient group of order 6. The three groups $G^{9}{}_{18}$ involve only one case of quotient group of this type, and this case gives rise to a single isomorphism. Thus by combination with the four groups G^{16}_{48} which involve cyclic quotient groups of order 6, the four groups of this sub-division are obtained.

(2) Non-cyclic quotient group of order 6. Such a quotient group gives rise to a single isomorphism. The three groups $G^{9}{}_{18}$ involve in all three distinct cases of quotient group of this type, and, by combination with the four groups G^{16}_{48} which involve non-cyclic quotient groups of order 6, the $3 \times 4 = 12$ groups of this sub-division are obtained.

Thus there are $4 + 12 = 16$ groups of division (g).

Division (*h*). $(G^{16}_{48} : G^{9}{}_{36})_{12}$. The two groups $G^{9}{}_{36}$ involve in all only one distinct case of quotient group of order 12; this is a dihedral quotient group. Only one case of quotient group of this type occurs among the ten groups G^{16}_{48} and this case gives rise to a single isomorphism. Hence there is one group of division (h).

Division (*i*). $(G^{16}_{48} : G^{9}{}_{72})_{24}$. As no group $G^{9}{}_{72}$ contains an invariant subgroup of order 3, there are no groups of division (i).

Division (*j*). $(G^{16}_{144} : G^{9}{}_{9k_2})_{9k_2}$. Every group in this division is simply isomorphic with its G^{16}_{144} and hence, to determine the number of groups of the division, it is sufficient to determine the number of transitive groups of degree 16 and order 144.

If G is a group which fulfills the above condition, it can contain no invariant subgroup of order 3 or 9. Consequently G can contain no subgroup of order 72, for every group of this order contains a characteristic subgroup of order 3 or 9 which would necessarily be invariant under G. Hence G can contain no invariant subgroup of order 24, since it can involve no quotient group of order six.

If G contains no subgroup of order 72, it must contain an invariant subgroup H of order 48. H can however contain no characteristic subgroup of order 3 or 24. Every group of order 48 which fulfills this condition contains a characteristic subgroup of order 16. Hence G contains an invariant subgroup K of order 16 and corresponds to a set of conjugate subgroups of order nine in the i-group of K. The only group of order 16 whose i-group contains subgroups of order nine is the abelian group of type (1, 1, 1, 1), and in the i-group in question the subgroups of order nine are Sylow subgroups. Hence there is only one group of division (j).

The number of groups of order 144 is thus

Division (a)	28
“ (b)	84
“ (c)	44
“ (d)	10
“ (e)	1
“ (f)	12
“ (g)	16
“ (h)	1
“ (j)	1
Total	197

There follows the list of the number of groups of every order (except 128) between 100 and 162 where this number exceeds one.

Order	Factors	Number of groups
102	$2 \cdot 3 \cdot 17$	4
104	$2^3 \cdot 13$	14
105	$3 \cdot 5 \cdot 7$	2
106	$2 \cdot 53$	2
108	$2^2 \cdot 3^3$	45
110	$2 \cdot 5 \cdot 11$	6
111	$3 \cdot 37$	2
112	$2^4 \cdot 7$	43
114	$2 \cdot 3 \cdot 19$	6
116	$2^2 \cdot 29$	5
117	$3^2 \cdot 13$	4
118	$2 \cdot 59$	2
120	$2^3 \cdot 3 \cdot 5$	47
121	11^2	2
122	$2 \cdot 61$	2
124	$2^2 \cdot 31$	4
125	5^3	5
126	$2 \cdot 3^2 \cdot 7$	16
128	2^7	not determined
129	$3 \cdot 43$	2
130	$2 \cdot 5 \cdot 13$	4
132	$2^2 \cdot 3 \cdot 11$	10
134	$2 \cdot 67$	2

Order	Factors	Number of groups
135	$3^3 \cdot 5$	5
136	$2^3 \cdot 17$	15
138	$2 \cdot 3 \cdot 23$	4
140	$2^2 \cdot 5 \cdot 7$	11
142	$2 \cdot 71$	2
144	$2^4 \cdot 3^2$	197
146	$2 \cdot 73$	2
147	$3 \cdot 7^2$	6
148	$2^2 \cdot 37$	5
150	$2 \cdot 3 \cdot 5^2$	13
152	$2^3 \cdot 19$	12
153	$3^2 \cdot 17$	2
154	$2 \cdot 7 \cdot 11$	4
155	$5 \cdot 31$	2
156	$2^2 \cdot 3 \cdot 13$	18
158	$2 \cdot 79$	2
160	$2^5 \cdot 5$	238

THE COVARIANTS OF TWO QUADRATIC FORMS IN n VARIABLES.

By J. WILLIAMSON.

1. *Introduction.* It is known that two quadratic forms in n variables possess $n+1$ projective covariants and that these $n+1$ covariants, together with the $n+1$ invariants, form a complete system of covariants and invariants. Of these covariants n are quadratic while the remaining one is the Jacobian of the other n. These n quadratic covariants play an important role in the determination by symbolic methods of the irreducible concomitants of the two quadratics.* Unfortunately the actual covariants are not very easily obtained from their symbolic expressions. It is our purpose here to determine another system of n quadratic covariants which can be expressed comparatively simply both symbolically and non-symbolically. Later we make use of these covariants to show how the factorization of the Jacobian, mentioned above, depends on the nature of the elementary divisors of the pencil of matrices defined by the two quadratics.

Let $f=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ and $g=\sum_{i,j=1}^{n} r_{ij}x_ix_j$ be two quadratic forms in n variables, whose matrices are A and R respectively, so that in matrix notation

$$f=X'AX \quad \text{and} \quad g=X'RX.$$

With the usual notation we shall write these two quadratics in the symbolic forms,

$$f=(ax)^2=(bx)^2=(cx)^2=\cdots, \qquad g=(rx)^2=(sx)^2=(tx)^2=\cdots,$$

where $a, b, c, \cdots;$ $r, s, t, \cdots$, are two sets of equivalent symbols. We shall denote the determinantal bracket factor containing i equivalent symbols $a, b, c, \cdots$ and $n-i$ equivalent symbols $r, s, t,$ by (A_iR_{n-i}). In particular we shall write α for A_{n-1} and ρ for R_{n-1}, so that the tangential forms of f and g are

$$(u\alpha)^2=\sum_{i,j=1}^{n}\alpha_{ij}u_iu_j, \quad \text{and} \quad (u\rho)^2=\sum_{i,j=1}^{n}\rho_{ij}u_iu_j,$$

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* J. Williamson, "A special prepared system for two quadratics in n variables," *American Journal of Mathematics*, vol. 52 (1930), pp. 399-412.

respectively. A simple calculation shows that, if A_{ij} is the cofactor of a_{ij} in A and R_{ij} the cofactor of r_{ij} in R,

$$(1) \qquad \alpha_{ij} = (n-1)!\,A_{ij}; \quad \rho_{ij} = (n-1)!\,R_{ij}; \qquad (i, j = 1, 2, \cdots, n).$$

If $A_i = aA_{i-1}$, that is, if the first column of A_i is a and the remaining $i-1$ columns are denoted by A_{i-1}, we define the bracket factor $(ix) = (A_iR_{n+1-i}x)$ by

$$(2) \qquad (ix) = (A_iR_{n+1-i}x) = \underset{a}{\Omega}(ax)(A_{i-1}R_{n+1-i}) \qquad (i = 2, 3, \cdots, n-1).$$

On the right of (2) is a series of i terms obtained from $a_x(A_{i-1}R_{n+1-i})$ by permuting determinantally all i symbols in A_i. For example, if $i = 3$ and $A_3 = abc$

$$\begin{aligned}(A_3R_{n-2}x) &= (abcR_{n-2}x)\\ &= a_x(bcR_{n-2}) - b_x(acR_{n-2}) + c_x(abR_{n-2}).\end{aligned}$$

Moreover, if $R_{n+1-i} = rR_{n-i}$, it follows from well known identities * that,

$$(3) \qquad (A_iR_{n+1-i}x) = \underset{a}{\Omega}\, a_x(A_{i-1}R_{n+1-i}) = (-1)^{n-1}\underset{r}{\Omega}\, r_x(A_iR_{n-i}).$$

Further, if we write $A_i = abA_{i-2}$ and $R_{n+1-i} = rsR_{n-1-i}$, since all the symbols appearing in A_i are equivalent, we have the following results:

$$(4) \qquad (ix)(iy) = i(ax)(A_{i-1}R_{n+1-i})(iy)$$

$$\begin{aligned}(5) \qquad &= i(ax)^2(A_{i-1}R_{n+1-i})^2\\ &\quad - i(i-1)(ax)(by)(bA_{i-2}R_{n+1-i})(aA_{i-2}R_{n+1-i})\end{aligned}$$

and similarly,

$$(6) \qquad (ix)(iy) = (-1)^{i-1}(n+1-i)(rx)(A_iR_{n-i})(iy)$$

$$\begin{aligned}(7) \qquad &= (n+1-i)(rx)(ry)(A_iR_{n-i})^2\\ &\quad - (n+1-i)(n-i)r_xs_y(A_isR_{n-i-1})(A_irR_{n-i-1}).\end{aligned}$$

A complete system of projective invariants for the two quadratics f and g is given by the set Δ_i, $(i = 0, 1, 2, \cdots, n)$ where Δ_i is the coefficient of λ^i in the expansion of the determinant $|A + \lambda R|$. The symbolic expression for Δ_i is,

$$(8) \qquad i!\,(n-i)!\,\Delta_i = (A_iR_{n-i})^2, \qquad (i = 0, 1, 2, \cdots, n).$$

One complete system of covariants consists of the n quadratic covariants $(ix)^2$,

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* H. W. Turnbull, *Determinants, Matrices and Invariants*, pp. 47-50.

where (ix) is defined by (3) for $(i = 2, \cdots, n-1)$ and $(1x) = (ax)$ and $(nx) = (rx)$, together with their Jacobian. Non-symbolically the covariant $(ix)^2$ is proportional to the coefficient of λ^{i-1} in the expansion of the determinant,

$$\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ x_1 & A_{11} + \lambda R_{11} & A_{12} + \lambda R_{12} & \cdots & A_{1n} + \lambda R_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_n & A_{n1} + \lambda R_{n1} & A_{n2} + \lambda R_{n2} & \cdots & A_{nn} + \lambda R_{nn} \end{vmatrix};$$

in other words $(ix)^2$ is proportional to the coefficient of λ^{i-1} in the point form of the pencil $(u\alpha)^2 + \lambda(u\rho)^2$.*

2. We now introduce other covariants which are linear combinations, with invariant coefficients, of the covariants mentioned above. Let

$$\phi_0 = (ax)^2,$$
$$\mu\phi_1 = (ax)(a\rho)(b\rho)(bx),$$
$$\mu^2\phi_2 = (ax)(a\rho)(b\rho)(b\sigma)(c\sigma)(cx),$$

and in general,

$$\mu^i\phi_i = (a_1x)(a_1\rho_1)(a_2\rho_1)(a_2\rho_2)\cdots(a_i\rho_i)(a_{i+1}\rho_i)(a_{i+1}x),$$

where $\mu = (n-1)\,!$, $a_1, a_2, \cdots, a_{i+1}$ are symbols equivalent to a and $\sigma, \rho_1, \rho_2, \cdots, \rho_i$ are symbols equivalent to ρ. But $\phi_0 = f = X'AX$, and

$$\begin{aligned} \mu\phi_1 &= \sum_{i,j,k,t=1}^{n} x_i a_{ij} \rho_{jk} a_{kt} x_t, \\ &= \mu \sum_{i,j,k,t=1}^{n} x_i a_{ij} R_{jk} a_{kt} x_t \quad \text{by (1)}, \\ &= \mu X'ASAX, \end{aligned}$$

where S is the matrix adjugate to R. Moreover

$$\begin{aligned} \mu^2\phi_2 &= \sum_{i,\ldots,p=1}^{n} x_i a_{ij} \rho_{jk} a_{kt} \rho_{ts} a_{sp} x_p, \\ &= \mu^2 X'A(SA)^2X, \end{aligned}$$

and similarly

$$\mu^i\phi_i = \mu^i X'A(SA)^iX.$$

Hence we have

$$\phi_i = X'A(SA)^iX, \qquad (i = 0, 1, 2, \cdots). \tag{9}$$

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* Turnbull and Williamson, "The minimum system of two quadratic forms," *Proceedings of the Royal Society of Edinburgh*, vol. 45 (1923), pp. 149-165.

We now proceed to express ϕ_i, $(i = 0, 1, 2, \cdots, n-1)$ in terms of the covariants $(ix)^2$. To do this we first consider the product, $(2x)(2\rho_2)$. If $(2x) = (A_2\rho_1 x)$ and $A_2 = a_1 a_2$, by (15)

$$(2x)(2\rho_2) = 2(a_1x)(a_1\rho_2)(a_2\rho_1)^2 - 2(a_1x)(a_1\rho_1)(a_2\rho_1)(a_2\rho_2)$$

or

$$(10) \qquad (a_1x)(a_1\rho_1)(a_2\rho_1)(a_2\rho_2) = (a_2\rho_1)^2(a_1x)(a_1\rho_2) - \tfrac{1}{2}(2x)(2\rho_2).$$

If we denote by $\{\rho_2\}$ the product of all factors to the right of $(a_2\rho_2)$ in the definition of $\mu^i\phi_i$, we derive from (10) the results

$$\mu^i\phi_i = (a_2\rho_1)^2\mu^{i-1}\phi_{i-1} - \tfrac{1}{2}(2x)(2\rho_2)\{\rho_2\} \qquad i \geqq 2,$$

and

$$\mu\phi_1 = (a_2\rho_1)^2\phi_0 - \tfrac{1}{2}(2x)^2,$$

or

$$(11) \qquad \mu^i\phi_i = \mu^i\Delta_{n-1}\phi_{i-1} - \tfrac{1}{2}(2x)(2\rho_2)\{\rho_2\} \qquad i \geqq 2,$$

and $\mu\phi_1 = \mu\Delta_{n-1}\phi_0 - \tfrac{1}{2}(2x)^2$ by (8).

Moreover, since

$$(ix)(i\rho) = (-1)^{i-1}(n-i+1)(ix)(A_iR_{n-i})(r\rho) \quad \text{by (6)},$$

$$\begin{aligned}(ix)(i\rho)(a\rho)(a\sigma) &= (-1)^{i-1}(n-i+1)(A_irR_{n-i}x)(A_iR_{n-i})(r\rho)(a\rho)(a\sigma),\\ &= (-1)^{i-1}\frac{n-i+1}{n}(A_i \cdot aR_{n-i}x)(A_iR_{n-i})(r\rho)^2(a\sigma),\end{aligned}$$

$$= (r\rho)^2\frac{n-i+1}{n}\Big[(A_iR_{n-i})^2(ax)(a\sigma) - \frac{1}{i+1}(i+1, x)(i+1, \sigma)\Big] \quad \text{by (4)},$$

$$\begin{aligned}(12) \qquad &= \mu\Delta_n\Big[(n-i+1)!\,i!\,\Delta_{n-i}(ax)(a\sigma)\\ &\qquad - \frac{n-i+1}{i+1}(i+1, x)(i+1, \sigma)\Big] \quad \text{by (8)}.\end{aligned}$$

If, then, when $i > j$,

$$\begin{aligned}(13) \qquad \mu^i\phi_i = \mu^i &\sum_{k=1}^{j-1}(-1)^{k-1}\Delta_n^{k-1}\Delta_{n-k}\phi_{i-k}\\ &+ (-1)^{j-1}\mu^{j-1}\Delta_n^{j-2}\frac{(jx)(j\rho_j)\{\rho_j\}}{j!\,(n-j+1)!},\end{aligned}$$

where $\{\rho_j\}$ denotes the product of all factors to the right of $(a_j\rho_j)$ in $\mu^i\phi_i$, by substitution from (12) with i replaced by j, ρ by ρ_j and σ by ρ_{j+1} we have

$$\begin{aligned}\mu^i\phi_i = \mu^i &\sum_{k=1}^{j-1}(-1)^{k-1}\Delta_n^{k-1}\Delta_{n-k}\phi_{i-k}\\ &+ (-1)^{j-1}\mu^j\Delta_n^{j-1}\Delta_{n-j}(ax)(a\rho_{j+1})\{\rho_{j+1}\}\\ &+ (-1)^j\mu^j\Delta_n^{j-1}\frac{(j+1, x)(j+1, \rho_{j+1})\{\rho_{j+1}\}}{(j+1)!\,(n-j)!},\end{aligned}$$

or

$$(14)\qquad \mu^i\phi_i = \mu^i \sum_{k=1}^{j} (-1)^{k-1} \Delta_n^{k-1}\Delta_{n-k}\phi_{i-k} + (-1)^j\mu^j\Delta_n^{j-1}\frac{(j+1,x)(j+1,\rho_{j+1})\{\rho_{j+1}\}}{(j+1)!\,(n-j)!},$$

since $(ax)(a\rho_{j+1})\{\rho_{j+1}\} = \mu^{i-j}\phi_{i-j}$. If, however, $i = j+1$, in (14), ρ_{j+1} must be replaced by x and $\{\rho_{j+1}\}$ by unity so that we have

$$(15)\qquad \phi_i = \sum_{k=1}^{i} (-1)^{k-1}\Delta_n^{k-1}\Delta_{n-k}\phi_{i-k} + (-1)^i\Delta_n^{i-1}(i+1,x)^2/(i+1)!\,(n-i)!, \quad (i = 1, 2, \cdots, n-2).$$

Since (14) could be obtained from (13) by replacing j by $j+1$ and since, when $j = 2$, (13) reduces to (11), by induction (14) and (15) are true for $(i = 1, 2, \cdots, n-2)$. Since $\phi_0 = (1x)^2$, from equations (15) we deduce that *ϕ_i is a linear combination with invariant coefficients of the $i+1$ covariants $(1x)^2, (2x)^2, \cdots, (i+1,x)^2$.*

But we may rewrite (15) in the form

$$(16)\qquad \Delta_n^{i-1}(i+1,x)^2 = (i+1)!\,(n-i)! \sum_{k=0}^{i} (-1)^{i-k}\Delta_n^{k-1}\Delta_{n-k}\phi_{i-k},$$

so that (16) gives an explicit formula for the expression of $\Delta_n^{i-1}(i+1,x)^2$ in terms of $\phi_0, \phi_1, \phi_2, \cdots, \phi_i$. In particular, if $\Delta_n \neq 0$, that is, if R is non-singular, equations (11) express $(i+1,x)^2$ as a linear combination of $\phi_0, \phi_1, \cdots, \phi_i$, whose coefficients have Δ_n^{i-1} in the denominator.

It is interesting to notice what equation (16) becomes when in it we give i the value $n-1$. We are at liberty to do this, if we replace $(nx)^2$ by $(A_n rx)^2 = (A_n)^2(rx)^2 = n!\,\Delta_0(rx)^2$. Equation (16) now becomes

$$\Delta_n^{n-2}\Delta_0(rx)^2 = \sum_{k=0}^{n-1} (-1)^{n-1-k}\Delta_n^{k-1}\Delta_{n-k}\phi_{n-1-k},$$

or in matrix notation

$$\Delta_n^{n-2}\Delta_0 R = \sum_{k=0}^{n-1} (-1)^{n-1-k}\Delta_n^{k-1}\Delta_{n-k}A(SA)^{n-1-k}.$$

But, if Δ_n is not zero, this last result takes the form

$$\Delta_0 R = \sum_{k=0}^{n-1} (-1)^{n-k-1}\Delta_{n-k}A(R^{-1}A)^{n-1-k},$$

or

$$(17)\qquad 0 = \sum_{k=0}^{n} (-1)^{n-k}\Delta_{n-k}(R^{-1}A)^{n-k},$$

and equation (17) is simply the Cayley Hamilton theorem for the matrix $R^{-1}A$.*

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* Cf. H. W. Turnbull, "Invariant theory of a general bilinear form," *Proceedings of the London Mathematical Society*, vol. 33 (1932), pp. 1-21.

3. In the sequel we shall assume that R is non-singular, so that Δ_n is not zero and that $(i+1, x)^2$ is a linear combination of $\phi_0, \phi_1, \cdots, \phi_i$. Accordingly the jacobian J of the n quadratic covariants $(ix)^2$ is k times J' the jacobian of $\phi_0, \phi_1, \cdots, \phi_{n-1}$ and g, where

$$k = [(n-1)!\,(n-2)!\cdots 3!\,2!\,]^2\,\Delta_n^{-[(n-2)(n-1)/2]}.$$

But,

$$\begin{aligned} J' &= (-1)^n \frac{\partial(g, \phi_0, \phi_1, \cdots, \phi_{n-1})}{\partial(x_1, x_2, x_3, \cdots, x_n)}, \\ &= (-1)^n 2^n \mid RX, AX, A(SA)X, \cdots, A(SA)^{n-2}X \mid, \\ &= (-1)^n 2^n \Delta_n \mid X, R^{-1}AX, R^{-1}A(SA)X, \cdots, R^{-1}A(SA)^{n-2}X \mid. \end{aligned}$$

On multiplying throughout by $\Delta_n^{-[(n-2)(n-1)/2]}$ and writing T for the matrix $R^{-1}A$ we have,

$$\Delta_n^{-[(n-2)(n-1)/2]}\,J' = (-1)^n \Delta_n \mid X, TX, T^2X, \cdots, T^{n-1}X \mid.$$

Accordingly,

$$J = q\Delta_n \mid X, TX, T^2X, \cdots, T^{n-1}X \mid, \tag{18}$$

where q is a numerical factor.

To consider the factors of J we take the two quadratic forms f and g in canonical form. Let the elementary divisors of $A + \lambda R$ be

$$(\lambda + \lambda_1)^{e_1}, \ (\lambda + \lambda_2)^{e_2}, \cdots, (\lambda + \lambda_t)^{e_t},$$

where $e_1 + e_2 + \cdots + e_t = n$. We may also suppose that $\lambda_i \neq \lambda_j$, if $i \neq j$, for otherwise T would satisfy an equation of degree $\leqq n-1$ and by (18) J would vanish identically. The matrix R is now the diagonal block matrix

$$[D_1, D_2, \cdots, D_t],$$

where D_i is a square matrix of order e_i, all of whose elements are zero except those in the counter diagonal, each of which is unity. It follows that $D_i^2 = E_i$, the unit matrix of order e_i, and that $R^2 = E$, so that $R = R^{-1}$. But A is also a diagonal block matrix

$$[M_1, M_2, \cdots, M_t],$$

where $M_i = D_i + \lambda_i F_i$ and F_i is the square matrix of order e_i, whose only non-zero elements are those in the diagonal immediately below the counter diagonal, all of which have the value unity. Accordingly,

$$\begin{aligned} T = R^{-1}A = RA &= [D_1M_1, D_2M_2, \cdots, D_tM_t], \\ &= [N_1, N_2, \cdots, N_t], \end{aligned}$$

so that T is simply the classical canonical form of a matrix whose elementary divisors are $(\lambda+\lambda_1)^{e_1}, (\lambda+\lambda_2)^{e_2}, \cdots, (\lambda+\lambda_t)^{e_t}$.*

If N_i is written in the form $\lambda_i E_i + H_i$,

$$N_i^r = \sum_{j=0}^{e_i-1} \binom{r}{j} \lambda_i^{r-j} H_i^j.$$

Hence, if G denote the matrix

$$(X, TX, T^2X, \cdots, T^{n-1}X),$$

G may be written in the form $G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_t \end{pmatrix}$, where G_i is a matrix of e_i rows and n columns. More explicitly, $G_i = (X_i, N_iX_i, \cdots, N_i^{n-1}X_i)$, where X_i is the corresponding vector of dimension e_i. To simplify the notation we shall now write q for e_i and denote the components of X_i by $x_1, x_2, \cdots, x_q$. If

$$(20) \qquad Y_i = \begin{pmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_{q-1} & x_q \\ x_2 & x_3 & \cdot & \cdot & \cdot & x_q & 0 \\ x_3 & x_4 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_q & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix},$$

and ϵ_1 denote the vector, whose only non-zero component is the first, which is unity, a simple calculation shows that

$$X_i = Y_i\epsilon_1 \quad \text{and} \quad N_iY_i = Y_iN'_i,$$

so that

$$N_i^rX_i = Y_i(N'_i)^r\epsilon_1.$$

Accordingly

$$\begin{aligned} G_i &= Y_i(\epsilon_1, N'_i\epsilon_1, N'^2_i\epsilon_1, \cdots, (N'_i)^{n-1}\epsilon_1) \\ (21) \qquad &= Y_iW_i, \end{aligned}$$

and

$$G = [Y_1, Y_2, \cdots, Y_t]W,$$

where W is the n rowed squared matrix whose first e_1 rows form the matrix W_1, etc. Hence

$$(22) \qquad |G| = |Y_1|\ |Y_2| \cdots |Y_t|\ |W|.$$

But

$$|W| = \prod_{\substack{i,j=1 \\ i<j}}^{t} (\lambda_i - \lambda_j)^{e_ie_j},\dagger$$

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* L. E. Dickson, *Modern Algebraic Theories*, pp. 105-131.

† Turnbull and Aitken, *Canonical Matrices*, p. 60.

so that $| W | \neq 0$, if $\lambda_i \neq \lambda_j$. Hence, if R is non-singular, J vanishes if and only if $R^{-1}A$ satisfies an equation of degree $\leqq n-1$. Now $| Y_i | = (-1)^{e_i} y_i^{e_i}$, $y_i = x_q$, so that

$$| G | = (-1)^n y_1^{e_1} y_2^{e_2} \cdots y_q^{e_q} | W |.$$

Since J is a covariant, from (18) we have the result: *If R is non-singular, the jacobian of the n quadratic covariants of f and g vanishes, if and only if $R^{-1}A$ satisfies an equation of degree $\leqq n-1$. Moreover, if the jacobian J does not vanish identically, and the elementary divisors of $A + \lambda R$ are $(\lambda + \lambda_1)^{e_1}$, $(\lambda + \lambda_2)^{e_2}, \cdots, (\lambda + \lambda_t)^{e_t}$, $J \equiv I k_1^{e_1} k_2^{e_2} \cdots k_t^{e_t}$, where I is an invariant, and $k_1, k_2, \cdots, k_t$ are t linearly independent linear forms in $x_1, x_2, \cdots, x_n$.**

Similar considerations may be applied to the mixed concomitants J_s, denoted symbolically by

$$J_s = (n, 1, 2, \cdots, s, P_{n-s-1})(nx)(1x)(2x)\cdots(sx), \qquad (s = 1, 2, \cdots, n-2, n-1). \tag{23}$$

In (23), P_{n-s-1} is the matrix $u_1 u_2 \cdots u_{n-s-1}$, where u_j is a vector of dimension n contragradient to the vector x, and represents a compound coördinate. It is apparent that J_s, apart from a numerical factor, is the jacobian of the $s+1$ quadratic forms $(nx)^2$, $(1x)^2, \cdots, (sx)^2$ and the $n-s-1$ linear forms $(u_j x)$. Let us write $| K_s |$ for this last jacobian so that

$$\begin{aligned} | K_s | &= | RX, AX, A(SA)X, \cdots, A(SA)^{s-1}X, R^{-1}P |, \\ &= \Delta_n^{[s(s-1)/2]+1} | X, TX, T^2X, \cdots, T^sX, R^{-1}P |, \end{aligned} \tag{24}$$

where P has been written for P_{n-s-1}. If we now take A, R, and T in the normal forms of the previous section, where, however, λ_i need no longer be distinct from λ_j, equation (24) takes the form

$$| K_s | = | H, R^{-1}P |,$$

where H is the matrix of the first $s+1$ columns of G (equation 19). Accordingly H may be subdivided into matrices H_j, where H_j is the matrix of the first $s+1$ columns of G_j. Hence $H_j = Y_j V_j$, where Y_j is defined by (20) and V_j is the matrix of the first $s+1$ columns of W_j (21).

By (24) we see that $| K_s |$ vanishes identically if T satisfies an equation of degree $\leqq s$. Let us therefore assume that the minimal equation for T is of degree $s+1$ and that $(\lambda+\lambda_1)^{e_1}(\lambda+\lambda_2)^{e_2}\cdots(\lambda+\lambda_k)^{e_k}$ is the n-th invariant factor of $A+\lambda R$. It follows that $e_1+e_2+\cdots+e_k=s+1$;[*] that $\lambda_i\neq\lambda_j$ if $i\neq j$ and both i and j are less than k, and that, if $p>k$, $\lambda_p=\lambda_q$, where $q\leqq k$ and $e_q\geqq e_p$. We now consider a Laplace expansion of the determinant $|\,H, R^{-1}P\,|$ in terms of the first $s+1$ columns, that is in terms of the columns of H. The first term in this expansion is $\begin{vmatrix} H_1 \\ H_2 \\ \vdots \\ H_k \end{vmatrix}\;|\,Q\,|$, where $|\,Q\,|$ is a determinant formed from $R^{-1}P$ and is independent of x. Since

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* For the particular case in which the elementary divisors are all linear, see Turnbull and Williamson, *op. cit.*, pp. 164-165.

$$\begin{vmatrix} H_1 \\ H_2 \\ \vdots \\ H_k \end{vmatrix} = c\,|\,Y_1\,|\;|\,Y_2\,|\cdots|\,Y_k\,|,$$

where

$$c=\prod_{\substack{i<j\\ i,j=1}}^{k}(\lambda_i-\lambda_j)^{e_ie_j},$$

J_s is not identically zero, so that J_s vanishes identically, if and only if T satisfies an equation of degree $\leqq s$. All other terms in the expansion of $|\,K_s\,|$ are obtained from the first term by a determinantal permutation of the rows. Moreover since, if $\lambda_p=\lambda_q$ where q is less than k, e_q is greater than or equal to e_p, any determinant of order $s+1$ containing H_q and any row of H_p is zero, non-zero terms in the expansion of $|\,K_s\,|$ can only arise when certain rows of H_q are replaced by rows of H_p. If in $\begin{vmatrix} H_1 \\ H_2 \\ \vdots \\ H_k \end{vmatrix}$ the i-th row of H_q is replaced by the j-th row of H_p the resulting determinant has the value $c\,|\,Y_1\,|\cdots|\,\bar{Y}_q\,|\cdots|\,Y_k\,|$, where $|\,\bar{Y}_q\,|$ is the determinant obtained from $|\,Y_q\,|$ by replacing the i-th row by the j-th row of $|\,Y_p\,|$. If $e_q-e_p=r_q$, it follows from the nature of Y_q that $|\,\bar{Y}_q\,|=0$, if $\bar{Y}_q$ is obtained from Y_q by replacing any of the first r_q rows of Y_q by and row of Y_p. Since $|\,Y_p\,|=(-1)^{e_q}y_q^{e_q}$, every term obtained by replacing i rows of Y_q by i rows of Y_p will have the factor $y_q^{r_q}$ and one term will not have $y_q^{r_q+1}$ as a factor. In particular, if $(\lambda+\lambda_q)^{e_q}$, $(\lambda+\lambda_q)^{e_p}$ are the two highest elementary divisors belonging to the basis λ_q, every term in the expansion of $| K_s |$ will contain $y_q^{r_q}$ as a factor. Accordingly we have shown that:

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[*] Turnbull and Aitken, *Canonical Matrices*, p. 48.

If $s+1$ *is the degree of the minimal equation satisfied by* T, *then* $J = J_{n-1}, J_{n-2}, \cdots, J_{s+1}$ *vanish identically but* J_s *does not vanish identically; if the quotient of the n-th invariant factor of* $A + \lambda R$ *by the* $(n-1)$*-st invariant factor is* $(\lambda + \lambda_1)^{r_1}, \cdots, (\lambda + \lambda_k)^{r_k}$, J_s *has the factor* $h_1^{r_1} h_2^{r_2} \cdots h_k^{r_k}$ *where* h_i *is a linear form in* $x_1, x_2, \cdots, x_n$.

It is possible in particular cases to proceed further and show how the nature of the concomitants J_s determines completely the exponents e_i of the different elementary divisors and conversely. However for a general value of n any such procedure is too complicated to be practicable.

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SYZYGIES FOR WEITZENBÖCK'S IRREDUCIBLE COMPLETE SYSTEM OF EUCLIDEAN CONCOMITANTS FOR THE CONIC, WITH AN ALGEBRAICALLY COMPLETE SYSTEM OF SUCH CONCOMITANTS.

By Thomas L. Wade, Jr.

1. *Introduction.* In Parts IV and V of *Über Bewegungsinvarianten* published in the *Wiener Berichte,* 122 (1913) and 123 (1914), Dr. Roland Weitzenböck established an irreducible complete system of Euclidean concomitants for the conic. This system consists of the eighteen concomitants given in Table I. In this paper we shall derive from this irreducibly complete system an algebraically complete system of nine concomitants; and, by means of the fundamental identities of the symbolic method, establish an algebraically complete set of nine syzygies connecting the eighteen irreducible concomitants.

An algebraically complete system of invariants and covariants for the conic was recently established by Professor MacDuffe through the medium of the Lie theory of continuous groups* ; and such a system was also established by Professor Franklin using the algebra of matrices and determinants.† But there is no record of an algebraically complete system of all types of concomitants, with a general method whereby any concomitant of the conic can be expressed in terms of the members of this fundamental system.

TABLE I

Weitzenböck's Irreducible Complete System for the Conic.

$$F(x)=(ax)^2=\cdots=a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2+2a_{13}x_1x_3+2a_{23}x_2x_3+a_{33}x_3^2=0.$$

No.	Type	Symbolic Expression	Degree in a_{ij}	x_i	u_i	No.
1	Invariants	$I_1=(abc)^2$	3	0	0	3
2		$I_2=(abl)^2$	2	0	0	
3		$I_3=(a\mid a)$	1	0	0	

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* C. C. MacDuffee, "Euclidean invariants of second degree curves," *The American Mathematical Monthly,* vol. 33 (1926), pp. 243-252.

† Philip Franklin, "The classification of quadrics in euclidean n-space by means of covariants," *The American Mathematical Monthly,* vol. 34 (1926), pp. 453-467.

			Degree in			
No.	Type	Symbolic Expression	a_{ij}	x_i	u_i	No.
4	Covariants	$F=(ax)^2$	1	2	0	4
5		$\mathcal{L}=(lx)$	0	1	0	
6		$C_1=(xa)(a\mid b)(bx)$	2	2	0	
7		$C_2=(abl)(ax)(b\mid c)(cx)$	3	2	0	
8	Contravariants	$\omega=(u\mid u)$	0	0	2	5
9		$K_1=(abu)^2$	2	0	2	
10		$K_2=(abu)(abl)$	2	0	1	
11		$K_3=(aul)(a\mid u)$	1	0	2	
12		$K_4=(a\mid u)^2$	1	0	2	
13	Mixed Concomitants	$U=(ux)$	0	1	1	6
14		$M_1=(u\mid a)(ax)$	1	1	1	
15		$M_2=(aul)(ax)$	1	1	1	
16		$M_3=(abu)(ax)(b\mid u)$	2	1	2	
17		$M_4=(abu)(ax)(b\mid c)(cx)$	3	2	1	
18		$M_5=(abl)(ax)(b\mid u)$	2	1	1	

It seems to the writer that a fundamental algebraically complete system, represented as a sub-group of an established irreducibly complete system, in conjunction with a complete set of syzygies connecting the members of the latter system, is of greater value and interest than a system of independent concomitants established by another method. The presentation of the two systems in union brings more clearly to light their differences. Moreover, by means of this information, a given concomitant can be more easily expressed in terms of members of the system of independent concomitants, if desired. For, on expressing the given concomitant symbolically, it can be expressed in terms of members of the irreducibly complete system by means of the identities of the symbolic method, and thence in terms of members of the algebraically complete system by means of the syzygies obtained in this paper.

2. *Identities.* The medium of investigation of this paper is the symbolic invariant notation of Aronhold and Clebsch. An exposition of this notation, with its adaptation to Euclidean concomitants, is given by Dr. Roland Weitzenböck in his book *Invariantentheorie,* and also in *Über Bewegungsinvarianten,* I, . . . , XV Mitteilungen, *Wiener Berichte,* 1913-1919. Briefly, the Euclidean invariant theory of the plane curve

$$F(x)\equiv(ax)^n\equiv(bx)^n\equiv\cdots\equiv(a_1x_1+a_2x_2+a_3x_3)^n=0$$

under the general Euclidean transformation is contained in two fundamental theorems. By the first fundamental theorem every Euclidean concomitant of the ground form $(ax)^n$ is expressible as a polynomial, the factors of whose terms are of the types

$$f_1 = (abc) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$f_2 = (abl) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix},$$

$$\begin{aligned} f_3 &= (a \mid b) = a_1 b_1 + a_2 b_2, \\ f_4 &= (ax) = a_1 x_1 + a_2 x_2 + a_3 x \\ f_5 &= (lx) = 0x_1 + 0x_2 + 1x_3 = x_3, \end{aligned}$$

together with numerical coefficients, it being understood that each symbol a, b, $c, \cdots$ is to appear just n times in a symbolic product. Conversely, every such polynomial is an Euclidean concomitant of $(ax)^n$. The symbol u in the universal concomitant

$$(ux) = u_1 x_1 + u_2 x_2 + u_3 x_3$$

behaves like the symbols $a, b, c, \cdots$, except that it may occur in a product any number of times.

The second fundamental theorem says that every identity satisfied by the concomitants of $(ax)^n$, formed according to the first fundamental theorem, can be expressed by means of the interchange of equivalent symbols, by means of the laws of ordinary algebra for the combination of the factor-types $f_1, \cdots, f_5$, and by means of the identities:

Iden. 1 $\quad (abc)(d \mid e) \equiv (dbc)(a \mid e) + (adc)(b \mid e) + (abd)(c \mid e)$

Iden. 1′ $\quad (abc)(dx) \equiv (dbc)(ax) + (adc)(bx) - (abd)(cx)$

Iden. 1″ $\quad (abc)(def) \equiv (dbc)(aef) + (adc)(bef) - (abd)(cef)$

Iden. 2 $\quad (abl)(cdl) \equiv \begin{vmatrix} (a \mid c) & (a \mid d) \\ (b \mid c) & (b \mid d) \end{vmatrix} \equiv (a \mid c)(b \mid d) - (a \mid d)(b \mid c).$

In order to make the subsequent work more explicit, we mention several identities to be used later, which are special cases or consequences of the above four identities, namely:

Iden. 3 $(abl)(c \mid d) \equiv (cbl)(a \mid d) + (acl)(b \mid d)$

Iden. 4 $(abl)(cx) \equiv (cbl)(ax) + (acl)(bx) + (abc)(lx)$

Iden. 5 $(abc)(del) \equiv (dbc)(ael) + (adc)(bel) + (abd)(cel)$

Iden. 6 $(abl)(cdl) \equiv (cbl)(adl) + (acl)(bdl)$

Iden. 7 $(xa)(a \mid b)(b \mid c)(cy)$
$$\equiv I_3(xa)(a \mid b)(by) - \tfrac{1}{2}I_2(ax)(ay) + \tfrac{1}{3}I_1(lx)(ly).$$

In Iden. 7 x and y may represent the same variable x, or one or both of them may be $a \mid$ or $u \mid$.

Identities 1 to 6 are established in "Über Bewegungsinvarianten," III Mitteilung, *Wiener Berichte*, vol. 122 (1913). Iden. 7 was established by an indirect method by Dr. Weitzenböck in "Über Bewegungsinvarianten," IV Mitteilung, *Wiener Berichte*, vol. 122 (1913). It may be established directly by means of the identity

Iden. 8
$$\begin{aligned}(ax)(b \mid \beta)(c \mid \gamma) \equiv{}& (ax)(b \mid \gamma)(c \mid \beta) + (a \mid \beta)(bx)(c \mid \gamma) \\ & - (a \mid \beta)(b \mid \gamma)(cx) - (a \mid \gamma)(bx)(c \mid \beta) \\ & + (a \mid \gamma)(b \mid \beta)(cx) + (abc)(\beta\gamma l)(lx).\end{aligned}$$

The truth of Iden. 8 can be confirmed in the following manner. Certainly,

$$(1) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & 0 \end{vmatrix} \equiv 0.$$

To both sides of (1) add $(abc)(\beta\gamma l)(lx)$. Then we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 & x_3 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & 0 \end{vmatrix} \equiv (abc)(\beta\gamma l)(lx),$$

which, expanded, is Iden. 8.

Applying Iden. 8 to $(xa)(a \mid b)(b \mid c)$ of

$$P = (xa)(a \mid b)(b \mid c)(cy),$$

we get *

$$\begin{aligned}P \equiv{}& (ax)(b \mid b)(c \mid a)(cy) + (a \mid a)(bx)(c \mid b)(cy) - (a \mid a)(b \mid b)(cx)(cy) \\ & - (a \mid b)(bx)(c \mid a)(cy) + (a \mid b)^2(cx)(cy) + (abc)(abl)(lx)(cy) \\ \equiv{}& I_3(xa)(a \mid c)(cy) + I_3(xa)\underset{a \leftrightarrow b}{(a \mid c)}(cy) - I_3^2(cx)(cy)\end{aligned}$$

---

* The fact that the concomitants $A(a, b, c, \cdots)$ and $B(a, b, c, \cdots)$ are equal on the interchange of a and b is symbolized by
$$A(a, b, c, \cdots) \underset{a \leftrightarrow b}{\equiv} B(a, b, c, \cdots).$$

$$-(ax)(a\,|\,b)\underset{a\leftrightarrow b}{}(b\,|\,c)(cy)+[I_3{}^2-I_2](cx)(cy)+(abc)(abl)(cy)(lx),$$

or

$$P\equiv I_3(xa)(a\underset{b\leftrightarrow c}{|}b)(by)-\tfrac{1}{2}I_2(ax)\underset{c\leftrightarrow a}{}(ay)+\tfrac{1}{2}(abc)(abl)(cy)(lx).$$

Let
$$P'=(abc)(abl)(cy).$$

Applying Iden. 4 to $(abl)(cy)$ of P' we get

$$\begin{aligned}P'&\equiv(abc)(cbl)(ay)+(abc)(acl)(by)+(abc)^2(ly)\\&\equiv-\underset{a\leftrightarrow c}{P'}-\underset{b\leftrightarrow c}{P'}+I_1(ly),\end{aligned}$$

or
$$P'\equiv\tfrac{1}{3}I_1(ly).$$

Therefore,

$$P\equiv I_3(xa)(a\,|\,b)(by)-\tfrac{1}{2}I_2(ax)(ay)+\tfrac{1}{6}I_1(lx)(ly),$$

which is Iden. 7.

3. *Establishment of the syzygies.* It is evident that the three irreducible invariants I_1, I_2, and I_3 are also algebraically independent. Hence there exists no syzygy wholly on the invariants of F.

The equation of F contains six coefficients and three variables, altogether nine (homogeneous) parameters. The general Euclidean transformation contains three parameters. These, eliminated between the nine equations expressive of the identity of F and its transform, leave six equations connecting old and new coefficients and variables. So F can have at most six algebraically independent (polynomial) invariants and covariants. Hence there must be at least one syzygy connecting the seven irreducible invariants and covariants. We shall proceed to find this syzygy; and also to show that there are, indeed, six independent invariants and covariants.

Since
$$C_2=(abl)(ax)(b\,|\,c)(cx),$$
We have

$$\begin{aligned}C_2{}^2&=(abl)(ax)(b\,|\,c)(cx)(del)(dx)(e\,|\,f)(fx)\\&\equiv[(a\,|\,d)(b\,|\,e)-(a\,|\,e)(b\,|\,d)](ax)(b\,|\,c)(cx)(dx)(e\,|\,f)(fx)\\&\hspace{20em}\text{(By Iden. 2)}\\&\equiv(a\,|\,d)(b\,|\,e)(ax)(b\,|\,c)(cx)(dx)(e\,|\,f)(fx)\\&\qquad\qquad-(a\,|\,e)(b\,|\,d)(ax)(b\,|\,c)(cx)(dx)(e\,|\,f)(fx)\\&\equiv C_1(b\,|\,c)(b\,|\,d)(\underset{\substack{f\leftrightarrow a\\ e\leftrightarrow d}}{d}\,|\,a)(cx)(ax)\\&\qquad\qquad-[(a\,|\,b)(b\,|\,c)(ax)\underset{\substack{b\leftrightarrow e\\ f\leftrightarrow c}}{(cx)}][(d\,|\,e)(e\,|\,f)(fx)(dx)]\\&\equiv C_1(xa)(a\,|\,b)(b\underset{\substack{d\leftrightarrow b\\ c\leftrightarrow d}}{|}c)(c\,|\,d)(dx)\\&\qquad\qquad-[(a\,|\,b)(b\,|\,c)(ax)(cx)][(d\,|\,e)(e\,|\,f)(fx)(dx)],\end{aligned}$$

or, for briefness let us say,

$$C_2{}^2 \equiv C_1 A - B^2. \tag{1}$$

Applying Iden. 7 to B we get

$$B \equiv I_3 (xa)(a \mid b)(bx) - \tfrac{1}{2} I_2 (ax)^2 + \tfrac{1}{6} I_1 (lx)^2,$$

or

$$B \equiv I_3 C_1 - \tfrac{1}{2} I_2 F + \tfrac{1}{6} I_1 \pounds^2. \tag{2}$$

The same identity applied to $(xa)(a \mid b)(b \mid c)(c \mid d)$ of A gives

$$A \equiv I_3 (xa)(a \mid b)(b \mid d)(dx) - \tfrac{1}{2} I_2 (xa)(a \mid d)(dx),$$

or

$$A \equiv I_3 B - \tfrac{1}{2} I_2 C_1. \tag{3}$$

In virtue of (1), (2), and (3) we have

$$\begin{aligned} (S_1) \qquad 36 C_2{}^2 \equiv & -18 I_2 C_1{}^2 - 9 I_2{}^2 F^2 - I_1{}^2 \pounds^2 \\ & + 18 I_2 I_3 C_1 F + 6 I_1 I_2 F \pounds^2 - 6 I_1 I_3 C_1 \pounds^2. \end{aligned}$$

Applying the well known theorem, that the necessary and sufficient condition that a set of functions be functionally independent is that their gradients be linearly independent, to the three covariants F, C_1, and $\pounds$, considered as functions of x_1, x_2, x_3, we find that they are functionally independent. And we know that I_1, I_2, and I_3, as functions of a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, and a_{23}, are likewise functionally independent. Hence the six concomitants I_1, I_2, I_3, F, C_1, and $\pounds$, as functions of the a's and the x's are functionally independent, i. e., these concomitants are algebraically independent. Hence there can exist only one independent syzygy on the seven irreducible invariants and covariants, and (S_1) may be chosen as this one.

The equation of F and the universal ternary concomitant

$$(ux) = u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

together contain nine coefficients of the x's. The three transformation parameters, eliminated between the nine equations expressive of the identity of old and new forms, leave six equations connecting old and new coefficients of the x's. Accordingly, there can be at most six algebraically independent invariants and contravariants of F. Hence there must be at least two independent syzygies connecting the eight irreducible invariants and contravariants.

We have

$$
\begin{aligned}
K_3^2 &= (aul)(a \mid u)(bul)(b \mid u) \\
&\equiv (a \mid u)(bul)[(bul)(a \mid u) + (abl)(u \mid u)] \quad \text{(By Iden. 3)} \\
&\equiv K_4 C + D\omega,
\end{aligned}
$$

where $$C \equiv I_3\omega - K_4 \qquad \text{by Iden. 2,}$$

and $$2D \equiv - I_2\omega \qquad \text{by Iden. 3.}$$

Hence

$$(S_2) \qquad 2K_3^2 \equiv 2I_3K_4\omega - 2K_4^2 - I_2\omega^2.$$

In like manner

$$
\begin{aligned}
K_2^2 &= (abu)(abl)(cdu)(cdl) \\
&= (abl)(cdu)[2(cbu)(adl) + (abc)(udl)] \equiv 2E + F,
\end{aligned}
$$

where $$2E \equiv I_2K_1 - \tfrac{1}{3} I_1(I_3\omega - K_4) \qquad \text{by Iden. 5,}$$

and $$F \equiv - \tfrac{1}{3} I_1(I_3\omega - K_4) \qquad \text{by Iden. 5.}$$

Therefore,

$$(S_3) \qquad 3K_2^2 \equiv 3I_2K_1 - 2I_1I_3\omega + 2I_1K_4.$$

By means of the syzygies (S_2) and (S_3) the five irreducible contravariants can be expressed as algebraic functions of the invariants and some three of them, such as ω, K_1, and K_2. Moreover, these three contravariants, as functions of u_1, u_2, and u_3, are independent.

Altogether the equation of F and the universal ternary concomitant $(ux) = 0$ contain twelve parameters (6 a's, 3 x's, and 3 u's). The three transformation parameters, eliminated between the twelve equations expressive of the identity of old and new forms, leave nine equations connecting old and new coefficients and variables. So there are at most nine algebraically independent concomitants of all types. However, we have found that there are, indeed, nine independent invariants, covariants, and contravariants. Hence there can exist no independent mixed concomitant. In other words, the mixed concomitants are expressible as algebraic functions of the remaining concomitants. A set of syzygies by means of which this can be done will now be found.

We have

$$
\begin{aligned}
M_2^2 &= (aul)(ax)(bul)(bx) \\
&\equiv (ax)(bul)[(bul)(ax) + (abl)(ux) + (aub)(lx] \quad \text{(By Iden. 4)} \\
&\equiv (I_3\omega - K_4)F + AU - B\pounds,
\end{aligned}
$$

where

$$(1) \qquad 2A \equiv K_2\pounds - I_2U \qquad \text{by Iden. 4,}$$

and

(2) $$2B \equiv K_1\pounds - K_2U \qquad \text{by Iden. 4.}$$

Consequently,

(S_4) $$2M_2^2 \equiv 2I_3\omega F - 2K_4F - I_2U^2 + 2K_2U\pounds - K_1\pounds^2.$$

In like manner,

$$\begin{aligned}(I_3\omega - K_4)M_1 &= (bul)(ax)(a \mid u)\\ &\equiv (bul)(ax)[(aul)(b \mid a) + (bal)(u \mid u)] \qquad \text{(By Iden. 3)}\\ &\equiv M_2K_3 - A\omega.\end{aligned}$$

Thence from (1) we have

(S_5) $$2(I_3\omega - K_4)M_1 \equiv 2M_2K_3 + I_2U\omega - K_2\omega\pounds.$$

By Iden. 3 we have

$$\begin{aligned}(cul)^2(abr)(ax)(b \mid u) &\equiv (abr)(ax)(cul)[(bul)(c \mid u) + (cbl)(u \mid u)]\\ &\equiv (abr)(bul)(ax)K_3 - E\omega,\end{aligned}$$

where, by Iden. 4,

$$2E \equiv -(abr)(abl)(cul)(cx) + (abc)(abr)(cul)\pounds.$$

But, applying Iden. 5 to $(abr)(cul)$ we find that

$$(abc)(abr)(cul) \equiv 0.$$

Therefore

$$(cul)^2(abr)(ax)(b \mid u) \equiv (abr)(bul)(ax)K_3 + \tfrac{1}{2}(abr)(abl)(cul)(cx)\omega.$$

Putting $r = l$ in this relation we get

$$\begin{aligned}2(I_3\omega - K_4)M_5 &\equiv 2(cul)^2(abl)(ax)(b \mid u)\\ &\equiv 2(abl)(bul)(ax)K_3 + (abl)^2(cul)(cx)\omega,\end{aligned}$$

or, by (1),

(S_6) $$2(I_3\omega - K_4)M_5 \equiv K_2K_3\pounds - I_2K_3U + I_2M_2\omega.$$

In like manner, putting $r = u$, we get

$$\begin{aligned}2(I_3\omega - K_4)M_3 &\equiv 2(cul)^2(abu)(ax)(b \mid u)\\ &\equiv 2(abu)(bul)(ax)K_3 + (abu)(abl)(cul)(cx)\omega,\end{aligned}$$

or, by (2),

(S_7) $$2(I_3\omega - K_4)M_3 \equiv K_1K_3\pounds - K_2K_3U + K_2M_2\omega.$$

Furthermore,

$$\begin{aligned}(I_3\omega - K_4)M_4 &\equiv (abu)(cx)(b \mid c)(dul)(ax)(dul)\\ &\equiv (abu)(cx)(b \mid c)(dul)[(dx)(aul)+(ux)(dal)+(cx)(dua)]\\ &\equiv \tfrac{1}{2}(abu)(cx)(dul)(dx)(u \mid c)(abl)\\ &\quad + \tfrac{1}{2}(abu)(cx)(dul)(ux)(d \mid c)(bal)\\ &\quad + \tfrac{1}{2}(abu)(cx)(dul)(lx)[(d \mid c)(bua)+(u \mid c)(dba)]\\ &\equiv \tfrac{1}{2}K_2M_2M_1 - \tfrac{1}{2}K_2U(d \mid c)(dcl)(cx)\\ &\quad + \tfrac{1}{2}K_1\pounds(dul)(d \mid c)(cx) + 0,\end{aligned}$$

or

$$(S_8) \qquad 2(I_3\omega - K_4)M_4 \equiv K_2M_2M_1 - (K_2U - K_1\pounds)(I_3M_2 - M_5).$$

By means of the syzygies (S_4), (S_5), (S_6), (S_7), and (S_8), we see that the mixed concomitants M_1, M_2, M_3, M_4, and M_5 are algebraically dependent on the invariants, covariants, contravariants, and the mixed concomitant U. That U, and as a consequence all mixed concomitants, are dependent on the invariants, covariants, and contravariants will now be shown.

We have

$$\begin{aligned}M_2{}^2 &= (aul)(ax)(bul)(bx)\\ &\equiv (ax)(bx)[(a \mid b)(u \mid u) - (a \mid u)(b \mid u)] \qquad \text{(By Iden. 2)}\\ &\equiv (a \mid b)(ax)(bx)(u \mid u) - (a \mid u)(b \mid u)(ax)(bx),\end{aligned}$$

or

$$(1) \qquad M_2{}^2 \equiv C_1\omega - M_1{}^2.$$

That the syzygy (1) is independent of $(S_4), \cdots, (S_8)$ is evident, since it is the only one of these syzygies in which C_1 appears. In order to see explicitly that U is dependent on the invariants, covariants, and contravariants, we choose as one of our independent syzygies not (1), but one derived from it and (S_4) and (S_5). Eliminating M_1 and M_2 from these three syzygies, we get as our desired syzygy

$$\begin{aligned}(S_9) \qquad &\{(I_3\omega - K_4)^2C_1\omega - \tfrac{1}{2}[(I_3\omega - K_4)^2 - K_4{}^2]\\ &\quad \times [2(I_3\omega - K_4)F - I_2U^2 + 2K_2U\pounds - K_1\pounds] - (K_2\pounds - I_2U)^2\}^2\\ &\equiv [2(I_3\omega - K_4)F - I_2U^2 + 2K_2U\pounds - K_1\pounds][K_2\pounds - I_2U]^2\omega^2K_5{}^2.\end{aligned}$$

4. *Summary.* The results obtained in this paper may be summarized in the two statements:

For the conic F there are nine algebraically independent concomitants, and these may be chosen as I_1, I_2, I_3, F, $\pounds$, C_1, ω, K_1, and K_2.

The eighteen irreducible concomitants for the conic F are connected by nine independent syzygies, and these may be chosen as:

$$
\begin{aligned}
&(S_1)\quad 36C_2^2 \equiv -18I_2C_1^2 - 9I_2^2F^2 - I_1^2\pounds^2\\
&\qquad\qquad + 18I_2I_3C_1F + 6I_1I_2F\pounds^2 - 6I_1I_3C_1\pounds^2.\\
&(S_2)\quad 2K_3^2 \equiv 2I_3K_4\omega - 2K_4^2 - I_2\omega^2.\\
&(S_3)\quad 3K_2^2 \equiv 3I_2K_1 - 2I_1I_3\omega + 2I_1K_4.\\
&(S_4)\quad 2M_2^2 \equiv 2I_3\omega F - 2K_4F - I_2U^2 + 2K_2U\pounds - K_1\pounds^2.\\
&(S_5)\quad 2(I_3\omega - K_4)M_1 \equiv 2M_2K_3 + I_2U\omega - K_2\omega\pounds.\\
&(S_6)\quad 2(I_3\omega - K_4)M_5 \equiv K_2K_3\pounds - I_2K_3U + I_2M_2\omega.\\
&(S_7)\quad 2(I_3\omega - K_4)M_3 \equiv K_1K_3\pounds - K_2K_3U + K_2M_2\omega.\\
&(S_8)\quad 2(I_3\omega - K_4)M_4 \equiv K_2M_2M_1 - (K_2U - K_1\pounds)(I_3M_2 - M_5).\\
&(S_9)\quad \{(I_3\omega - K_4)^2C_1\omega - \tfrac{1}{2}[(I_3\omega - K_4)^2 - K_4^2]\\
&\qquad\qquad \times [2(I_3\omega - K_4)F - I_2U^2 + 2K_2U\pounds - K_1\pounds] - (K_2\pounds - I_2U)^2\}^2\\
&\qquad\quad \equiv [2(I_3\omega - K_4)F - I_2U^2 + 2K_2U\pounds - K_1\pounds][K_2\pounds - I_2U]^2\omega^2K_5^2.
\end{aligned}
$$

ON THE CANONICAL FORM OF A NON-SINGULAR PENCIL OF HERMITIAN MATRICES.

G. RICHARD TROTT.

Introduction. In this paper we shall be interested in finding a set of normal forms to one of which a non-singular pencil of Hermitian matrices can be reduced by a conjunctive transformation. The result may be used immediately for finding a necessary and sufficient condition that two non-singular pencils of Hermitian matrices be equivalent under conjunctive transformations. As has been done in previous cases we carry out the work simultaneously for a non-singular pencil of real symmetric matrices under real congruent transformations and a non-singular pencil of Hermitian matrices under conjunctive transformations.

1. In order to simplify our later arguments we first define certain notations and prove some preliminary Lemmas on matrices. If R is a matrix of order n, we may consider R as a matrix of matrices and write

$$R = (R_{ij}), \qquad (i, j = 1, 2, \cdots, t)$$

where R_{ij} is a matrix of r_i rows and r_j columns and

$$r_1 + r_2 + r_3 + \cdots + r_t = n.$$

In particular, if $R_{ij} = 0$, where $i \neq j$, we shall call the matrix a diagonal block matrix and denote it by

$$R = [R_{11}, R_{22}, \cdots, R_{tt}].$$

If S is another square matrix of order n, and S is written as a matrix of matrices

$$S = (S_{ij}),$$

where S_{ij} is a matrix of r_i rows and r_j columns, we shall say that S and R are similarly partitioned.

We shall represent by $(f_0, f_1, \cdots, f_{j-1})_i{}^j$ the matrix of i rows and j columns, where f_0 is the element in each place down the main diagonal, and f_i is the element in each place down the i-th diagonal above the main diagonal, with zeros elsewhere.

We shall denote by $\{f_0, f_1, \cdots f_{i-1}\}_i^j$ the matrix of i rows and j columns, where f_0 is the element in each place down the counter diagonal and f_i is the element in each place down the ith sub-counter diagonal, with zeros elsewhere. The counter diagonal in a rectangular matrix will always be taken as that diagonal which has its first element in the upper right hand corner of the matrix. If the matrices are square, we shall omit the subscript i and the superscript j. For example

$$(f_0, f_1, f_2, f_3) = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ 0 & f_0 & f_1 & f_2 \\ 0 & 0 & f_0 & f_1 \\ 0 & 0 & 0 & f_0 \end{pmatrix} \qquad \{0, f_1, f_2\}_2^3 = \begin{pmatrix} 0 & 0 \\ 0 & f_1 \\ f_1 & f_2 \end{pmatrix}.$$

In the sequel we shall also have to consider matrices of these two types in which every f_i is a matrix of order two. For convenience we shall use the same notation for both.

We shall employ the notation B^* to represent the conjugate-transposed of the matrix B, i. e. $B^* = \bar{B}'$, and in particular if the elements of B are real, $B^* = B'$. We shall say that a matrix A is transformed into a matrix D by the matrix B, if

$$B^*AB = D.$$

(A transformation of this type is congruent if $B^* = B'$, and is conjunctive if $B^* = \bar{B}'$).

Let
$$N_s = (p_j, e, 0, \cdots, 0)$$

be a matrix of order s, and let

$$N_q = (p_j, e, 0, \cdots, 0)$$

be a matrix of order q, where $e = 1$, and p_j is a scalar or else $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and p_j is the real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $b \neq 0$.

LEMMA (1). *If* $RN_q = N'_sR$, *where* R *is a matrix of* s *rows and* q *columns, then*

$R = \{f_0, f_1, \cdots, f_{s-1}\}$ *if* $s = q$, *and* $R = \{0, 0, \cdots, f_{s-q}, \cdots, f_{s-1}\}_s^q$ *if* $s > q$. *Moreover if* $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *then* f_i *is a matrix of type* $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$.

Proof. The Lemma has been proved for $e = 1$.† We wish to prove it for $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and in doing so we shall use an induction proof.

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† C. E. Cullis, *Matrices and Determinoids*, vol. 3, Part I, pp. 456-459.

Let $R = (r_{ij})$ $(i = 1, 2, \cdots, s)$ and $(j = 1, 2, \cdots, q)$

where r_{ij} is a matrix of order two. Since $RN_q = N'_sR$, we have on equating the elements in the i-th rows and j-th columns

$$(1) \qquad r_{ij}p_j - p'_jr_{ij} = r_{i-1,j} - r_{i,j-1},$$

where $r_{0,t} = r_{p,0} = 0$ $(t = 1, 2, \cdots, s)$ and $(p = 1, 2, \cdots, q)$.

Let us assume that r_{ij} is of the type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$ for all values of i and j for which $i + j \leq k$. Under this assumption it follows from (1) that

$$r_{ij}p_j - p'_jr_{ij} \text{ is of type } \begin{pmatrix} d & c \\ c & -d \end{pmatrix}, \text{ if } i + j = k + 1.$$

If we write $r_{ij} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, then

$$r_{ij}p_j - p'_jr_{ij} = \begin{pmatrix} (g_3 - g_2)b & (g_1 + g_4)b \\ -(g_1 + g_4)b & (g_3 - g_2)b \end{pmatrix},$$

and, since $b \neq 0$, this matrix is only of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$, if $g_1 = -g_4$ and $g_3 = g_2$; i. e. if r_{ij} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$. Hence $r_{ij}p_j - p'_jr_{ij} = 0$, and accordingly

$$(2) \qquad r_{i,j-1} = r_{i-1,j}, \qquad i + j \leq k.$$

Since $r_{11}p_j - p'_jr_{11} = 0$, r_{11} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$. Hence our assumption is true for $i + j = 2$ and by induction r_{ij} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$ for all admissible values of i and j, and (2) is true for $i + j \leqq s + q$.

In particular, if $i + j \leqq$ the greater of q and s, since

$$r_{1,j-1} = r_{0j} = 0 \quad (j = 1, 2, \cdots, q) \quad \text{and} \quad r_{i-1,1} = r_{i0} = 0 \quad (i = 1, 2, \cdots, s)$$

it follows that $r_{ij} = 0$.

COROLLARY. *If $R = R^*$, then R is a real matrix.*

For, if $R = R^*$, $f_i = f^*_i$ and since f_i is either a scalar or a matrix of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$, f_i is real.

Let $R = (R_{ij})$ $(i, j = 1, 2, \cdots, t)$, where $R = R^*$ and R_{ij} is a square matrix of type $\{f_0, f_1, \cdots, f_t\}$, and if each f_i is a two rowed square matrix, it is of type $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

LEMMA (2). *If in the above matrix* $|R_{11}| = 0$ *and* $|R_{ii}| = 0$, *and if* R_{1i} *has a non-zero element in the upper right-hand corner, then there exists a non-singular matrix* Q *such that* $Q^*RQ = G$, *where* G *is partitioned similarly to* R *and* $|G_{11}| \neq 0$.

Proof. Let us assume the contrary. Without loss of generality we may take R_{1i} to be R_{12}, since by interchanging the 2nd and the i-th columns, and 2nd, and i-th rows, we may move R_{1i} into the place of R_{12}. (Transformations of this type are conjunctive).[†] Let

$$H = \left[\begin{pmatrix} E & 0 \\ E & E \end{pmatrix}, E, \cdots, E\right],$$

where E is the unit matrix whose order is the same as the order of R_{ij}. Then $H^*RH = K$, where $K_{11} = R_{11} + R_{12} + R^*_{12} + R_{22}$.

Now K_{11} is of type $\{f_0, f_1, \cdots, f_t\}$, and the element in the upper right-hand corner of K_{11} is $r_1 + r_2 + r^*_2 + r_3$ where r_1, r_2, r^*_2 and r_3 are the elements in the upper right-hand corners of R_{11}, R_{12}, R^*_{12}, and R_{22}, respectively. By the corollary to Lemma (1), R_{11} and R_{22} are real matrices, hence if $|R_{11}| = |R_{22}| = 0$, then $r_1 = r_3 = 0$. Since by our assumption $|K_{11}| = 0$, then $|r_2 + r^*_2| = 0$, and as $r_2 + r^*_2$ is real, $r_2 + r^*_2 = 0$. In a similar manner let

$$P = \left[\begin{pmatrix} E & 0 \\ iE & E \end{pmatrix}, E, \cdots, E\right].$$

Then $P^*RP = L$, where $L_{11} = R_{11} - iR_{12} + iR^*_{12} + R_{22}$.

Proceeding in exactly the same manner as above, we have $r_2 - r^*_2 = 0$. Hence $r_2 = 0$, contrary to our hypothesis.

LEMMA (3). *Let* $R = (R_{ij})$ $(i, j = 1, 2, \cdots, t)$, *where* $R = R^*$ *and* R_{ij} *is a matrix with* r_i *rows and* r_j *columns. If* $|R_{11}| \neq 0$ *there exists a matrix* W, *such that*

$$W^*RW = \begin{pmatrix} R_{11} & 0 & 0 & \cdots & 0 \\ 0 & Z_{22} & Z_{23} & \cdots & Z_{2t} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & Z_{t2} & Z_{t3} & \cdots & Z_{tt} \end{pmatrix}.$$

Proof. Let

$$Y_i = \begin{pmatrix} E_1 & 0 & \cdots & -R_{11}^{-1}R_{1i} & \cdots & 0 \\ 0 & E_2 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & E_i & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \cdots & E_t \end{pmatrix},$$

[†] Cf. Turnbull and Aitken, *An Introduction to the Theory of Canonical Matrices*, p. 11.

where E_i is the unit matrix of order r_i. Then

$$Y^*_t Y^*_{t-1} \cdots Y^*_1 R Y_1 Y_2 \cdots Y_t = \begin{pmatrix} R_{11} & 0 \\ 0 & F_1 \end{pmatrix},$$

where $F_1 = (Z_{ij})$ $(i, j = 2, 3, \cdots, t)$.

LEMMA (4). *If F is the real matrix $\{f_0, f_1, \cdots, f_t\}$, there exists a real non-singular matrix*

$$S = (e, s_1, \cdots, s_t)$$

such that

$$S^*FS = \{f_0, 0, \cdots, 0\},$$

where s_i is of type $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, if each f_i is a two rowed square matrix of the type $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$.

Proof. Now

$$F = \sum_{i=0}^{t} f_i T_{i+1},$$

where

$$T_r = \sum_{a=r}^{k} e_{a,k+r-a}, \qquad (r = 1, 2, \cdots, t),$$

and e_{ij} is the matrix all of whose elements are zero except that element in the i-th row and j-th column which is e. Likewise

$$S = I_0 + \sum_{i=1}^{t} I_i s_i,$$

where

$$I_j = \sum_{a=1}^{t} e_{a,a+j}, \qquad (j = 0, 1, 2, \cdots, t).$$

Thus it follows that

$$T_r I_j = I'_j T_r = T_{r+j}.$$

Let us assume that there exists a matrix S_{r-1} of the same type as S, such that

$$S^*_{r-1} F S_{r-1} = G = f_0 T_1 + g_{r+1} T_{r+1} + \sum_{a=r+2}^{t+1} g_a T_a,$$

where each g_i is of the same type as f_i. Now

$$(I_0 + s^*_r I'_r) G (I_0 + s_r I_r) = f_0 T_1 + g_{r+1} T_{r+1} + s^*_r f_0 T_{r+1} + f_0 s_r T_{r+1} + \sum_{a=r-2}^{t+1} h_a T_a.$$

Hence on equating the coefficients of T_{r+1} to zero, we deduce that

$$(3) \qquad g_{r+1} + s^*_r f_0 + f_0 s_r = 0.$$

But, from the nature of f_i and g_i, it follows that (3) is satisfied if

$$s_r = -\tfrac{1}{2} f_0^{-1} g_{r+1},$$

and that s_r is a matrix of type $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, if f_0 and g_{r+1} are two rowed square matrices. If

$$S_{r-1}(I_0 + s_r I_r) = S_r,$$

where s_r has the value, $s_r = -\tfrac{1}{2} f_0^{-1} g_{r+1}$, then S_r is of the same type as S and

$$S^*_r R S_r = f_0 T_1 + k_{r+2} T_{r+2} + \sum_{a=r+3}^{t+1} k_a T_a,$$

where if f_0 and g_{r+1} are two rowed square matrices then k_i is a matrix of type $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$. The truth of our Lemma follows immediately by induction, since this formula holds for $r = 1$ $(s_0 = e)$.

2. We consider the non-singular pencil of matrices $A + \lambda B$, in which $|B| \neq 0$, and where $A = A^*$ and $B = B^*$, so that either A or B are both real symmetric matrices or both Hermitian matrices. If $(\lambda + \lambda_s)^\rho$ appears among the elementary divisors of $A + \lambda B$ and if λ_s is complex, then $(\lambda + \bar{\lambda}_s)^\rho$ is also an elementary divisor. Hence we may arrange the elementary divisors of $A + \lambda B$ in such a way that the real elementary divisors are given by

$$(\lambda + \lambda_i)^{\eta_{i1}} (\lambda + \lambda_i)^{\eta_{i2}} \cdots (\lambda + \lambda_i)^{\eta_{ik_i}}, \qquad (i = 1, 2, \cdots, t)$$

and the complex elementary divisors by

$$(\lambda + \lambda_i)^{\eta_{i1}} (\lambda + \bar{\lambda}_i)^{\eta_{i1}} \cdots (\lambda + \lambda_i)^{\eta_{ik_i}} (\lambda + \bar{\lambda}_i)^{\eta_{ik_i}}, \quad (i = t+1, \cdots, s),$$

where $k_1 + k_2 + \cdots + k_t + 2k_{t+1} + \cdots + 2k_s = n$. As is customary we assume that $\eta_{jp} \geqq \eta_{jp+1}$ for all values of j and p.

Let $$M = [M_1, M_2, \cdots, M_s]$$

be a real canonical form † for the matrix AB^{-1}, where

$$M_j = [N_{j1}, N_{j2}, \cdots, N_{jk_j}]$$

and $$N_{ji} = (p_j, e, 0, 0, \cdots, 0)$$

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† Although this differs slightly from the canonical forms usually listed, it is obviously a real canonical form, since the elementary divisors of $M + \lambda E$ are the same as those of $AB^{-1} + \lambda E$. Cf. Turnbull and Aitken, *An Introduction to the Theory of Canonical Matrices*, p. 72.

is a matrix of order η_{ji}. Hence $e = 1$ and $p_j = \lambda_j$ if λ_j is real, and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $p_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$ if $\lambda_j = a_j + ib_j$. Thus there exists a non-singular matrix K such that

$$K^{-1}(AB^{-1} + \lambda E)K = M + \lambda E, \tag{4}$$

where, if A and B are real, K is also real. From (4) it follows that

$$K^{-1}(A + \lambda B)B^{-1}K = M + \lambda E$$

and, accordingly, that

$$K^*(B^{-1})^*(A + \lambda B)B^{-1}K = K^*(B^{-1})^*K(M + \lambda E),$$

or more concisely

$$P^*(A + \lambda B)P = F(M + \lambda E), \tag{5}$$

where $P = B^{-1}K$ and $F = K^*(B^{-1})^*K$. Since $B = B^*$, it follows that $F = F^*$, and from (5) that $FM = M^*F$. If $F = (F_{ij})$ is a partition of F similar to that of M, since $FM = M^*F$ we have the result

$$F_{ij}M_j = M^*{}_iF_{ij}, \qquad (i, j = 1, 2, \cdots, s).$$

Since no latent root of $M^*{}_i$ is the same as a latent root of M_j,

$$F_{ij} = 0, \qquad i \neq j.\dagger$$

Thus the matrix F reduces to the diagonal block matrix $[F_{11}, \cdots, F_{ii}, \cdots, F_{ss}]$, where $F^*{}_{ii} = F_{ii}$ and $F_{ii}M_i = M^*{}_iF_{ii}$.

We now consider the matrix equation $F_{ii}M_i = M^*{}_iF_{ii}$, and to simplify our notation, we write $N_j = N_{ij}$ and $k = k_i$. Hence

$$F_{ii}M_i = F_{ii}[N_1, N_2, \cdots, N_k]. \tag{6}$$

If $F_{ii} = (D_{rj})$ is a partition of F_{ii} similar to that of M_i we deduce from (6) that

$$D_{rj}N_j = N^*{}_rD_{rj} = N'{}_rD_{rj}.$$

Hence by Lemma (1) D_{rj} is a matrix of type $\{f_0, f_1, \cdots, f_{r-1}\}_r{}^j$, where f_i is of the type $\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$ if every f_i is a two rowed square matrix. In particular D_{jj} is real and of type $\{f_0, f_1, \cdots, f_r\}$.

If the order of N_1 is greater than the order of N_2, D_{11} is non-singular, since F_{ii} is non-singular, and by Lemma (1) D_{1t} $(t = 1, 2, \cdots, k)$ has a

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† C. E. Cullis, *loc. cit.*

zero element in the upper right-hand corner. If the order of N_1 is equal to the order of N_2, i. e. if $k_1 = k_2$, let

$$k_1 = k_2 \cdots = k_q > k_{q+1}.$$

If $|D_{11}| = 0$, there is at least one matrix D_{1j} $(1 < j \leq q)$ which has a non-zero element in the upper right-hand corner. If $|D_{11}| = 0$ and $|D_{jj}| \neq 0$ $(j = 2, 3, \cdots, q)$, then by a simple interchange of rows and columns we may move D_{jj} into the place of D_{11}. Again if $|D_{11}| = 0$, and $|D_{jj}| = 0$, then by Lemma (2) there exists a matrix X_i permutable with M_i such that

$$X^*_i F_{ii} X_i = T_i,$$

where $|T_{11}| \neq 0$. Hence without loss of generality we may assume that $|D_{11}| \neq 0$. Now by Lemma (3) there exists a matrix L_i permutable with M_i such that

$$L^*_i F_{ii} L_i = \begin{pmatrix} D_{11} & 0 \\ 0 & D_1 \end{pmatrix}$$

where $D_1 = (H_{ij})$ $(i, j = 2, 3, \cdots, k)$. Since $\begin{pmatrix} D_{11} & 0 \\ 0 & D_1 \end{pmatrix} M_i$ is Hermitian

$$H_{ij} N_j = N^*_i H_{ij} \qquad (i, j = 2, 3, \cdots, k).$$

Thus, we can treat D_1 in the same manner as F_{ii} and in k steps deduce the existence of a non-singular matrix G_i such that

$$(7) \qquad G^*_i F_{ii} M_i G_i = G^*_i F_{ii} G_i M_i = [K_1, K_2, \cdots, K_k] M_i.$$

Thus we have found a non-singular matrix G_i permutable with M_i such that (7) is true, where $K^*_j = K_j$, and $K_j N_j = N^*_j K_j$.

Hence by the corollary to Lemma (1), K_j is a real matrix of type $\{f_0, f_1, \cdots, f_r\}$. Therefore by Lemma (4) there exists a matrix S_{ij} permutable with N_j such that

$$(8) \qquad S^*_{ij} K_j N_j S_{ij} = S^*_{ij} K_j S_{ij} N_j = (f_0 T_1) N_j.$$

We have thus shown that there exists a matrix Q permutable with M such that

$$Q^* P^* (A + \lambda B) P Q = Q^* F (M + \lambda E) Q = Q^* F Q (M + \lambda E)$$

where $Q^* F Q = J$ and J is the diagonal block matrix

$$J = [J_1, J_2, \cdots, J_s],$$

and J_p is a matrix of the diagonal block type

$$J_p = [J_{p1}, J_{p2}, \cdots, J_{pk_p}]$$

and

$$J_{pj} = f_{pj} T_1 \tag{9}$$

is a matrix of η_{pj} rows and columns.

It might be well to note here that Q is a product of matrices of the type

$$GS, \quad \text{where} \quad G = [G_1, G_2, \cdots, G_s] \quad \text{as in (7)}$$

and $S = [S_1, S_2, \cdots, S_s]$; where $S_i = [S_{i1}, S_{i2}, \cdots, S_{ik}]$ as in (8).

If λ_i is real, then in (9) f_{ij} is a real number different from zero. On transforming the matrix $(f_{ij}T_1)N_j$ by the matrix $V_{ij} = (|f_{ij}|^{-\frac{1}{2}} I_0)$ we derive

$$V^*_{ij}(f_{ij}T_1)N_j V_{ij} = V^*_{ij}(f_{ij}T_1) V_{ij} N_j = (\epsilon_{ij} T_1) N_j$$

where $\epsilon_{ij} = \pm 1$. If λ_i is complex, f_{ij} is of the type $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. If $a \neq 0$, on transforming the matrix $(f_{ij}T_1)N_j$ by the matrix $U_{ij} = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix} I_0$, where $r = -[b \pm (b^2 + a^2)^{\frac{1}{2}}]/a$,

$$U^*_{ij}(f_{ij}T_1)N_j U_{ij} = U^*_{ij}(f_{ij}T_1) U_{ij} N_j = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1 N_j, \tag{10}$$

where k is a positive or negative real number. If $a = 0$, we see that the matrix $(f_{ij}T_1)N_j$ is already in the form (10) with $k = b$. If k is negative we transform $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1 N_j$ by the matrix $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I_0$ and this transformation gives

$$W^*_{ij} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1 N_j W_{ij} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1 N_j, \qquad k > 0.$$

If we now transform $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1 N_j$ by the matrix $X_{ij} = \begin{pmatrix} k^{-\frac{1}{2}} & 0 \\ 0 & k^{-\frac{1}{2}} \end{pmatrix} I_0$, we have

$$X^*_{ij} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} N_j X_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T_1 N_j.$$

Thus we have found a matrix Y permutable with M, such that

$$Y^*Q^*P^*(A + \lambda B)PQY = Y^*J(M + \lambda E)Y = Y^*JY(M + \lambda E),$$

where $Y^*JY = Z$, and Z is the diagonal block matrix

$$Z = [Z_1, Z_2, \cdots, Z_s] \quad \text{with} \quad Z_i = [Z_{i1}, Z_{i2}, \cdots, Z_{ik_i}]$$

and $Z_{ij} = \epsilon_{ij} T_1$, where $\epsilon_{ij} = \pm 1$ if λ_i is real and $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex. Here

$$Y = [Y_1, Y_2, \cdots, Y_s] \quad \text{where} \quad Y_i = [V_{i1}, V_{i2}, \cdots, V_{ik_i}]$$

if $i \leqq t$, and $Y_i = [Y_{i1}, Y_{i2}, \cdots, Y_{ik_i}]$ if $i > t$. If $a \neq 0$ $Y_{ij} = U_{ij}X_{ij}$ $(j = l_1, l_2, \cdots, l_r)$, and $Y_{ij} = U_{ij}W_{ij}X_{ij}$ $(j = l_s, l_{s+1}, \cdots, l_v)$. If $a = 0$,

$$Y_{ij} = W_{ij}X_{ij} \ (j = t_1, t_2, \cdots, t_s), \quad \text{and} \quad Y_{ij} = X_{ij} \ (j = t_{s+1}, \cdots, t_p),$$

where $l_1, \cdots, l_r, l_s, \cdots, l_v, t_1, \cdots, t_s, \cdots, t_p$ is some permutation of $1, 2, \cdots, k_i$.

Summing up our results we may state: Given a non-singular pencil of matrices $A + \lambda B$, having the properties stated in paragraph 2, we can find a non-singular matrix H, where $HM = MH$ with elements independent of λ such that

$$(11) \qquad H^*P^*(A + B)PH = Z(M + \lambda E) = [Z_1, Z_2, \cdots, Z_s](M + \lambda E),$$

where $Z_i = [Z_{i1}, Z_{i2}, \cdots, Z_{ik_i}]$ and $Z_{ij} = \epsilon_{ij}T_1$, where $\epsilon_{ij} = \pm 1$ if λ_i is real and $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex.

3. However, it is quite possible that $A + \lambda B$ be equivalent to two different normal forms (11). Let $V(M + \lambda E)$ and $Z(M + \lambda E)$ be two normal forms both equivalent to $A + \lambda B$, where $V = [V_1, V_2, \cdots, V_s]$ and $V_i = [V_{i1}, V_{i2}, \cdots, V_{ik_i}]$, where $V_{ij} = \delta_{ij}T_1$, where $\delta_{ij} = \pm 1$ if λ_i is real and $\delta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex. Then there exists a non-singular matrix R such that $R^*VMR = ZM$, and, since in obtaining the normal forms only transformations permutable with M were employed, it follows that

$$(12) \qquad RM = MR.$$

Hence

$$R^*VR = Z.$$

If the matrices R, V, and F are similarly partitioned, since $RM = MR$, R is a diagonal block matrix and

$$R^*_{ii}V_iR_{ii} = Z_i.$$

Since $V_i = Z_i$ for the complex roots, we need only consider the cases where $i \leq t$. If for simplicity we write R, V, Z and M for R_{ii}, V_i, Z_i and M_i respectively, M is the diagonal block matrix

$$M = [N_1, N_2, \cdots, N_k],$$

where $N_{ij} = N_j$ and N_j is a square matrix of order η_j. Let $R = (S_{ij})$ be a similar partition of R. Then, on equating the elements in the i-th row and j-th column of $R^*VR = Z$, we find that

$$\sum_{j=1}^{k} S^*_{ji}V_jS_{ji} = Z_i.$$

It follows from (12) that †

$$S_{ij} = (r_{ij}, x, y, \cdots, w),$$

where r_{ij} is zero if $\eta_i > \eta_j$. Thus the element in the upper right-hand corner of $S^*_{ji}V_jS_{ji}$ is zero unless $\eta_i = \eta_j$. Hence the element in the upper right-hand corner of $\sum_{j=1}^{k} S^*_{ji}V_jS_{ji}$ is

$$(13) \qquad \sum \bar{r}_{ji}\delta_j r_{ji},$$

where j ranges over all values for which $\eta_i = \eta_j$. Let

$$\eta_{i-1} > \eta_i = \eta_{i+1} = \cdots = \eta_f > \eta_{f+1}.$$

It follows from (13) that

$$(14) \qquad \sum_{t=i}^{f} \bar{r}_{tp}\delta_t r_{tp} = \epsilon_p \quad \text{and} \quad \sum_{t=i}^{f} \bar{r}_{tp}\delta_t r_{tq} = 0 \qquad (p, q = i, i+1, \cdots, f).$$

The two sets of equations in (14) can be written in the following manner

$$\begin{pmatrix} \bar{r}_{ii} & \bar{r}_{i+1,i} & \cdots & \bar{r}_{fi} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \bar{r}_{if} & \bar{r}_{i+1,f} & \cdots & \bar{r}_{ff} \end{pmatrix} \begin{pmatrix} \delta_i & & \cdots & 0 \\ 0 & \delta_{i+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \delta_f \end{pmatrix} \begin{pmatrix} r_{ii} & r_{i,i+1} & \cdots & r_{if} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{fi} & r_{f,i+1} & \cdots & r_{ff} \end{pmatrix} = \begin{pmatrix} \epsilon_i & & \cdots & 0 \\ 0 & \epsilon_{i+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \epsilon_f \end{pmatrix}.$$

Thus the two matrices $[\delta_i, \delta_{i+1}, \cdots, \delta_f]$ and $[\epsilon_i, \epsilon_{i+1}, \cdots, \epsilon_f]$ are equivalent under conjunctive transformation, and therefore

$$\delta_i + \delta_{i+1} + \cdots + \delta_f = \epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_f.$$

Accordingly, if $(\lambda + \lambda_i)^{\eta_{ij}}$ be a real elementary divisor of $A + \lambda B$, where $|B| \neq 0$, in the normal form there is associated with this elementary divisor an $\epsilon_{ij} = \pm 1$. If $(\lambda + \lambda_i)^{\eta_{ij}}$ is repeated exactly p times among the elementary divisors of $A + \lambda B$, we denote the ϵ_{ij} associated with these p elementary divisors by ${}^1\epsilon_{ij}, {}^2\epsilon_{ij}, \cdots, {}^p\epsilon_{ij}$ and write

$$(15) \qquad \sigma_{ij} = \sum_{\alpha=1}^{p} {}^\alpha\epsilon_{ij}.$$

Now if s of the ${}^\alpha\epsilon_{ij}$ have the value $+1$ and $p - s$ have the value -1, in the normal forms there are $p!/s!(p-s)!$ possible arrangements of the ${}^\alpha\epsilon_{ij}$, all of which are equivalent by our last result. We choose for *the normal form* that one in which the first s are positive. Thus two non-singular pencils of matrices $A + \lambda B$ and $C + \lambda D$, both having the properties of paragraph 2, are equivalent under congruent or conjunctive transformation if and only if they have the same normal form, i. e. if

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† C. E. Cullis, *loc. cit.*

(1) they have the same elementary divisors,

(2) the integers σ_{ij} defined by (15) are the same for both pencils.[†]

If in the pencil of matrices $A+\lambda B$ both A and B are singular, but $|A+\lambda B| \not\equiv 0$, we consider the non-singular pencil $\lambda_1 A+\lambda_2 B$. Let

$$\lambda_1 = g\lambda - g' \quad \text{and} \quad \lambda_2 = h\lambda - h',$$

where g, h, g' and h' are so chosen that

$$|gA+hB| \neq 0 \quad \text{and} \quad gh'-g'h = 1.$$

Hence $\lambda_1 A+\lambda_2 B = \lambda(gA+hB)-(g'A+h'B) = A_1+\lambda B_1.$

In a similar manner

$$\lambda_1 C+\lambda_2 D = \lambda(gC+hD)-(g'C+h'D) = C_1+\lambda D_1.$$

Now $\lambda_1 A+\lambda_2 B$ and $\lambda_1 C+\lambda_2 D$ are equivalent if and only if $A_1+\lambda B_1$ and $C_1+\lambda D_1$ are equivalent. Thus we may state:

A necessary and sufficient condition that two non-singular pencils of matrices $\lambda_1 A+\lambda_2 B$ *and* $\lambda_1 C+\lambda_2 D$ *(where* $A=A^*$, $B=B^*$, $C=C^*$ *and* $D=D^*$ *so that either both A and B, also both C and D, are real symmetric matrices or Hermitian matrices) be equivalent under congruent or conjunctive transformations is that*

(1) *they have the same elementary divisors,*

(2) *the integers* σ_{ij} *be the same for both pencils.* Here σ_{ij} is defined for the pencil $A_1+\lambda B_1$ and $C_1+\lambda D_1$ in a manner analogous to that in (15).

At the time of the completion of this paper, an abstract by K. W. Wegner [‡] appeared as follows:

"A necessary and sufficient condition that two pairs of Hermitian matrices $A+\lambda B$ and $C+\lambda D$, $|B| \neq 0$ and $|D| \neq 0$, be equivalent under conjunctive transformations is that

(α) they have the same elementary divisors,

(β) the indices of $B(B^{-1}A+\lambda E)^m$ and $D(D^{-1}C+\lambda E)^m$ be the same for every real value of λ and every positive integral value of m."

We wish to show here that the results obtained by Wegner and the results obtained in this paper are equivalent. From (4) we have

$$K^{-1}(AB^{-1}+\lambda E)K = M+\lambda E.$$

Hence $$K^{-1}(AB^{-1}+\lambda E)^m K = (M+\lambda E)^m.$$

[†] In the real case, this result is the same as the result derived by P. Muth, "Ueber reele Aquivalenz von Scharen reeler quadratischer Formen," *Crelle's Journal*, p. 314.

[‡] *Bulletin of the American Mathematical Society*, January, 1934.

Thus we can write

$$K^{-1}(AB^{-1}+\lambda E)^m BB^{-1}K = (M+\lambda E)^m$$

and accordingly

$$P^*(AB^{-1}+\lambda E)^m BP = F(M+\lambda E)^m,$$

where P has the same meaning as in (5). We may proceed exactly as we did in the derivation of our normal form and deduce that

$$Q^*P^*(AB^{-1}+\lambda E)^m BPQ = Q^*FQ(M+\lambda E)^m = Z(M+\lambda E)^m.$$

It is obvious that if the two pencils have the same normal form, the results (α) and (β) are true.†

For a fixed value of λ and a fixed value of m, it is apparent that there exists a non-singular matrix K, such that

$$K^*(AB^{-1}+\lambda E)^m BK = H, \quad \text{where} \quad H = [H_1, H_2, \cdots, H_s] \quad \text{and}$$
$$H_i = [H_{i1}, H_{i2}, \cdots, H_{ik_i}] \quad \text{and} \quad H_{ij} = \epsilon_{ij}(\lambda+\lambda_i)^m T_1.$$

If $i > t$, the signature of H_{ij} is obviously zero. If $i \leqq t$ and $\lambda + \lambda_i$ is positive, the signature of H_{ij} is independent of the value of m. Let us suppose λ_1 so chosen that if $\lambda = -\lambda_1$, $\lambda_j + \lambda > 0$ $(j = 2, 3, \cdots, t)$. Let s_i be the signature of $Z_i(M_i - \lambda_1 E)^m$, and s'_i be the signature of $V_i(M_i - \lambda_1 E)^m$ $(i = 2, 3, \cdots, t)$. If $m = \eta_{11}$,

$$Z_1(M_1 - \lambda_1 E)^m = 0,$$

and, since the signature of $Z(M+\lambda E)^m$ and $V(M+\lambda E)^m$ are the same for all real values of λ and every integral value of m, we have

$$s_2 + s_2 + \cdots + s_t = s'_2 + s'_3 + \cdots + s'_t.$$

If $m = \eta_{11} - 1$, the signature of $Z_1(M_1 - \lambda_1 E)^m$ is $\sum^a \epsilon_{11} = \sigma_{11}$, and the signature of $V_i(M_1 + \lambda_1 E)^m$ is $\sum^a \delta_{11} = \sigma'_{11}$. Therefore

$$\sum{}^a \epsilon_{11} + s_2 + \cdots s_t = \sum{}^a \delta_{11} + s'_2 \cdots + s'_t.$$

Hence $$\sum{}^a \epsilon_{11} = \sum{}^a \delta_{11}, \quad \text{i. e.} \quad \sigma_{11} = \sigma'_{11}.$$

On continuing this process, we easily see that $\sigma_{ij} = \sigma'_{ij}$ for every value of i and j. Thus if (α) and (β) are true, the two pencils have the same normal form.

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† The matrix which we transform here differs from that in (β) only as to the situation of B, hence a change of notation would be necessary to alter this slight difference; but it is obvious that the work carries through in the same manner, so we refer to (β), keeping this difference in mind.

A GENERALIZATION OF THE VERONESE AND STEINER SURFACES.*

By John R. Mayor.

Introduction. If $f_i(x_j)$ are linearly independent quadratic forms, the equations

$$(1) \qquad y_i = f_i(x_j), \; i = 0, \cdots, R = r(r+3)/2; \; j = 0, \cdots, r,$$

map the points of S_r, hereafter called π, upon a locus of dimension r in S_R. It is called the Veronese variety. In this paper properties of this variety and of its projections are studied by means of a $(1,1)$ correspondence that may be set up between the hyperquadrics of π of the system

$$(2) \qquad \sum_i \lambda_i f_i(x_j) = 0$$

and the hyperplanes of S_R of the system

$$(3) \qquad \Sigma \lambda_i y_i = 0.$$

If (2) is the system of all hyperquadrics of π, it is simple, and the variety given by (1) is of order 2^r.

Properties of the Veronese $G_r^{2^r}$ are determined in Article 1. In the second, third, and fourth articles two projections of the $G_r^{2^r}$ upon an S_{r+1} are considered. The first projection, given in Article 2, is that for which the space of projection S_c, $c = (r^2 + r - 4)/2$, is chosen in a general way in order that the hypersurface, $G_r^{2^r}$, obtained by the projection is one which can be represented rationally by general quadratic forms. Articles 3 and 4 are devoted to the second projection; for it the S_c of projection is chosen in a special way in order that the hypersurface resulting from the projection may have the equation

$$\Sigma y_i^{1/2} = 0.$$

$G_r^{2^r}$ is used to represent only this Veronese variety in S_R given by (1) or its projection upon S_{r+1} from a general S_c of S_R. The locus in S_{r+1}, which has a rational representation with each f_i the square of a linear form, is denoted by $F_r^{2^r}$ to distinguish it from that for which the f_i are general quadratic forms. In the discussion a pair of points P, P' always refer to a pair of points which forms a composite hyperplane hyperquadric of a linear system corresponding to the space of projection, under the correspondence which is established.

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* Presented to the Society, April 14, 1933.

The well-known Veronese surface is given by (1) for $r = 2$; Veronese[*] showed that the Steiner surface is a projection of the Veronese surface from a general line upon an S_3. In this case, $r = 2$, there is no distinction between the $G_2{}^4$ and $F_2{}^4$ in S_3, since in a general linear system of ∞^3 point conics, there are four conics which consist of a line counted twice. Either of the two projections of the Veronese variety $G_r{}^{2^r}$ which are given in this paper might be considered as generalizations of the Steiner surface. The author has determined in detail the properties of the Veronese variety and these two projections of it for the cases $r = 3$ and $r = 4$ but the results are not included here.

Several references to a generalization of the Veronese surface occur in the literature. That generalization, for $r = 3$, is implied by Reye[†] in a discussion of linear systems of quadrics. Use of the Veronese $G_3{}^8$ is made by Del Pezzo[‡] in his study of surfaces of n-th order in S_n for the case $n = 8$. Bertini[§] first rigorously proved that the Veronese surface is the only surface of S_r, $r > 4$, not a cone, whose tangent planes meet in pairs in a point. This theorem has been extended for varieties of dimension 3 and 4 by Scorza.[¶]

In a paper devoted to varieties whose plane sections are elliptic, Scorza[‖] first referred to the variety given by (1) as the variety of Veronese and mentioned that the $G_3{}^8$ is a double locus of a certain $M_6{}^{10}$. In a generalization of an argument recorded by Bertini,[**] he showed that two $G_3{}^8$'s coincide if they have two hyperplane sections in common and that any variety whose general hyperplane section is a $G_3{}^8$ is a cone. It is stated in a footnote that the argument could be generalized. This last property has also been given in a more general theorem by Tanturri.[††]

Cosserat[‡‡] suggested a method for studying a certain type of surface as a hyperplane section of the $G_3{}^8$ in S_4. He mentioned the special variety,

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[*] Veronese, *Reale Accademia dei Lincei, Memorie* (3), vol. 19 (1884), pp. 344-71.

[†] Reye, *Journal für Mathematik*, vol. 82 (1877), pp. 54-83.

[‡] Del Pezzo, *Rendiconti di Palermo*, vol. 1 (1887), pp. 261-71.

[§] Bertini, *Introduzione alla geometria proiettiva degli iperspazi* (Pisa, 1907), p. 315.

[¶] Scorza, *Rendiconti di Palermo*, vol. 25 (1908), pp. 193-204; also, vol. 27 (1909), pp. 148-78.

[‖] Scorza, *Annali di Matematica* (3), vol. 15 (1908), pp. 217-73.

[**] Bertini, *loc. cit.*, p. 342.

[††] Tanturri, *Giornale di Matematiche di Battaglini*, vol. 45 (1907), pp. 291-7. (14° della 2° serie.)

[‡‡] Cosserat, *Comptes Rendus*, vol. 124 (1897), pp. 1004-8.

$F_3{}^8$, in this connection and noted that his variety in that case is one of the "ipersteineriana" of Brambilla.* The $F_r{}^{2r}$ of this paper is a special case, $n = 2$, of these "ipersteineriana," hypersurfaces of S_{r+1} which can be represented rationally by the n-th powers of linear forms. Brambilla's method is strictly analytic, while properties of the $F_r{}^{2r}$ are found in this paper by means of its projection from the Veronese variety. Segre † made an application of a cubic variety, which is a particular case of the Veronese variety for $r = 3$, to a study of systems of lines in S_3 and surfaces in S_3. Palatini ‡ has defined a variety representable in S_r by all the varieties of dimension $r - 1$ of S_r of a given order and considered some special cases in a study of forms (particularly ternary) which can be represented by sums of powers of linear forms.

1. *Properties of the $G_r{}^{2r}$ in S_R.* Let all the point hyperquadrics of π of a linear ∞^{R-1} system correspond to the ∞^{R-1} hyperplanes through a point of S_R. If one makes this point of S_R correspond to the hyperplane hyperquadric of π apolar to the point hyperquadrics of the ∞^{R-1} system, by means of the (1, 1) correspondence between the point hyperquadrics of π and the hyperplanes of S_R given by values of λ_i in (2) and (3), one obtains a (1, 1) correspondence between the hyperplane hyperquadrics of π and the points of S_R. It can be shown that this correspondence is unique in both directions. The conditions that a point of S_R lie on a hyperplane of S_R is that the corresponding hyperplane hyperquadric and point hyperquadric of π be apolar. Requiring that the hyperquadrics of (2) pass through a point x in π is requiring that they be apolar to a hyperplane hyperquadric which consists of a point counted twice, the point x. Thus each point of $G_r{}^{2r}$ corresponds to a point of π, considered as a hyperplane hyperquadric consisting of a point counted twice. The points of hyperplane sections of $G_r{}^{2r}$ correspond to the points of π lying on a hyperquadric in π. The 2^r points in which $G_r{}^{2r}$ is met by an S_{R-r} correspond to the 2^r base points of a linear ∞^{r-1} system of point hyperquadrics.

In S_r are ∞^{2r} point pairs, so that to all point pairs of π correspond an $M^k{}_{2r}$, $k = \frac{1}{2}\binom{2r}{r}$. *The $G_r{}^{2r}$ is double on the locus $M^k{}_{2r}$.* The complete system of hyperplane hyperquadrics of π which consist of a pair of points is of dimension $2r$; since by $2r$ hyperplanes are determined $\frac{1}{2}\binom{2r}{r}$ pairs of points, the degree of the system is $\frac{1}{2}\binom{2r}{r}$. Consequently in any linear system of hyperplane hyperquadrics of dimension $R-2r$ are $\frac{1}{2}\binom{2r}{r}$ hyperquadrics which consist of a pair of points; or the variety $M^k{}_{2r}$ is met by an $S_{r(r-1)/2}$ in S_R in k points.*

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* Brambilla, *Atti dell'Accademia delle scienze fisiche e matematische, Napoli* (2), IX (1899), paper no. 14. Also other papers by the same author, the results of which have been combined and generalized in this paper.

† Segre, *R. Accad. delle scienze di Torino, Memorie* (2), vol. 39 (1888), pp. 1-48.

‡ Palatini, *R. Accad. dei Lincei, Rendiconti* (5), vol. 12 (1903), pp. 378-84.

To an i-dimensional locus of order n in π *corresponds one of dimension i and order* $2^i n$ *on* $G_r{}^{2^r}$ *in* S_R. *On* $G_r{}^{2^r}$ *the only loci of dimension i are of order* $2^i n$. In S_R, a variety $V_i{}^{2^i n}$, is met by an S_{R-i} in $2^i n$ points. These points in π correspond to the $2^i n$ points in which a locus $V_i{}^n$ in π is met by i hyperquadrics. By the correspondence an S_{R-i} corresponds to i hyperquadrics in π, since it is the intersection of i hyperplanes. Suppose on $G_r{}^{2^r}$ were a locus $V_i{}^m$, $m \neq 2^i n$. Then it is met by an S_{R-i} in m points. But these points, since the correspondence is $(1,1)$, correspond to the m points in which a variety of dimension i in π is met by i hyperquadrics. This variety is necessarily of order $m/2^i$, which is a contradiction.

The system of hyperplane hyperquadrics consisting of a fixed point with every point of an S_i is an ∞^i linear system. *On* $M^k{}_{2r}$ *are linear spaces* S_i, $(i=1,\cdots,r)$. From the correspondence in π, it is seen that each S_r of $M^k{}_{2r}$ has a point in common with $G_r{}^{2^r}$ at which it is tangent. Two S_r's of $M^k{}_{2r}$ always have a point in common which is not a point of $G_r{}^{2^r}$.† There are two different types of planes on $M^k{}_{2r}$. Those already described are referred to as planes of the second species. A plane of the first species corresponds in π to a linear system of hyperplane hyperquadrics consisting of every point pair on a line. Such a plane of the first species does not meet any of the other linear spaces of $M^k{}_{2r}$ in more than a line. Two planes of the first species may have a point in common, in which case it is a point of $G_r{}^{2^r}$. The planes of the first species meet $G_r{}^{2^r}$ in a conic. *On each point of* $M^k{}_{2r}$ *is a plane of the first species and two* S_r*'s.* The two S_r's through a point have only the one point in common, but the S_{2r} determined by them contains the plane of the first species through the point. Each S_r has a line in common with the plane. *Through a point of* $G_r{}^{2^r}$ *are* ∞^{r-1} *planes of the first species and just one* S_r.

Two S_r's of $M^k{}_{2r}$ determine an S_{2r}, which is met by a third in an S_{3r-1}; r of the S_r's determine a hyperplane in S_R. This hyperplane is tangent to $G_r{}^{2^r}$ at r points and from the correspondence is seen also to contain the conics, cut out by planes of the first species, joining these points in pairs. Such a hyperplane in general corresponds to a hyperquadric of π with r double points, points of an S_{r-1}, and the lines on the hyperquadric joining these. Such a hyperquadric is necessarily a hyperplane squared, so that the hyperplane of S_R corresponding to this hyperquadric meets the $G_r^{2^r}$ in a $V_{r-1}^{2^{r-1}}$ counted twice. It is tangent along this variety, as is seen from the correspondence in π; it contains all tangent hyperplanes to $G_r^{2^r}$ along this $V_{r-1}^{2^{r-1}}$. *The* $G_r^{2^r}$ *has hyperplanes tangent along a* $V_{r-1}^{2^{r-1}}$.

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* The order k of the variety may also be obtained from formulas given by Segre, *Atti della Reale Accademia dei Lincei* (5), vol. 9 (1900), p. 258; this case requires that a symmetric determinant of order $r+1$ be of rank 2. Scorza noted, *Annali di Matematica, loc. cit.*, that properties of the Veronese variety could be determined with the aid of these formulas of Segre.

† Scorza stated that the tangent hyperplanes of the Veronese variety, for the cases $r=3$ and $r=4$, meet in pairs in a point (see the introduction).

2. *Projection of* $G_r^{2^r}$ *from a general* S_c. The projection of the Veronese $G_r^{2^r}$ upon S_{r+1} from a general S_c, a $G_r^{2^r}$ in S_{r+1}, is represented by (1), where $i = 0, \cdots, r+1$; $j = 0, \cdots, r$. The S_c of projection meets the $M^k{}_{2r}$ in a $C^k{}_{r-2}$ corresponding to the locus of point pairs P, P' which are hyperplane hyperquadrics in the linear system of ∞^c hyperplane hyperquadrics corresponding to the S_c. The conics of $G_r^{2^r}$ which lie in a plane of the first species on a point of $C^k{}_{r-2}$ correspond to the lines PP' in π. Under the projection for each point of the $C^k{}_{r-2}$ is a double line on $G_r^{2^r}$ in S_{r+1} since on each point of $C^k{}_{r-2}$ is a plane of the first species which meets $G_r^{2^r}$ in a conic. This conic projects into a double line; every S_{c+1} through the S_c of projection and a point of the conic will necessarily meet the conic in two points. *On* $G_r^{2^r}$ *in* S_{r+1} *is a double locus,* $V_{r-1}^{2^{r-2}}$, *which is a locus of double lines.*

The points of tangency of lines from a point of $C^k{}_{r-2}$ to the conic on its plane of the first species project into cuspidal points on $G_r^{2^r}$ (in S_{r+1}) with the properties of cuspidal points of a Steiner surface. *On* $G_r^{2^r}$ *there is a cuspidal locus* $V^m{}_{r-2}$, which corresponds in π to the locus of points P, P'. These multiple loci and others, arising from the intersections of the double $V_{r-1}^{2^{r-2}}$'s may always be determined by a consideration of the properties of a linear ∞^{r+1} system of point hyperquadrics. Triple loci on $G_r^{2^r}$ in S_{r+1} correspond to loci of intersections of two lines PP' as is the case on the Steiner surface; and k-fold loci correspond to loci of intersections of $k-1$ lines PP'. If two lines PP' of π intersect, the corresponding conics in planes of the first species associated with points of $C^k{}_{r-2}$ intersect in a point; an S_{c+1} determined by the S_c of projection and a point of intersection of two of these conics meets each of the conics in a second point, and, as the S_{c+1} meets the $G_r^{2^r}$ in S_R in three points, the projection of $G_r^{2^r}$ upon S_{r+1} will have a triple point.

Since part of the projection in this general case may be discussed in terms of the projection of the Veronese surface to obtain the Steiner surface, it might seem possible that the projection of the Veronese $G_r^{2^r}$, for a given r, from a general S_c could be described in terms of that for the preceding value of r as is precisely the situation for the special case to be considered in

Article 3. This however is not the case. In the general linear system of ∞^{r+1} point hyperquadrics in π, for $r > 3$, there are no hyperquadrics which consist of two hyperplanes. The order of the system of hyperplane hyperquadrics apolar to ∞^{r+1} point hyperquadrics is c. $[(r^2+r-4)/2]+1$ conditions must be satisfied by the coefficients of a hyperquadric apolar to this system. This is $> 2r$, for $r > 3$.* For the projection of the $G_r^{2^r}$ from an S_c in S_R to be described in terms of the projection of the preceding case, that is as the projection of ∞^1 $G^{2^{r-1}}_{r-1}$'s from an $S_{[(r-1)^2+(r-1)-4]/2}$ in $S_{(r+2)(r-1)/2}$, it is necessary that some space of the space of projectivity meet the $M^k{}_{2r}$ in a locus like that in which the projection space meets the variety on which the Veronese variety is double of the preceding case. This is possible only if in a general linear system of ∞^c hyperplane hyperquadrics there is a linear system of dimension $[(r-1)^2+(r-1)-4]/2$ which contains a locus of point pairs just like that in an $\infty^{[(r-1)^2+(r-1)-4]/2}$ linear system of hyperplane hyperquadrics of S_{r-1}. If such a system existed there would be imposed upon the hyperplane which contains it conditions sufficient that another hyperplane could be determined such that the product of the pair would be a hyperquadric of the system in π apolar to the ∞^c system. If a hyperquadric is composite, with a hyperplane containing this configuration of pairs of points which are hyperplane hyperquadrics of a system of dimension c as a component of it, it satisfies $\{[(r-1)^2+(r-1)-4]/2\}+1$ of the conditions that a hyperquadric must satisfy in order to be in the system apolar to a system of dimension c. A hyperquadric belonging to such a system must satisfy $c+1$ conditions. But $[(r^2+r-4)/2]+1-\{[(r-1)^2+(r-1)-4]/2\}-1=r$. A unique hyperplane could be determined for the other component. But if this ∞^c system is general, the system apolar to it is also and could not have a hyperquadric degenerated to a pair of hyperplanes.

It follows that *on $G_r^{2^r}$ of S_{r+1} there is no $G^{2^{r-1}}_{r-1}$ for $r > 3$.*

3. *Projection of $G_r^{2^r}$ when the S_c is specially chosen.* When the space of projection is chosen to meet the $M^k{}_{2r}$, on which $G_r^{2^r}$ is double, in such a way that the linear system of hyperplane hyperquadrics corresponding to it in π is a linear system with an apolar $(r+2)-S_{r-1}$, the $G_r^{2^r}$, which is the projection is denoted by $F_r^{2^r}$.† The locus of point pairs P, P' of the system

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* Note that for $r=2$, the number of conditions is two so that given any line in the plane another can be determined such that the pair form a conic of an ∞^3 linear system of point conics. This accounts for the fact that each tangent plane to a Steiner surface meets the surface in a pair of conics.

† A linear system of hyperplane hyperquadrics is said to have an apolar $(r+2)-S_{r-1}$ if there is associated with the system $r+2$ hyperplanes such that the S_i's, $(i < r-1)$, determined by $r-i$ of the $r+2$ S_{r-1}'s are apolar to the

of hyperplane hyperquadrics is the $(r+2)(r+1)/2$ S_{r-2}'s of intersection of the $r+2$ hyperplanes of the $(r+2)-S_{r-1}$. The points P, P' in a pair are so arranged that each point of an S_i determined by $r-i$ of the $r+2$ hyperplanes, with every point on the S_{r-2-i} determined by the remaining $i+2$ of these hyperplanes is a point pair forming a hyperplane hyperquadric of the system. From the correspondence between the point pairs of π and the points of $M^k{}_{2r}$ it is seen that *the S_c of projection must be chosen to meet the $M^k{}_{2r}$, if r is odd, in $(r+2)(r+1)/2$ linear spaces S_{r-2} and $\binom{r+2}{r-p}$ $V^{k_p}_{r-2}$'s for $p=1, \cdots, (r-3)/2$. If r is even the configuration in which the space of projection meets $G_r{}^{2r}$ has $(r+2)(r+1)/2$ linear spaces S_{r-2}, $\binom{r+2}{r-p}$ $V^{k_p}_{r-2}$'s for $p=1, \cdots, (r-4)/2$, and $\binom{r+2}{p+2}/2$ $V^{k_p}_{r-2}$'s for $p=r-p-2$. k_p is the order of the system of hyperplane hyperquadrics made up of a pair of points, one from an S_p of S_r and the other from an S_{r-p-2} not meeting S_p.*

On each point of this composite $C^k{}_{r-2}$, as in the general case, there is a plane of the first species which meets $G_r{}^{2r}$ in a conic and this conic will be projected into a double line. From the correspondence in π it is seen that all of the planes of the first species on a point of one of the $(r+2)(r+1)/2$ S_{r-2}'s of $C^k{}_{r-2}$ meet in a point so that *part of the locus of double lines on $F_r{}^{2r}$ consists of $(r+2)(r+1)/2$ V^{2r-2}_{r-1}'s*, [case $r=2$, there are $(r+2)\times(r+1)/2\times 2$], *on each of which there are ∞^{r-2} lines through a point.* Further the points of tangency of lines of a plane of the first species, from the point of the $C^k{}_{r-2}$ to the conic on the plane, project into cuspidal points. Since these points correspond to P, P' in π, the cuspidal locus will correspond to the S_{r-2}'s of the $(r+2)-S_{r-1}$. Therefore *there are $(r+2)(r+1)/2$ V^{2r-2}_{r-2}'s on $F_r{}^{2r}$, the points of which have the properties of the cuspidal points of a Steiner surface.* In fact, in the next article it will be shown that *the V^{2r-2}_{r-2}'s are actually the loci of cuspidal points of the ∞^{r-2} Steiner surfaces which are on $F_r{}^{2r}$.* It follows that *the V^{2r-2}_{r-2} is an F^{2r-2}_{r-2}* from the fact that an S_i of the $(r+2)-S_{r-1}$ is met by the other spaces of the $(r+2)-S_{r-1}$ in an $(i+2)-S_{i-1}$. By an S_i of the $(r+2)-S_{r-1}$ is meant an S_i determined as the intersection of $r-i$ of the $r+2$ hyperplanes.

As in the more general case, all the properties of the $F_r{}^{2r}$ may be determined by means of a consideration of the properties of an ∞^c linear system of hyperplane hyperquadrics with an apolar $(r+2)-S_{r-1}$. Where two lines, PP', meet in two, or r in a point the conics on the planes of the first species of $M^k{}_{2r}$ meet 2 or r in a point, and in the resulting projection triple or $(r+1)$-fold points arise.

S_{r-i-2}'s determined by the remaining S_{r-1}'s in regard to the point hyperquadrics of the system apolar to this system of hyperplane hyperquadrics.

4. *$F_i^{2^i}$'s on $F_r^{2^r}$.* A consideration of the properties of an $(r+2)-S_{r-1}$ in S_r is sufficient to show not only the existence of $F_i^{2^i}$'s on $F_r^{2^r}$, but also how each projection may be described in terms of that of next lower dimension. In an $(r+2)-S_{r-1}$ in S_r are $\binom{r+2}{i+2}$ S_i's, $(i<r)$. In each hyperplane of the $(r+2)-S_{r-1}$ are $\binom{r+1}{i+2}$ S_i's; in each S_k $(i<k<r)$ are $\binom{k+2}{i+2}$ S_i's. Through each S_k are $\binom{r-k}{r-i}$ S_i's $(r>i>k)$. In an $(r+2)-S_{r-1}$ of S_r, the configuration of spaces on any S_k is that of an $(k+2)-S_{k-1}$ in an S_k. In the $(m+2)-S_{m-1}$ of an S_m, belonging to an $(r+2)-S_{r-1}$ in S_r, the S_i's $(i<m-1)$ of S_m, intersection of S_m with an S_{r-m+i} of the $(r+2)-S_{r-1}$ in S_r, are opposite in the $(m+2)-S_{m-1}$ of S_m to the S_{m-i-2} in the $(m+2)-S_{m-1}$ of S_m, which are opposite to the S_{r-m+i} in the $(r+2)-S_{r-1}$ of S_r. Thus, in the configuration of a $(k+2)-S_{k-1}$ of an S_k which is an S_k of an $(r+2)-S_{r-1}$ of S_r, $(k<r)$, the point pairs which are on opposite spaces, are also on opposite spaces in the $(r+2)-S_{r-1}$. These properties may be used to prove the theorem on hyperquadrics: The ∞^{k-2} point pairs consisting of the pairs made up of a point of an S_i of an $(k+2)-S_{k-1}$, $(i<k-1)$, of S_k with a point from the opposite S_{k-2-i} are hyperplane hyperquadrics in a linear $\infty^{(k^2+k-4)/2}$ system of hyperplane hyperquadrics in S_r, $(r \geqq k)$.

Any $\bar{S}_{r-1}$ through an S_{r-2} of the $(r+2)-S_{r-1}$ is met by the spaces of the $(r+2)-S_{r-1}$ in an $(r+1)-S_{r-2}$, and points of the $(r+1)-S_{r-2}$ of $\bar{S}_{r-1}$ on opposite spaces are on opposite spaces on the $(r+2)-S_{r-1}$. By these theorems it is seen that the components of $C^k{}_{r-2}$ meeting any of the linear components of it, are met by ∞^1 $S_{[(r-1)^2+(r-1)-4]/2}$'s in the points corresponding to an $(r+1)-S_{r-2}$. *The projection of the $G_r^{2^r}$ from an S_c, for this special case,* actually *reduces to the projection from ∞^1 $S_{[(r-1)^2+(r-1)-4]/2}$'s.* All of the multiple loci on the projection of the Veronese variety arise from the manner in which the conics on $G_r^{2^r}$ intersect; but the conics on $G_r^{2^r}$ intersect as they do on $G^{2^{r-1}}_{r-1}$. *On $F_r^{2^r}$ are $(r+2)(r+1)/2$ sets of ∞^1 $F^{2^{r-1}}_{r-1}$'s with a cuspidal $F^{2^{r-2}}_{r-2}$ common to all of one pencil, which is a trope for each of the $F^{2^{r-1}}_{r-1}$'s of the pencil. On $F_r^{2^r}$ are ∞^{r-i} varieties, $F_i^{2^i}$.*

On $F_r^{2^r}$ are ∞^r $V^{2^{r-1}}_{r-1}$'s corresponding to the hyperplanes of S_r. For $r=2$, any V_1^2 is an F_1^2 so that on the Steiner surface are ∞^2 conics. The existence of the pencils of $F^{2^{r-1}}_{r-1}$'s follows from the property of the $\bar{S}_{r-1}$'s through an S_{r-2} of the $(r+2)-S_{r-1}$ just stated. The cuspidal property of the locus corresponding to the S_{r-2} itself is shown below.

The $F^{2^{r-1}}_{r-1}$'s on $F_r^{2^r}$ correspond to ∞^1 hyperplanes on an S_{r-2} of the $(r+2)-S_{r-1}$ but a pair of these hyperplanes form a hyperquadric of the system corresponding to the S_c of projection. *A necessary condition for an*

S_{r-1} in S_r to correspond to an $F^{2^{r-1}}_{r-1}$ is that it be one of a pair whose product is a hyperquadric of the system corresponding to the S_c of projection. This follows from the same argument used in the general case of Article 2. *The condition is sufficient for all but those $(r+2)(r+1)/2$ exceptional hyperplanes, joining an S_{r-2} of the $(r+2)-S_{r-1}$ to the opposite vertex, which correspond to double $V^{2^{r-2}}_{r-1}$'s.*

In a linear system of ∞^{r+1} point hyperquadrics which is apolar to a linear system of ∞^c hyperplane hyperquadrics with an apolar $(r+2)-S_{r-1}$, are $r+2$ hyperplanes squared. If the space of projection, S_c, corresponds to such a system of hyperplane hyperquadrics, through S_c are $r+2$ hyperplanes in S_R, each tangent to $G_r^{2^r}$ along a $V^{2^{r-1}}_{r-1}$. These appear on $F_r^{2^r}$ after projection, as $V^{2^{r-1}}_{r-1}$'s with properties similar to those of a conic trope of a Steiner surface. Also since hyperquadrics of the system which consist of a pair of hyperplanes or hyperplanes counted twice, are hyperplanes through an S_{r-2} of the $(r+2)-S_{r-1}$, these $V^{2^{r-1}}_{r-1}$'s are $F^{2^{r-1}}_{r-1}$'s. *On $F_r^{2^r}$ are $r+2$ $F^{2^{r-1}}_{r-1}$'s with properties of a conic trope for a Steiner surface. They are the locus of conic tropes for the ∞^{r-2} F_2^{4}'s of $F_r^{2^r}$. The class of $F_r^{2^r}$ is $r+2$, and its hyperplane equation is $\Sigma 1/u_i=0$.*

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LINEARLY CONNECTED SPACES AND ENNUPLES OF CURVES.

By HARRY LEVY.

1. In a Riemannian space with a positive definite fundamental form

$$ds^2 = g_{ij} dx^i dx^j \tag{1.1}$$

n mutually orthogonal unit congruences of curves defined by

$$dx^1/{}_\alpha\lambda^1 = dx^2/{}_\alpha\lambda^2 = \cdots = dx^n/{}_\alpha\lambda^n \qquad (\alpha = 1, 2, \cdots, n) \tag{1.2}$$

and consequently satisfying the conditions

$$g_{ij}\, {}_\alpha\lambda^i\, {}_\beta\lambda^j = \delta_{\alpha\beta} = \begin{matrix} 0 & \alpha \neq \beta, \\ 1 & \alpha = \beta, \end{matrix} \tag{1.3}$$

determine a set of invariants, which we denote by $\gamma^\alpha{}_{\beta\gamma}$ defined by

$$\gamma^\alpha{}_{\beta\gamma} = {}_\beta\lambda^i{}_{,j}\, {}^\alpha\lambda_i\, {}_\gamma\lambda^j. \tag{1.4}$$

It is immaterial whether we adopt the customary definition

$${}^\alpha\lambda_i = g_{ij}\, {}_\alpha\lambda^j$$

or whether we define ${}^\alpha\lambda_i$ to be the cofactor of ${}_\alpha\lambda^i$ in the determinant $\lambda = |\, {}_\beta\lambda^j \,|$ divided by λ itself. ${}_\beta\lambda^i{}_{,j}$ is the covariant derivative of ${}_\beta\lambda^i$ with respect to the form (1.1)

$${}_\beta\lambda^i{}_{,j} = \partial\, {}_\beta\lambda^i / \partial x^j + {}_\beta\lambda^h \{{}_h{}^i{}_j\}. \tag{1.5}$$

Here $\{{}_h{}^i{}_j\}$ are the Christoffel symbols of the second kind formed with respect to the form (1.1).

It is readily seen that it is not necessary in the formulation of these ideas to restrict ourselves to Riemannian geometry as did Ricci when he first introduced the γ's.* For example, in a linearly connected space, that is one in which parallelism is defined by means of n^3 arbitrary functions $\Gamma^i{}_{jk}$ instead of $\frac{1}{2}n^2(n+1)$ functions which are the Christoffel symbols of a quadratic form, we can readily define invariants of an ennuple essentially by the above process.*

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* L. P. Eisenhart in his *Riemannian Geometry*, Princeton University Press (1926), gives a detailed treatment of ennuples and their *coefficients of rotation*, the name introduced by Ricci of the functions $\gamma^\alpha{}_{\beta\gamma}$. We shall refer to this book with the notation R. G.

In the development of both Riemannian geometry and the geometry of a linearly connected space ennuples of curves and their invariants have played an important rôle. It seems to the writer however that the possibilities in this direction have not been fully exploited, partially because the methods applied depend on Ricci's absolute calculus rather than on a calculus in which the ennuples themselves play the fundamental part. One result of this is that in the main the invariants $\gamma^{\alpha}{}_{\beta\gamma}$ have been utilized in the study of properties of configurations within the given space rather than those of the space itself. We shall develop and apply a theory in which the ennuple and its invariants are fundamental, in which for example, our process of differentiation, analogous to covariant differentiation, enables us to obtain from a given *intrinsic* tensor (see § 3) a new intrinsic tensor.

From another point of view our work will tie up with Graustein's study of invariant methods in classical differential geometry in which he points out that a powerful theory may possibly be built by combining the methods of Ricci and the intrinsic methods of Cesaro.†

2. We begin with a space V_n of n dimensions, to every point of which is assigned coördinates $x^1, x^2, \cdots, x^n$. n independent but otherwise arbitrary contravariant vector fields are given by a set of functions ${}_{\alpha}\lambda^i$ where $\alpha, i = 1, 2, \cdots, n$ and α is fixed for a fixed vector field. We associate with the ennuple ${}_{\alpha}\lambda^i$ a set of n^3 arbitrary functions $\gamma^{\beta}{}_{\alpha\gamma}$ which we postulate to be invariant under transformations of coördinates. We shall speak of these γ's as the *invariants* of the ennuple ${}_{\alpha}\lambda^i$.

If in place of the n given vector fields we introduce n others we can express their components ${}_{\alpha}\bar{\lambda}^i$ linearly in terms of the components of the given fields ‡

$$(2.1) \qquad {}_{\alpha}\bar{\lambda}^i = t_{\alpha}{}^{\beta}\, {}_{\beta}\lambda^i$$

and we postulate that the invariants of the new ennuple are obtained from those of the old by means of the equations

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* See the author's "Congruences of curves in the geometry of paths," *Rendiconti del circolo Mathematico di Palermo*, vol. 51 (1927), pp. 304-311, for the case of a symmetric linear connection, and Eisenhart's "Non-Riemannian geometry," *American Mathematical Society Colloquium Publications*, vol. 8 (1927), for the general case. We shall use the abbreviation N-R. G. to refer to the latter work.

† *Bulletin of the American Mathematical Society*, vol. 36 (1930), pp. 489-521.

‡ As usual an index repeated, once as a subscript and once as a superscript, is summed from 1 to n.

$$\bar{\gamma}^\beta{}_{\alpha\gamma} = \gamma^b{}_{ac} t_\alpha{}^a t_\gamma{}^c T_b{}^\beta + (\partial t_\alpha{}^a/\partial x^j)\, {}_b\lambda^j\, T_a{}^\beta t_\gamma{}^b. \tag{2.2}$$

Here $T_b{}^\beta$ is the cofactor of $t_\beta{}^b$ in t, the determinant $| t_\beta{}^a |$, divided by t itself, so that

$$T_a{}^\beta t_\beta{}^\gamma = \delta_a{}^\gamma, \qquad T_a{}^\beta t_\gamma{}^a = \delta_{\gamma\beta}, \tag{2.3}$$

where $\delta_\alpha{}^\gamma$ is Kronecker's delta

$$\delta_\alpha{}^\gamma = \begin{cases} 0 & \alpha \neq \gamma \\ 1 & \alpha = \gamma. \end{cases} \tag{2.4}$$

We define the associate covariant vector fields ${}^\alpha\lambda_i$ by

$${}^\alpha\lambda_i\, {}_\beta\lambda^i = \delta_\beta{}^\alpha, \tag{2.5}$$

so that

$${}^\alpha\lambda_i\, {}_\alpha\lambda^j = \delta_j{}^i. \tag{2.6}$$

It is readily verified that the transformation (2.1) induces on ${}^\alpha\lambda_i$ the transformation

$${}^\alpha\bar{\lambda}_i = T_\beta{}^\alpha\, {}^\beta\lambda_i. \tag{2.7}$$

We shall denote by $\partial/\partial s_\alpha$ the operator ${}_\alpha\lambda^i\, \partial/\partial x^i$ so that

$$\partial/\partial s_\alpha = {}_\alpha\lambda^i\, \partial/\partial x^i, \qquad \partial/\partial x^i = {}^\alpha\lambda_i\, \partial/\partial s_\alpha. \tag{2.8}$$

Through differentiation we find that integrability conditions take the form

$$\frac{\partial}{\partial s_\beta}\left(\frac{\partial}{\partial s_\alpha}\right) - \frac{\partial}{\partial s_\alpha}\left(\frac{\partial}{\partial s_\beta}\right) = \left(\frac{\partial\, {}_\alpha\lambda^i}{\partial s_\beta} - \frac{\partial\, {}_\beta\lambda^i}{\partial s_\alpha}\right)\frac{\partial}{\partial x^i}. \tag{2.9}$$

From (2.6) we obtain by differentiation

$$(\partial\, {}_\alpha\lambda^i/\partial s_\beta)\, {}^\alpha\lambda_j + {}_\alpha\lambda^i\, \partial\, {}^\alpha\lambda_j/\partial s_\beta = 0, \tag{2.10}$$

whence

$$\partial\, {}_\alpha\lambda^i/\partial s_\beta = -(\partial\, {}^\gamma\lambda_j/\partial s_\beta)\, {}_\gamma\lambda^i\, {}_\alpha\lambda^j. \tag{2.11}$$

Equations (2.9) may be rewritten by means of (2.11)

$$\frac{\partial}{\partial s_\beta}\left(\frac{\partial}{\partial s_\alpha}\right) - \frac{\partial}{\partial s_\alpha}\left(\frac{\partial}{\partial s_\beta}\right) = \left(\frac{\partial\, {}^\gamma\lambda_i}{\partial s_\alpha}\, {}_\beta\lambda^i - \frac{\partial\, {}^\gamma\lambda_i}{\partial s_b}\, {}_\alpha\lambda^i\right)\frac{\partial}{\partial s_\gamma}. \tag{2.12}$$

It will be useful to insert here also the equations obtained from (2.10) by solving for $\partial\, {}^\alpha\lambda_j/\partial s_\beta$

$$\partial\, {}^\alpha\lambda_j/\partial s_\beta = -(\partial\, {}_\gamma\lambda^i/\partial s_\beta)\, {}^\gamma\lambda_j\, {}^\alpha\lambda_i. \tag{2.13}$$

In a similar manner we obtain from (2.3) that

$$\partial T_{a}{}^{\beta}/\partial s_{\gamma} = -(\partial t_{a}{}^{b}/\partial s_{\gamma})\, T_{a}{}^{a} T_{b}{}^{\beta}. \tag{2.14}$$

Equations (2.2) may be written

$$\bar{\gamma}^{a}{}_{\beta\gamma} = \gamma^{a}{}_{bc} T_{a}{}^{a} t_{\beta}{}^{b} t_{\gamma}{}^{c} + (\partial t_{\beta}{}^{c}/\partial s_{c})\, T_{a}{}^{a} t_{\gamma}{}^{c}. \tag{2.15}$$

If we solve these for the derivatives of $t_{b}{}^{a}$ we obtain

$$\partial t_{b}{}^{a}/\partial s_{c} = \bar{\gamma}^{a}{}_{b\gamma} t_{a}{}^{a} T_{c}{}^{\gamma} - \gamma^{a}{}_{\beta c} t_{b}{}^{\beta}. \tag{2.16}$$

From (2.14) we obtain by means of (2.16) that

$$\partial T_{\beta}{}^{a}/\partial s_{\gamma} = -\gamma^{a}{}_{bc} T_{\beta}{}^{b} T_{\gamma}{}^{c} + \gamma^{a}{}_{\beta\gamma} T_{a}{}^{a}. \tag{2.17}$$

If we denote by $\Gamma^{i}{}_{jk}$ the functions defined by

$$\Gamma^{i}{}_{jk} = \gamma^{\beta}{}_{\alpha\gamma}\, {}_{\beta}\lambda^{i}\, {}^{\alpha}\lambda_{j}\, {}^{\gamma}\lambda_{k} - (\partial\, {}_{\alpha}\lambda^{i}/\partial x^{k})\, {}^{\alpha}\lambda_{j} \tag{2.18}$$

and by $\bar{\Gamma}^{i}{}_{jk}$ the corresponding functions for the ennuple defined by (2.1), we find by virtue of (2.15) together with (2.1), (2.3), and (2.7) that $\bar{\Gamma}^{i}{}_{jk} = \Gamma^{i}{}_{jk}$, so that these are independent of the choice of ennuple; moreover, we obtain from (2.8) that with respect to arbitrary transformations of coördinates

$$x'^{i} = x'^{i}(x^{1}, x^{2}, \cdots, x^{n}) \tag{2.19}$$

the functions $\Gamma^{i}{}_{jk}$ undergo the following transformations

$$\Gamma'^{i}{}_{jk} = \Gamma^{r}{}_{st} \frac{\partial x^{s}}{\partial x'^{j}} \frac{\partial x^{t}}{\partial x'^{k}} \frac{\partial x'^{i}}{\partial x^{r}} + \frac{\partial^{2} x^{r}}{\partial x'^{j} \partial x'^{k}} \frac{\partial x'^{i}}{\partial x^{r}} \tag{2.20}$$

so that they may be taken as the components of a linear connection.*

3. If $A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}$ is a set of functions on which the transformation (2.1) induces the transformation

$$\bar{A}^{a_1 \cdots a_\lambda}_{\beta_1 \cdots \beta_\mu} = A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}\, t^{b_1}_{\beta_1} \cdots t^{b_\mu}_{\beta_\mu}\, T^{a_1}_{a_1} \cdots T^{a_\lambda}_{a_\lambda}. \tag{3.1}$$

We say that the functions $A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}$ are the components of an *intrinsic* tensor of order $\lambda + \mu$, covariant in the subscripts $b_1 \cdots b_\mu$ and contravariant in the superscripts $a_1 \cdots a_\lambda$. A tensor of order zero will, as usual, be said to be an invariant, and one of order one a vector. In particular ${}_{\alpha}\lambda^{i}$ $(\alpha = 1, 2, \cdots n)$ are components of an intrinsic covariant vector, while ${}^{\alpha}\lambda_{i}$ $(\alpha = 1, 2, \cdots n)$

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* Hessenberg in his paper "Vektorielle Begrundung der Differentialgeometrie," *Mathematische Annalen*, vol. 78 (1917-18), pp. 187-217, defined a connection in terms of an ennuple in such a way that $\gamma^{\beta}{}_{\alpha\gamma} = 0$.

are components of an intrinsic contravariant vector. When necessary to emphasize the difference we shall refer to a set of functions which undergo the ordinary tensor laws of transformation with respect to transformations of coördinates as the components of a *scalar* tensor.

If we express the components μ^i $(i = 1, 2, \cdots n)$ of a scalar contravariant vector linearly in terms of the components of a given ennuple

$$\mu^i = A^a {}_a\lambda^i \tag{3.2}$$

the coefficients A^a are scalar invariants and the components of an intrinsic contravariant vector.

Clearly the algebraic results of ordinary tensor analysis may be extended to include the theory of intrinsic tensors.*

Just as an algorithm for obtaining scalar tensors from scalar tensors may be established by differentiation and through the specification of a set of functions (for example, the components of a linear connection) so can we establish a similar algorithm. If we differentiate (3.1) and eliminate the derivatives of the t's by means of (2.16) and (2.17), we find that

$$\bar{A}^{a_1 \cdots a_\lambda}_{\beta_1 \cdots \beta_\mu, \gamma} = A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, c}\, t^{b_1}_{\beta_1} \cdots t^{b_\mu}_{\beta_\mu}\, t_\gamma{}^c T^{a_1}_{\alpha_1} \cdots T^{a_\lambda}_{\alpha_\lambda} \tag{3.3}$$

where

$$A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, c} = \frac{\partial A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}}{\partial s_c} + \sum_{i=1}^{\lambda} A^{a_1 \cdots h \cdots a_\lambda}_{b_1 \cdots b_\mu} \gamma^{a_i}_{hc} - \sum_{i=1}^{\mu} A^{a_1 \cdots a_\lambda}_{b_1 \cdots h \cdots b_\mu} \gamma^{h}_{b_i c} \tag{3.4}$$

and $\bar{A}^{a_1 \cdots a_\lambda}_{\beta_1 \cdots \beta_\mu, c}$ is obtained from $\bar{A}^{a_1 \cdots a_\lambda}_{\beta_1 \cdots \beta_\mu}$ by similar equations in $\bar{s}$ and the $\bar{\gamma}$'s. We shall speak of $A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, c}$ as the *intrinsic* covariant derivative of $A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}$. Again it is clear that intrinsic covariant differentiation of the sum, difference, or product of tensors obeys the same laws as does ordinary differentiation.†

We shall obtain in section 7 the analogues of Ricci's identities for the interchange of the order of differentiation in the case of a second covariant derivative.

We may observe here that if the components of an intrinsic tensor are scalar invariants its intrinsic covariant derivatives are also scalar invariants.

4. If we subject an ennuple ${}_a\lambda^i$ with invariants $\gamma^a{}_{\beta\gamma}$ to a transformation (2.1) we obtain a new ennuple ${}_a\bar{\lambda}^i$ whose invariants $\bar{\gamma}^a{}_{\beta\gamma}$ are given by (2.15). Under a general transformation of coördinates given by (2.19) we obtain

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* Cf. R. G., pp. 1-17.

† Cf. R. G., pp. 26-30.

$$(4.1) \qquad {}_{a}\lambda'^{i} = {}_{a}\bar{\lambda}^{j}\, \partial x'^{i}/\partial x^{j}$$

and

$$(4.2) \qquad \gamma'^{a}{}_{\beta\gamma} = \bar{\gamma}^{a}{}_{\beta\gamma}$$

where the primes denote the analogous functions in the x' coördinate system. If we eliminate the $\bar{\lambda}$'s and $\bar{\gamma}$'s from (4.1) and (4.2) by means of (2.15) the resulting equations are reducible to

$$(4.3) \qquad \partial x'^{i}/\partial s_{\beta} = {}_{a}\lambda'^{i}\, T_{\beta}{}^{a}$$

and

$$(4.4) \qquad \partial t_{\beta}{}^{a}/\partial s_{\gamma} = \gamma'^{a}{}_{\beta c} t_{a}{}^{a} T_{\gamma}{}^{c} - \gamma^{a}{}_{b\gamma} t_{\beta}{}^{b}.$$

If we differentiate (4.3) with respect to s_{γ} eliminate the derivatives of x'^{i}, $t_{\beta}{}^{a}$ and $T_{\beta}{}^{a}$ by means of (2.17), (4.2), (4.3), and (4.4), interchange β and γ, subtract and make use of (2.12) we obtain

$$(4.5) \qquad L'^{a}{}_{\beta\gamma} = L^{a}{}_{bc} T_{a}{}^{a} t_{\beta}{}^{b} t_{\gamma}{}^{c}$$

where

$$(4.6) \qquad L^{a}{}_{bc} = \gamma^{a}{}_{bc} - \gamma^{a}{}_{cb} + {}_{b}\lambda^{i}\, \partial\, {}^{a}\lambda_{i}/\partial s_{c} - {}_{c}\lambda^{i}\, \partial\, {}^{a}\lambda_{i}/\partial s_{b}$$

and $L'^{a}{}_{bc}$ is the same expression in the γ''s, λ''s and s'. For reasons which will appear later we shall call $L^{a}{}_{bc}$ the *torsion invariant* of the space.

Under the transformations of coördinates (2.19), the λ's transform in accordance with (4.1). By differentiation it follows that

$$(4.7) \qquad \frac{\partial\, {}_{a}\lambda'^{i}}{\partial x'^{k}} = \frac{\partial\, {}_{a}\lambda^{j}}{\partial x^{l}} \frac{\partial x'^{i}}{\partial x^{j}} \frac{\partial x^{l}}{\partial x'^{k}} + {}_{a}\lambda^{j} \frac{\partial^{2} x'^{i}}{\partial x^{j} \partial x^{l}} \frac{\partial x^{l}}{\partial x'^{k}}.$$

If we multiply on the left by ${}_{\beta}\lambda'^{k}$ and on the right by its equal ${}_{\beta}\lambda^{j}\, \partial x'^{k}/\partial x^{j}$ and sum on k we obtain

$$(4.8) \qquad \frac{\partial\, {}_{a}\lambda'^{i}}{\partial s'_{\beta}} = \frac{\partial\, {}_{a}\lambda^{j}}{\partial s_{\beta}} \frac{\partial x'^{i}}{\partial x^{j}} + {}_{a}\lambda^{j}\, {}_{\beta}\lambda^{k} \frac{\partial^{2} x'^{i}}{\partial x^{j} \partial x^{k}}.$$

From this it follows that

$${}_{\beta}\lambda^{k}\, \partial\, {}^{a}\lambda_{k}/\partial s_{\gamma} - {}_{\gamma}\lambda^{k}\, \partial\, {}^{c}\lambda_{k}/\partial s_{\beta}$$

is a scalar invariant. Consequently $L^{a}{}_{bc}$ is a scalar invariant and an intrinsic tensor.

Similarly we can differentiate (4.4) and apply the integrability conditions (2.12), after having eliminated the derivatives of the x''s, the t's, and the T's by means of (4.3) and (4.4). We obtain that

$$(4.10) \qquad P'^{a}{}_{\beta\gamma\delta} = P^{a}{}_{bcd} T_{a}{}^{a} t_{\beta}{}^{b} t_{\gamma}{}^{c} t_{\delta}{}^{d}$$

where

$$P^a{}_{bcd} = \partial\gamma^a{}_{bc}/\partial s_d - \partial\gamma^a{}_{bd}/\partial s_c + \gamma^a{}_{\mu d}\gamma^\mu{}_{bc} - \gamma^a{}_{\mu c}\gamma^\mu{}_{bd} + \gamma^a{}_{b\mu}({}_c\lambda^k\,\partial\,{}^\mu\lambda_k/\partial s_d - {}_d\lambda^k\,\partial\,{}^\mu\lambda_k/\partial s_c). \tag{4.11}$$

and $P'^a{}_{\beta\gamma\delta}$ is a similar expression in γ', λ', and s'. As was the case with $L^a{}_{bc}$ the P's are scalar invariants and intrinsic tensors. We shall designate them briefly as the *curvature invariants.*

By intrinsic covariant differentiation of (4. 5) and (4. 6) we obtain the intrinsic tensors of orders four and five respectively, invariants with respect to transformations of coördinates,

$$L'^a{}_{\beta\gamma,\delta} = L^a{}_{bc,d}T_a{}^a t_\beta{}^b t_\gamma{}^c t_\delta{}^d \tag{4.12}$$

and

$$P'^a{}_{\beta\gamma\delta,\epsilon} = P^a{}_{bcd,e}T_a{}^a t_\beta{}^b t_\gamma{}^c t_d{}^d t_\epsilon{}^e \tag{4.13}$$

where

$$L^a{}_{bc,d} = \partial L^a{}_{bc}/\partial s_d + L^\mu{}_{bc}\gamma^a{}_{\mu d} - L^a{}_{\mu c}\gamma^\mu{}_{bd} - L^a{}_{b\mu}\gamma^\mu{}_{cd} \tag{4.14}$$

$$P^a{}_{bcd,e} = \partial P^a{}_{bcd}/\partial s_e + P^\mu{}_{bcd}\gamma^a{}_{\mu e} - P^a{}_{\mu cd}\gamma^\mu{}_{be} - P^a{}_{b\mu d}\gamma^\mu{}_{ce} - P^a{}_{bc\mu}\gamma^\mu{}_{de} \tag{4.15}$$

and $L'^a{}_{bc,d}$ and $P'^a{}_{bcd,e}$ are the same expressions in γ', L', P', and s'.

If we denote the intrinsic covariant derivative of ${}^a\lambda_k$ by ${}^a\lambda_{(k),\beta}$ so that

$${}^a\lambda_{(k),\beta} = \partial\,{}^a\lambda_k/\partial s_\beta + {}^\mu\lambda_k\gamma^a{}_{\mu\beta} \tag{4.16}$$

it follows from (4. 5) that

$$L^a{}_{\beta\gamma} = {}^a\lambda_{(k),\gamma}\,{}_\beta\lambda^k - {}^a\lambda_{(k),\beta}\,{}_\gamma\lambda^k. \tag{4.17}$$

5. In a linearly connected space there does not exist, in general, a coördinate system in which the coefficients of the linear connection are zero at a given point.* Only if the connection is symmetric can such a coördinate system be established. This condition arises from the fact that the law of transformation of the coefficients of the linear connection involves the second derivatives. Since, the law of transformation of the γ's involves only the first derivatives of $t_\beta{}^a$ and since $\frac{\partial}{\partial s_\beta}\,t_\gamma{}^a$ unlike $\frac{\partial}{\partial x^\beta}\,\frac{\partial x'^a}{\partial x^\gamma}$ need not be symmetric in β and γ we expect and we shall proceed to prove that *there exist (infinitely many) ennuples which at a given point have given directions and whose invariants at that point are zero.* For if we impose on the functions $t_a{}^\beta$ the following conditions to be satisfied at a preassigned point P_0: $({}_0x^1 \cdots {}_0x^n)$

$$(t_a{}^\beta)_0 = \delta_a{}^\beta \tag{5.1}$$

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* Cf. N-R. G., p. 53.

$$(\partial t_\alpha{}^\beta/\partial x^i)_0 = -(\gamma^\beta{}_{\alpha\mu}{}^\mu\lambda_i)_0 \tag{5.2}$$

so that

$$t_\beta{}^\alpha = \delta_\beta{}^\alpha - (\gamma^\alpha{}_{\beta\gamma}{}^\gamma\lambda_i)_0\,(x^i - {}_0x^i) + \cdots \tag{5.3}$$

we find that the invariants of the new ennuple are zero at P_0. Because of (5.1) the directions of the two ennuples coincide at P_0. We shall say an ennuple is a *canonical* ennuple at a point if its invariants vanish at that point.

Fermi's extension of local Riemannian coördinates to local coördinates along a curve* is valid here too. For let a curve C be given by $x^2 = x^3 = \cdots = x^n = 0$. We define functions $d_\beta{}^\alpha$ of x^1 by the differential equations

$$(d/dx^1)\,d_\beta{}^\alpha = -(\gamma^\alpha{}_{\mu\nu}{}^\nu\lambda_1)_c\,d_\beta{}^\mu \tag{5.4}$$

in which the γ's and λ's are evaluated along C. We now define t's as functions of the x's by means of a power series in $x^2, x^3, \cdots x^n$ whose coefficients are functions of x^1,

$$t_\beta{}^\alpha = d_\beta{}^\alpha - (\gamma^\alpha{}_{cb}{}^b\lambda_i)_c\,d_\beta{}^\alpha\,x^i + \cdots. \tag{5.5}$$

The index i in (5.5) is summed from 2 to n. If we differentiate (5.5) and apply (5.4) we obtain

$$(\partial t_\beta{}^\alpha/\partial x^1)_c = -(\gamma^\alpha{}_{\alpha b}{}^b\lambda_1)_c\,d_\beta{}^\alpha \tag{5.6}$$

$$(\partial t_\beta{}^\alpha/\partial x^i)_c = -(\gamma^\alpha{}_{\alpha b}{}^b\lambda_i)_c\,d_\beta{}^\alpha \tag{5.7}$$

and by substitution of these values in (2.2) we find that $\gamma'^\alpha{}_{\beta\gamma}$ is zero along C.

Along C, $t_\beta{}^\alpha = d_\beta{}^\alpha$ and the two ennuples are related by the equations

$${}_\alpha\lambda'^i = d_\alpha{}^\beta\,{}_\beta\lambda^i \tag{5.8}$$

so that

$$d\,{}_\alpha\lambda'^i/dx^1 = d_\alpha{}^\beta(d\,{}_\beta\lambda^i/dx^1 - \gamma^\alpha{}_{\beta\gamma}\,{}_\alpha\lambda^i\,{}^\gamma\lambda_1). \tag{5.9}$$

It then follows by virtue of (2.18) that

$$d\,{}_\alpha\lambda'^i/dx^1 + \Gamma^i{}_{jk}\,{}_\alpha\lambda'^j\,dx^k/dx^1 = 0. \tag{5.10}$$

The directions ${}_\alpha\lambda'^i$ accordingly are parallel along C in the space whose coefficients of linear connection are given by (2.18). Conversely starting from (5.10) we can retrace our steps so that *if a canonical ennuple is displaced by infinitesimal parallelism along a curve its invariants are zero along the curve.*

In general it will not be possible to select an ennuple whose invariants are zero at all points of a surface for in that case we must have in place of (5.4) a set of partial differential equations which do not necessarily admit solutions.

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* Cf. N-R. G., pp. 64-67.

6. The functions $\gamma^a{}_{\beta\gamma\delta}$ defined by

$$(6.1) \quad \gamma^a{}_{\beta\gamma\delta} = \partial\gamma^a{}_{\beta\gamma}/\partial s_\delta - \partial\gamma^a{}_{\beta\delta}/\partial s_\gamma - \gamma^a{}_{\beta\mu}(\gamma^\mu{}_{\gamma\delta} - \gamma^\mu{}_{\delta\gamma}) + \gamma^\mu{}_{\beta\gamma}\gamma^a{}_{\mu\delta} - \gamma^\mu{}_{\beta\delta}\gamma^a{}_{\mu\gamma}$$

are in the case of an orthogonal ennuple in a Riemannian space with a positive definite form identical with Ricci's four index symbols.* We take (6.1) as their definition in every case. If we compare (6.1) with (4.6) and (4.11) we find that

$$(6.2) \qquad P^a{}_{\beta\gamma\delta} = \gamma^a{}_{\beta\gamma\delta} + \gamma^a{}_{\beta\mu}L^\mu{}_{\gamma\delta}.$$

Since the P's and L's are intrinsic tensors but $\gamma^a{}_{\beta\gamma}$ is not, it is clear that in general $\gamma^a{}_{\beta\gamma\delta}$ is not an intrinsic tensor.

When the γ's are the invariants of an ennuple in a linearly connected space whose coefficients of connection are given by (2.18) it follows from (4.6) and (2.10) that

$$(6.3) \qquad L^a{}_{\beta\gamma} = (\Gamma^i{}_{jk} - \Gamma^i{}_{kj})\,{}^a\lambda_i\,{}_\beta\lambda^j\,{}_\gamma\lambda^k.$$

If we denote by $B^i{}_{jkl}$ the curvature tensor of the space †

$$(6.4) \qquad B^i{}_{jkl} = \partial\Gamma^i{}_{jl}/\partial x^k - \partial\Gamma^i{}_{jk}/\partial x^l + \Gamma^i{}_{hk}\Gamma^h{}_{jl} - \Gamma^i{}_{hl}\Gamma^h{}_{jk}$$

it follows by substitution in (6.4) of the values of $\Gamma^i{}_{jk}$ and of their derivatives as obtained from (2.18) and simplification by means of (2.5) that

$$(6.5) \qquad P^a{}_{\beta\gamma\delta} = -B^i{}_{jkl}\,{}^a\lambda_i\,{}_\beta\lambda^j\,{}_\gamma\lambda^k\,{}_\delta\lambda^l + \frac{\partial\,{}^a\lambda_i}{\partial s_\delta}\,\frac{\partial\,{}_\beta\lambda^i}{\partial s_\gamma} - \frac{\partial\,{}^a\lambda_i}{\partial s_\gamma}\,\frac{\partial\,{}_\beta\lambda^i}{\partial s_\delta}\,.$$

We see then that in general the invariants determined by the curvature tensor are not intrinsic tensors.

7. From equations (4.6) and (4.11) it follows at once that $L^a{}_{\gamma\delta}$ and $P^a{}_{\beta\gamma\delta}$ are skew symmetric in γ and δ, so that

$$(7.1) \qquad L^a{}_{\gamma\delta} = -L^a{}_{\delta\gamma}, \qquad P^a{}_{\beta\gamma\delta} = -P^a{}_{\beta\gamma\delta}.$$

In terms of a canonical ennuple with origin at a fixed point P_0, $\gamma^a{}_{\beta\gamma}$ will be zero at P_0 and from (4.6) and (4.11) it follows that at P_0

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* Cf. R. G., p. 98.

† Cf. N-R. G., p. 5.

$$L^{\alpha}{}_{\beta\gamma} = {}_{\beta}\lambda^{k}\, \partial\, {}^{\alpha}\lambda_{k}/\partial s_{\gamma} - {}_{\gamma}\lambda^{k}\, \partial\, {}^{\alpha}\lambda_{k}/\partial s_{\beta} \tag{7.2}$$

and

$$P^{\alpha}{}_{\beta\gamma\delta} = (\partial/\partial s_{\delta})\gamma^{\alpha}{}_{\beta\gamma} - (\partial/\partial s_{\gamma})\gamma^{\alpha}{}_{\beta\delta}. \tag{7.3}$$

From (4.14) and (7.1) we obtain

$$L^{\alpha}{}_{\beta\gamma,\delta} = \frac{\partial}{\partial s_{\delta}}(\gamma^{\alpha}{}_{\beta\gamma} - \gamma^{\alpha}{}_{\gamma\beta}) + \frac{\partial\, {}_{\beta}\lambda^{i}}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}} - \frac{\partial\, {}_{\gamma}\lambda^{i}}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\beta}} + {}_{\beta}\lambda^{i}\frac{\partial}{\partial s_{\delta}}\left(\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}}\right) - {}_{\gamma}\lambda^{i}\frac{\partial}{\partial s_{\delta}}\left(\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\beta}}\right). \tag{7.4}$$

Let us denote by C the sum of the terms obtained by cyclic permutation of the subscripts β, γ, δ so that, for example,

$$CL^{\alpha}{}_{\beta\gamma,\delta} \equiv L^{\alpha}{}_{\beta\gamma,\delta} + L^{\alpha}{}_{\gamma\delta,\beta} + L^{\alpha}{}_{\delta\beta,\gamma}. \tag{7.5}$$

We observe moreover that the relations

$$C\, A_{\beta\gamma\delta} \equiv C\, A_{\gamma\delta\beta} \equiv C\, A_{\delta\beta\gamma}$$

hold for any set of functions $A_{\beta\gamma\delta}$. By virtue of (2.11) and this cyclic property we find that

$$C\left(\frac{\partial\, {}_{\beta}\lambda^{i}}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}} - \frac{\partial\, {}_{\gamma}\lambda^{i}}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\beta}}\right) = C\,\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}}\left(\frac{\partial\, {}^{\alpha}\lambda_{j}}{\partial s_{\delta}}\, {}_{\alpha}\lambda^{i}\, {}_{\delta}\lambda^{j} - \frac{\partial\, {}^{\alpha}\lambda_{j}}{\partial s_{\delta}}\, {}_{\alpha}\lambda^{i}\, {}_{\beta}\lambda^{j}\right)$$

Similarly by means of (2.12)

$$C\left({}_{\beta}\lambda^{i}\frac{\partial}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}} - {}_{\gamma}\lambda^{i}\frac{\partial}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\beta}}\right) = C\, {}_{\beta}\lambda^{i}\left(\frac{\partial}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\gamma}} - \frac{\partial}{\partial s_{\gamma}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\delta}}\right) = C\left({}_{\beta}\lambda^{i}\, {}_{\delta}\lambda^{j}\frac{\partial\, {}^{\alpha}\lambda_{j}}{\partial s_{\gamma}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\alpha}} - {}_{\beta}\lambda^{i}\, {}_{\gamma}\lambda^{j}\frac{\partial\, {}^{\alpha}\lambda_{j}}{\partial s_{\delta}}\frac{\partial\, {}^{\alpha}\lambda_{i}}{\partial s_{\alpha}}\right)$$

Making use of these values in (7.4) we find by virtue of (7.1) and (7.2) that

$$CL^{\alpha}{}_{\beta\gamma,\delta} = CP^{\alpha}{}_{\beta\gamma\delta} - CL^{\alpha}{}_{\beta\alpha}L^{\alpha}{}_{\gamma\delta}.$$

Since these are tensor equations they hold for every ennuple and at every point, so that

$$P^{\alpha}{}_{\beta\gamma\delta} + P^{\alpha}{}_{\gamma\delta\beta} + P^{\alpha}{}_{\delta\beta\gamma} \equiv L^{\alpha}{}_{\beta\gamma,\delta} + L^{\alpha}{}_{\gamma\delta,\beta} + L^{\alpha}{}_{\delta\beta,\gamma} + L^{\alpha}{}_{\beta\mu}L^{\mu}{}_{\gamma\delta} + L^{\alpha}{}_{\gamma\mu}L^{\mu}{}_{\delta\beta} + L^{\alpha}{}_{\delta\mu}L^{\mu}{}_{\beta\gamma}. \tag{7.6}$$

We shall see in the next section that the vanishing of $L^{\alpha}{}_{\beta\gamma}$ is a necessary and sufficient condition that the linear connection be symmetric. In that case $L^{\alpha}{}_{\beta\gamma,\delta}$ is also zero, and (7.6) reduce to the well known relations

$$\gamma^{\alpha}{}_{\beta\gamma\delta} + \gamma^{\alpha}{}_{\gamma\delta\beta} + \gamma^{\alpha}{}_{\delta\beta\gamma} = 0. \tag{7.7}$$

In a Riemannian space these equations are equivalent to the well known relations

$$B^{i}{}_{jkl} + B^{i}{}_{klj} + B^{i}{}_{ljk} = 0.$$

By means of a canonical ennuple we can obtain another identity in the P's and L's, one which may be regarded as the generalization to an assymetric connection of the identity of Bianchi. In fact in terms of a canonical ennuple

$$\begin{aligned} P^{\alpha}{}_{\beta\gamma\delta,\epsilon} = \frac{\partial}{\partial s_\epsilon} P^{\alpha}{}_{\beta\gamma\delta} &= \frac{\partial}{\partial s_\epsilon}\left(\frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\delta} - \frac{\partial \gamma^{\alpha}{}_{\beta\delta}}{\partial s_\gamma}\right) \\ &+ \frac{\partial \gamma^{\alpha}{}_{\beta\mu}}{\partial s_\epsilon}\left(\frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\delta}\, {}_{\gamma}\lambda^{j} - \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\gamma}\, {}_{\delta}\lambda^{j}\right). \end{aligned} \tag{7.8}$$

If we advance γ, δ, ϵ cyclically and if we here denote by C the resulting sum, we obtain

$$\begin{aligned} CP^{\alpha}{}_{\beta\gamma\delta,\epsilon} &= C\left(\frac{\partial^2 \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\epsilon\, \partial s_\delta} - \frac{\partial^2 \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\delta\, \partial s_\epsilon}\right) \\ &+ C\, \frac{\partial \gamma^{\alpha}{}_{\beta\mu}}{\partial s_\gamma}\left(\frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\epsilon}\, {}_{\delta}\lambda^{j} - \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\delta}\, {}_{\epsilon}\lambda^{j}\right). \end{aligned} \tag{7.9}$$

By means of (2.9) the first parenthesis on the right becomes

$${}^{\mu}\lambda_j \left\{ \frac{\partial\, {}_{\delta}\lambda^{j}}{\partial s_\epsilon}\, \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu} - \frac{\partial\, {}_{\epsilon}\lambda^{j}}{\partial s_\delta}\, \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu} \right\}.$$

Because of (2.6) this reduces to

$${}_{\epsilon}\lambda^{j}\, \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\delta}\, \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu} - {}_{\delta}\lambda^{j}\, \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\epsilon}\, \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu}.$$

If we substitute in (7.9) and make a slight modification in the dummy indices of the second parenthesis we obtain

$$CP^{\alpha}{}_{\beta\gamma\delta,\epsilon} = C\, \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\epsilon}\, {}_{\delta}\lambda^{j} \left(\frac{\partial \gamma^{\alpha}{}_{\beta\mu}}{\partial s_\gamma} - \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu}\right) + C\, {}_{\epsilon}\lambda^{j}\, \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\delta} \left(\frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu} - \frac{\partial \gamma^{\alpha}{}_{\beta\mu}}{\partial s_\gamma}\right)$$

or

$$CP^{\alpha}{}_{\beta\gamma\delta,\epsilon} = C\left(\frac{\partial \gamma^{\alpha}{}_{\beta\mu}}{\partial s_\gamma} - \frac{\partial \gamma^{\alpha}{}_{\beta\gamma}}{\partial s_\mu}\right)\left(\frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\epsilon}\, {}_{\delta}\lambda^{j} - \frac{\partial\, {}^{\mu}\lambda_j}{\partial s_\delta}\, {}_{\epsilon}\lambda^{j}\right)$$

consequently the relations

$$P^{\alpha}{}_{\beta\gamma\delta,\epsilon} + P^{\alpha}{}_{\beta\delta\epsilon,\gamma} + P^{\alpha}{}_{\beta\epsilon\gamma,\delta} = P^{\alpha}{}_{\beta\mu\gamma} L^{\mu}{}_{\delta\epsilon} + P^{\alpha}{}_{\beta\mu\delta} L^{\mu}{}_{\epsilon\gamma} + P^{\alpha}{}_{\beta\mu\epsilon} L^{\mu}{}_{\gamma\delta} \tag{7.10}$$

are valid at any point in terms of a canonical ennuple and since they are intrinsic tensor equations they must hold throughout space for every ennuple. In the case of a symmetric connection they reduce to the well known identity of Bianchi and Veblen,*

$$B^h{}_{ijk,l} + B^h{}_{ikl,j} + B^h{}_{ilj,k} = 0. \tag{7.11}$$

In a similar manner we can obtain the generalization of Ricci's identity for covariant differentiation. For in terms of a canonical ennuple the second covariant derivatives of an intrinsic tensor $A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu}$ are given by (cf. § 3)

$$\begin{aligned} A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, \alpha\beta} = \frac{\partial}{\partial s_\beta} \frac{\partial}{\partial s_\alpha} A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu} + \sum_{i=1}^{\lambda} A^{a_1 \cdots h \cdots a_\lambda}_{b_1 \cdots b_\mu} \frac{\partial}{\partial s_\beta} \gamma^{a_\iota}_{h\alpha} \\ - \sum_{i=1}^{\mu} A^{a_1 \cdots a_\lambda}_{b_1 \cdots h \cdots b_\mu} \frac{\partial}{\partial s_\beta} \gamma^{h}_{b_\iota \alpha} . \end{aligned} \tag{7.12}$$

Consequently

$$\begin{aligned} & A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, \alpha\beta} - A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, \beta\alpha} \\ & = - L^h{}_{\alpha\beta} A^{a_1 \cdots a_\lambda}_{b_1 \cdots b_\mu, h} + \sum_{i=1}^{\lambda} A^{a_1 \cdots h \cdots a_\lambda}_{b_1 \cdots b_\mu} P^{a_\iota}_{h\alpha\beta} - \sum_{i=1}^{\mu} A^{a_1 \cdots a_\lambda}_{b_1 \cdots h \cdots b_\mu} P^{h}_{b_\iota \alpha\beta} . \end{aligned} \tag{7.13}$$

In particular, if A is an intrinsic invariant

$$A_{,\alpha\beta} - A_{,\beta\alpha} = - A_{,h} L^h{}_{\alpha\beta} \tag{7.14}$$

and if A_α is an intrinsic covariant vector

$$A_{\alpha,\beta\gamma} - A_{\alpha,\gamma\beta} = - A_{\alpha,h} L^h{}_{\beta\gamma} - A_h P^h{}_{\alpha\beta\gamma}. \tag{7.15}$$

8. In section 4 we found that under transformations of coördinates and of ennuples the functions $L^\alpha{}_{\beta\gamma}$ and $P^\alpha{}_{\beta\gamma\delta}$ were transformed in accordance with (4.5) and (4.10). Conversely if we are given two spaces each defined by n^2 functions ${}_\alpha\lambda^i$ and n^3 functions $\gamma^\alpha{}_{\beta\gamma}$ of the corresponding coördinates they determine the same space provided that equations (4.3) and (4.4) admit a solution. Consequently if (4.5) and (4.10) are identically satisfied equations (4.3) and (4.4) are completely integrable and the solution will contain $n^2 + n$ arbitrary constants. In particular for a Euclidean space in terms of n mutually orthogonal normal congruences of straight lines the γ's of Ricci are zero and $L^\alpha{}_{\beta\gamma}$ and $P^\alpha{}_{\beta\gamma\delta}$ are zero. Accordingly *a necessary and sufficient condition that a space defined by an ennuple* ${}_\alpha\lambda^i$ *and invariants* $\gamma^\beta{}_{\alpha\gamma}$ *be Euclidean is that* $L^\alpha{}_{\beta\gamma} = 0$ *and* $P^\alpha{}_{\beta\gamma\delta} = 0$.

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* Veblen, "Normal coördinates for the geometry of paths," *Proceedings of the National Academy of Sciences*, vol. 8 (1922), pp. 192-197.

Clearly the n^2+n constants of integration correspond to the fact that the coördinate system and the ennuple may be subjected to a rigid motion, and that independently of each other. Hence it follows that *if two ennuples in Euclidean space have equal invariants, they are congruent.*

From (6.3) it follows that $L^\alpha{}_{\beta\gamma}$ are zero if and only if $\Gamma^i{}_{jk}$ are symmetric in j and k. Hence it follows that *a necessary and sufficient condition that a space defined by an ennuple* ${}_\alpha\lambda^i$ *and invariants* $\gamma^\alpha{}_{\beta\gamma}$ *be a space with a symmetric linear connection is that the torsion invariants vanish.*

We shall say that a space is *pseudo-euclidean* if $P^\alpha{}_{\beta\gamma\delta}=0$. An example of such a space is one in which the coefficients of the linear connection are defined in terms of an ennuple by *

$$(8.1) \qquad \Gamma^i{}_{jk}={}_\alpha\lambda^i\,\partial\,{}^\alpha\lambda_j/\partial x^k$$

for it then follows from (2.18) that $\gamma^\beta{}_{\alpha\gamma}$ are zero, and consequently that $P^\alpha{}_{\beta\gamma\delta}$ are zero for all ennuples. We proceed to show that every space for which $P^\alpha{}_{\beta\gamma\delta}=0$ admits ∞^{n^2} ennuples whose invariants $\gamma^\alpha{}_{\beta\gamma}$ are zero throughout the space, and whose coefficients of connection consequently satisfy (8.1). Since the coefficients of connection are invariant under transformations of the ennuple we must show that the equations

$$(8.2) \qquad {}_\alpha\lambda'^i\,\partial\,{}^\alpha\lambda'_j/\partial x^k=\gamma^\alpha{}_{\beta\gamma}\,{}_\alpha\lambda^i\,{}^\beta\lambda_j\,{}^\gamma\lambda_k-(\partial\,{}_\alpha\lambda^i/\partial x^k)\,{}^\alpha\lambda_j$$

admit as solutions n^2 functions ${}_\alpha\lambda'^i$. These equations may be solved for $\partial\,{}^\alpha\lambda'_j/\partial x^k$ and it is readily found that the integrability conditions are identically satisfied by virtue of the hypothesis that $P^\alpha{}_{\beta\gamma\delta}=0$. Clearly if ${}^\alpha\lambda'_i$ are solutions, $A_\alpha\,{}^\alpha\lambda'_i$ where the A's are constants, is the general solution. Since the invariants of the ennuple ${}_\alpha\lambda'^i$ are zero, such an ennuple is analogous to the ennuple determined by a cartesian coördinate system in euclidean space. We shall call such an ennuple a *fundamental* ennuple so that *in a pseudo-euclidean space there exist* ∞^{n^2} *fundamental ennuples, whose invariants are zero throughout the space, and the components of one fundamental ennuple are linearly dependent on the components of any other with coefficients of dependence which are constants.*

Since, under a transformation of coördinates, the components of an ennuple are transformed in accordance with

$$(8.3) \qquad {}_\alpha\lambda'^i={}_\alpha\lambda^j\,\partial x'^i/\partial x^j$$

there exists a coördinate system x' in which the λ''s are constants if and only if the equations obtained from (8.3) by differentiation, namely

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* Cf. Hessenberg, *loc. cit.*, p. 190, or N-R. G., p. 48.

(8.4) $$\frac{\partial^2 x'^i}{\partial x^j\,\partial x^k} = -{}^a\lambda_j \frac{\partial\,{}_a\lambda^h}{\partial x^k}\,\frac{\partial x'^i}{\partial x^h}$$

admit a solution. One can verify easily that these equations admit a solution only if the coefficients of the connection are symmetric and then they are completely integrable. Accordingly *a pseudo-euclidean space admits a coördinate system in which the components of a fundamental ennuple are constants if and only if the space is euclidean.*

From another point of view the connection (8.1) is symmetric if and only if

(8.5) $$(\partial\,{}^a\lambda_j/\partial x^k)\,{}_a\lambda^i = (\partial\,{}^a\lambda_k/\partial x^j)\,{}_a\lambda^i.$$

These equations are equivalent to

(8.6) $$\partial\,{}^a\lambda_j/\partial x^k = \partial\,{}^a\lambda_k/\partial x^j$$

and hence there exist n functions, f^a, such that

(8.7) $${}^a\lambda_j = \partial f^a/\partial x^j.$$

If we put

(8.8) $$y^a = f^a(x^1, x^2, \cdots x^n)$$

and regard these equations as defining the x's as functions of the y's it follows that

(8.9) $${}_a\lambda^i = \partial x^i/\partial y^a$$

so that the curves of the congruences can be taken to be coördinate curves. Accordingly with any n congruences of curves we can associate by means of the linear connection (8.1) a pseudo-euclidean geometry which is euclidean if and only if every set of congruences generate a family of hypersurfaces. This point of view may be of value in the study of Riemannian manifolds which admit n-tuply orthogonal systems of hypersurfaces.

9. We recall that if the coefficients of two linear connections are related by

(9.1) $$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta_j^i\psi_k + \delta_k^i\psi_j + N^i_{jk}$$

where ψ_j is an arbitrary covariant vector and N^i_{jk} an arbitrary tensor, skew symmetric in j and k,

(9.2) $$N^i_{jk} = -N^i_{kj}$$

contravariant in the index i and covariant in the indices j and k the two spaces have the same paths. Conversely if two linearly connected spaces have the same paths there exist a vector ψ_j and a tensor $N^i{}_{jk}$ such that (9.1) hold.* If we define invariants $^*\gamma^\alpha{}_{\beta\gamma}$ of an ennuple $_\alpha\lambda^i$ by

$$^*\gamma^\alpha{}_{\beta\gamma} = {}_\beta\lambda^i{}_{.j}\, {}^\alpha\lambda_i\, {}_\gamma\lambda^j \tag{9.3}$$

where

$$_\alpha\lambda^i{}_{.j} = {}_\alpha\lambda^i{}_{,j} - \frac{1}{n+1}\left({}_\alpha\lambda^i\, {}_\beta\lambda^h{}_{,j}\, {}^\beta\lambda_h + \delta_j{}^i\, {}_\alpha\lambda^h\, {}_\beta\lambda^k{}_{,h}\, {}^\beta\lambda_k\right) \tag{9.4}$$

and $_\alpha\lambda^i{}_{,j}$ is the covariant derivative with respect to the Γ's, and if we denote by $^*\bar{\gamma}^\alpha{}_{\beta\gamma}$ the corresponding functions determined by the $\bar{\Gamma}$'s it follows that †

$$^*\bar{\gamma}^\alpha{}_{\beta\gamma} = {}^*\gamma^\alpha{}_{\beta\gamma} + N^i{}_{jk}\, {}^\alpha\lambda_i\, {}_\beta\lambda^j\, {}_\gamma\lambda^k. \tag{9.5}$$

Since $^*\gamma^\alpha{}_{\beta\gamma}$ can be regarded as the invariants of an ennuple in a space whose coefficients of linear connection are

$$\Gamma^i{}_{jk} + \delta_j{}^i\psi_k + \delta_k{}^i\psi_j \quad \text{where} \quad \psi_k = \frac{-1}{n+1}\, {}_\alpha\lambda^i{}_{,k}\, {}^\alpha\lambda_i$$

it follows that with respect to transformations (2.1) $^*\gamma^\alpha{}_{\beta\gamma}$ transforms in accordance with (2.2). Moreover since $N^i{}_{jk}$ is an intrinsic invariant $^*\bar{\gamma}^\alpha{}_{\beta\gamma}$ does likewise. We shall normalize $^*\gamma^\alpha{}_{\beta\gamma}$ by so selecting $N^i{}_{jk}$ that $^*\gamma^\alpha{}_{\beta\gamma}$ is symmetric in β and γ for a particular ennuple. Although in the main the results of the preceding sections are valid here, one important exception occurs in § 8, for in the consideration of the equivalence problem one must here take into consideration a change in the value of the invariants due to the arbitrariness of $N^i{}_{jk}$.

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* Cf. N-R. G., p. 31.

† Cf. Levy, *loc. cit.*, p. 310

HEAT CONDUCTION IN A SEMI-INFINITE RADIOACTIVE SOLID.

By Arnold N. Lowan.

The boundary $x = 0$ of a semi-infinite radioactive solid is assumed to be kept at, or to radiate into, a medium at variable temperature $\phi(t)$. If the temperature at time $t = 0$ is $f(x)$ what is the subsequent temperature distribution in the solid?

The above problem is the generalization of a problem treated in an earlier paper.* To solve it we shall first derive the solution for the case of a finite slab extending from $x = 0$ to $x = a$, and shall obtain the limit of the solution $T(x, t)$ for $a \rightarrow \infty$. For the finite slab, the temperature $T(x, t)$ must satisfy the following differential equation initial and boundary conditions:

$$(\partial/\partial t)T(x, t) - k(\partial^2/\partial x^2)T(x, t) = \phi(x, t) \tag{1}$$

$$\lim_{t \to 0} T(x, t) = f(x) \tag{2}$$

$$T(0, t) = \phi(t) \tag{3}$$

$$T(a, t) = 0 \tag{4}$$

where $k =$ Ratio between the thermal conductivity K and the product of the density d and the specific heat c; $\phi(x, t) = (1/cd) \times$ heat generated per unit time per unit volume, and we have assumed that the boundary $x = 0$ is kept at temperature $\phi(t)$. The function $\phi(x, t)$ will be referred to as the "radioactivity function." It is clear from physical considerations that $f(x)$ may be assumed to be bounded, piecewise continuous and twice differentiable. It is also clear that $\phi(x, t)$ and $\phi(t)$ are bounded, and that $\phi(t)$ is differentiable. To solve the system (1) to (5) we make the substitution

$$T(x, t) = [(a - x)/a]\, \phi(t) + u(x, t). \tag{5}$$

The function $u(x, t)$ must then satisfy the system

$$\frac{\partial}{\partial t} u(x, t) - k \frac{\partial^2}{\partial x^2} u(x, t) = \phi(x, t) + \frac{x - a}{a} \phi'(t) = \psi(x, t) \quad \text{(say)} \tag{6}$$

$$\lim_{t \to 0} u(x, t) = f(x) + \frac{x - a}{a} \phi(0) = F(x) \quad \text{(say)} \tag{7}$$

$$u(0, t) = 0 \tag{8}$$

$$u(a, t) = 0. \tag{9}$$

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* Arnold N. Lowan, "On the cooling of a radioactive sphere," *Physical Review*, November 1, 1933.

The system (6) to (9) is formally similar to the system (1) to (4), and may be said to characterize the thermal history of a finite slab initially at temperature $F(x)$, the boundaries of which are permanently at 0°, and for which the radioactivity function is $\psi(x,t)$.

The solution of the latter system may be obtained by the method presented in the paper above mentioned, and may be written down at once in the form

$$(10)\quad u(x,t)=\int_0^a F(\xi)\Gamma(x,\xi,t)\,d\xi+\int_0^a d\xi\int_0^t \psi(\xi,\eta)\Gamma(x,\xi,t-\eta)\,d\eta$$

where

$$(11)\qquad \Gamma(x,\xi,t)=\sum_{n=0}^{\infty} e^{-k\lambda_n^2 t}\,y_n(x)\,y_n(\xi)$$

the summation extending over the characteristic values, and the corresponding normalized characteristic functions of the system

$$(12)\qquad \begin{aligned} y''(x)+\lambda^2 y(x)&=0\\ y(0)=y(a)&=0. \end{aligned}$$

Clearly

$$(13)\qquad \lambda_n=n\pi/a;\qquad y_n(x)=(2/a)^{1/2}\sin\,(n\pi/a)x.$$

In view of the significance of the functions $F(x)$ and $\psi(x,t)$, our solution (5) becomes

$$\begin{aligned}(14)\quad T(x,t)=&\frac{a-x}{a}\phi(t)\\ &+\int_0^a f(\xi)\Gamma(x,\xi,t)\,d\xi-\phi(0)\int_0^a\frac{a-\xi}{a}\Gamma(x,\xi,t)\,d\xi\\ &+\int_0^a\frac{\xi-a}{a}\left\{\int_0^t\frac{d}{d\eta}\phi(\eta)\Gamma(x,\xi,t-\eta)\,d\eta\right\}d\xi\\ &+\int_0^a d\xi\int_0^t\phi(\xi,\eta)\Gamma(x,\xi,t-\eta)\,d\eta.\end{aligned}$$

If the second integral in the third term is integrated by parts, and if we make use of the identity

$$(15)\qquad \frac{a-x}{a}=\sum_{n=0}^{\infty}y_n(x)\int_0^a\frac{a-\xi}{a}y_n(\xi)\,d\xi$$

our solution (14) becomes

$$\begin{aligned}(16)\quad T(x,t)=&\int_0^a f(\xi)\Gamma_a(x,\xi,t)\,dt+\int_0^a d\xi\int_0^t\phi(\xi,\eta)\Gamma_a(x,\xi,t-\eta)\,d\eta\\ &+\int_0^t\phi(\eta)G_a(x,t-\eta)\,d\eta\end{aligned}$$

where we have put

$$G_a(x, t) = \sum_{n=0}^{\infty} \frac{2k}{a} \lambda_n \sin \lambda_n x \cdot \exp(-k\lambda_n{}^2 t)$$

$$= -\frac{k}{a} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \exp [(n\pi x/a)i - k(n^2\pi^2/a^2)t] = -\frac{k}{a} \frac{\partial}{\partial x} \vartheta(x, t)$$

the subscript a being a reminder that the functions $G_a(x, t)$ and $\Gamma_a(x, \xi, t)$ depend on a.

The problem has now reduced to finding the limits of the functions $G_a(x, t)$ and $\Gamma_a(x, \xi, t)$, as $a \rightarrow \infty$.

Consider the identity *

$$(18) \qquad \sum_{n=-\infty}^{\infty} \exp(2n\pi vi - n^2\pi^2\tau) \equiv \sum_{n=-\infty}^{\infty} \frac{\exp\{-[(v+n)^2/\tau]\}}{(\pi\tau)^{1/2}}.$$

Replacing v by $x/2a$ and τ by kt/a^2 we get

$$(19) \qquad \vartheta_a(x, t) = a \sum_{n=-\infty}^{\infty} \frac{\exp\{-[(x+2an)^2/4kt]\}}{(\pi kt)^{1/2}}$$

whence evidently

$$(20) \qquad \lim_{a\rightarrow\infty} \left\{ \frac{1}{a} \vartheta_a(x, t) \right\} = \frac{1}{(\pi kt)^{1/2}} \exp[-(x^2/4kt)]$$

and

$$(21) \qquad \lim_{a\rightarrow\infty} \left\{ \frac{1}{a} \frac{\partial}{\partial x} \vartheta_a(x, t) \right\} = -\frac{1}{2kt} \cdot \frac{1}{(\pi kt)^{1/2}} \exp[-(x^2/4kt)].$$

In view of (21) it is clear that

$$(22) \qquad G(x, t) = \lim_{a\rightarrow\infty} G_a(x, t) = [x/2t(\pi kt)^{1/2}] \exp[-(x^2/4kt)].$$

For the function $\Gamma_a(x, \xi, t)$ we may write successively

$$(23) \qquad \Gamma_a(x, \xi, t) = (2/a) \sum \sin(n\pi x/a) \sin(n\pi\xi/a) \exp[-k(n^2\pi^2/a^2)t]$$

$$= (1/a) \sum_{n=0}^{\infty} \cos(2n\pi/a)(x-\xi) \cdot \exp[-k(n^2\pi^2/a^2)t]$$

$$- (1/a) \sum_{n=0}^{\infty} \cos(2n\pi/a)(x+\xi) \exp[-k(n^2\pi^2/a^2)t]$$

$$= (1/2a) \sum_{n=-\infty}^{\infty} \exp[(2n\pi/a)(x-\xi)i - k(n^2\pi^2/a^2)t]$$

$$- (1/2a) \sum_{n=-\infty}^{\infty} \exp[2n\pi/a)(x+\xi)i - k(n^2\pi^2/a^2)t].$$

In view of (20), the latter equation yields

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* G. Doetsch, *Mathematische Zeitschrift*, vol. 22 (1925), p. 290.

$$(24)\quad \Gamma(x,\xi,t) = \operatorname{Lim} \Gamma_a(x,\xi,t)$$
$$= [1/2(\pi k t)^{1/2}]\,[\exp\{-[(x-\xi)^2/4kt]\} - \exp\{-[(x+\xi)^2/4kt]\}].$$

In view of (22) and (24) our solution (16) finally becomes

$$(25)\quad T(x,t) = [1/2(\pi k t)^{1/2}] \int_0^\infty f(\xi)$$
$$\times [\exp\{-[(x-\xi)^2/4kt]\} - \exp\{-[(x+\xi)^2/4kt]\}]\,d\xi$$
$$+ [x/2(\pi k)^{1/2}] \int_0^t \phi(\eta)\,(t-\eta)^{-3/2} \exp\{-[x^2/4k(t-\eta)]\}\,d\eta$$
$$+ [1/2(\pi k)^{1/2}] \int_0^\infty d\xi \int_0^t \phi(\xi,\eta)$$
$$\times [\exp\{-[(x-\xi)^2/4k(t-\eta)]\} - \exp\{-[(x+\xi)^2/4k(t-\eta)]\}]$$
$$\times (t-\eta)^{-1/2}\,d\eta.$$

For $\phi(x,t) = 0$ (non-radioactive solid), (25) yields a well-known result.*

It is evident that the third term of (25), say $T_3(x,t)$ satisfies the condition

$$(26)\qquad T_3(0,t) = 0.$$

In view of the boundedness of $\phi(x,t)$, it may be shown, without difficulty, that $T(x,t)$ may be differentiated termwise, once with respect to t and twice with respect to x, that it satisfies the differential equation (1) and the initial condition

$$(27)\qquad \operatorname*{Lim}_{t\to 0} T_3(x,t) = 0.$$

Thus (25) satisfies all the conditions (1) to (4), and, therefore, represents the complete solution of our problem for the case where the boundary $x = 0$ is kept at temperature $\phi(t)$.

Case where radiation takes place at the boundary $x = 0$ into a medium at temperature $\phi(t)$.

The temperature $T(x,t)$ must then satisfy the system:

$$(28)\qquad \frac{\partial}{\partial x} T(x,t) - k(\partial^2 T/\partial x^2)(x,t) = \phi(x,t) \qquad 0 < x < \infty,\ t > 0$$

$$(29)\qquad \operatorname*{Lim}_{t\to 0} T(x,t) = f(x)$$

$$(30)\qquad -\frac{\partial}{\partial t} T(x,t) + hT(x,t) = h\phi(t) \qquad x = 0.$$

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* See Carslaw, Art. 81.

Let us make the substitution

$$v(x,t)=T(x,t)-\frac{1}{h}\frac{\partial}{\partial x}T(x,t). \tag{31}$$

The function $v(x,t)$ must then satisfy the system

$$\frac{\partial}{\partial t}v(x,t)-k\frac{\partial^2}{\partial x^2}v(x,t)=\phi(x,t)-\frac{1}{h}\frac{\partial}{\partial x}\phi(x,t) \tag{32}$$

$$\lim_{t\to 0}v(x,t)=f(x)-\frac{1}{h}\frac{d}{dx}f(x) \tag{33}$$

$$v(0,t)=\phi(t). \tag{34}$$

It is clear that the solution of (32) to (34) may be obtained at once from (25) by replacing $\phi(x,t)$ and $f(x)$ by $\phi(x,t)-\frac{1}{h}\frac{d}{dx}\phi(x,t)$ and $f(x)-\frac{1}{h}\frac{d}{dx}f(x)$, respectively. Furthermore (31) yields

$$T(x,t)=h\int_0^\infty v(x+\rho,t)\exp(-h\rho)\,d\rho. \tag{35}$$

The complete solution of the system (28) to (30) is, therefore

$$\begin{aligned}T(x,t)&=\frac{h}{2(\pi kt)^{1/2}}\int_0^\infty\{f(\xi)-\frac{1}{h}\frac{d}{d\xi}f(\xi)\}d\xi\\&\times\int_0^\infty e^{-h\rho}[\exp\{-[(x+\rho-\xi)^2/4kt]\}-\exp\{-[(x+\rho+\xi)^2/4kt]\}]d\rho\\&+\frac{h}{2(\pi k)^{1/2}}\int_0^t\phi(\eta)(t-\eta)^{-3/2}\,d\eta\\&\times\int_0^\infty(x+\rho)\exp\{-[(x+\rho)^2/4k(t-\eta)]\}\,e^{-h\rho}\,d\rho\\&+\frac{h}{2(\pi k)^{1/2}}\int_0^\infty d\xi\int_0^t(t-\eta)^{-1/2}\{\phi(\xi,\eta)-\frac{1}{h}\frac{\partial}{\partial\xi}\phi(\xi,\eta)\}d\eta\\&\times\int_0^\infty e^{-h\rho}[\exp\{-[(x+\rho-\xi)^2/4k(t-\eta)]\}\\&-\exp\{-[(x+\rho+\xi)^2/4k(t-\eta)]\}]d\rho.\end{aligned} \tag{36}$$

For $\phi(x,t)=0$, (36) yields, after some simple transformations, the result obtained by Carslaw in Article 83.

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ON THE ASYMPTOTIC DIFFERENTIAL DISTRIBUTION OF ALMOST-PERIODIC AND RELATED FUNCTIONS.

By Aurel Wintner.

Let H denote the class of those real-valued continuous bounded functions $x(t)$, $-\infty < t < +\infty$, for which the time-averages

$$(1) \qquad \mathfrak{M}(x^n) = \lim_{T=+\infty} \int_{-T}^{T} [x(t)]^n \, dt/2T \qquad (n = 0, 1, 2, \cdots)$$

exist. The most important particular cases of these H-functions are the almost-periodic functions of Bohr on the one hand and the functions representable as Fourier-Stieltjes transforms of continuous functions of bounded variation on the other hand. For a given H-function $x(t)$, let $[T;\xi]$ denote the set of those values t in the interval $-T \leqq t \leqq T$ for which $x(t) < \xi$ where ξ is a real number so that

$$\sigma_T(\xi) = \operatorname{meas}[T;\xi] : \operatorname{meas}[T;+\infty] \qquad (\operatorname{meas}[T;+\infty] = 2T)$$

is for every T a monotone function of ξ, $-\infty < \xi < +\infty$, and represents the probability of the inequality $x(t) < \xi$ when t is restricted to the range $-T \leqq t \leqq T$. It has been proven † that there exists a monotone function $\sigma(\xi)$, $-\infty < \xi < +\infty$, such that $\sigma_T(\xi) \to \sigma(\xi)$, $T \to +\infty$, at every continuity point ξ of $\sigma(\xi)$, and that this σ is a solution of the momentum problem

$$(2) \qquad \mu_n(\sigma) = \mathfrak{M}(x^n) \qquad (n = 0, 1, 2, \cdots),$$

where

$$(3) \qquad \mu_n(\phi) = \int_{-\infty}^{+\infty} \xi^n \, d\phi(\xi).$$

It is clear from $\sigma_T \to \sigma$ that $\sigma(\xi)$ may be termed the asymptotic distribution function of the H-function $x(t)$, describing the repartition of the values ξ attained by $\xi = x(t)$ when $t \to \pm\infty$.

In the present note there will be proven the existence of another kind of distribution function, which also will be associated with every time-function $x(t)$ of class H. Whereas $\sigma(\xi)$, $-\infty < \xi < +\infty$, describes the asymptotic distribution of the possible *values* of $x(t)$, $-\infty < t < +\infty$, the distribution function to be considered, which will be denoted by $\rho(\xi)$, $-\infty < \xi < +\infty$, describes the asymptotic distribution of the *fluctuations* of $x(t)$. In some applications, for instance from the point of view of the Einstein theory of the Brownian movement, not σ but ρ is of interest. There is, however, an essential difference between our problem and the Wiener theory of "differential space,"[†] inasmuch as in the theory of the Brownian movement the Gaussian distribution is *a priori the* distribution function of the differentials or rather fluctuations, whereas we shall associate with every *given* $x(t)$ of class H a distribution function $\rho(\xi)$ describing the *actual* asymptotic distribution of the fluctuations of $x(t)$. Thus if $x(t) = \cos t$, the function $\rho(\xi)$ is built up by a direct consideration of $\eta = \operatorname{arc}\cos\xi$ and has therefore nothing to do with *a priori* or Gaussian probabilities in the function space. The proof for the existence of $\rho(\xi)$ yields also a rather simple relation between the asymptotic distribution function $\rho(\xi)$ of the differentials of $x(t)$ and the asymptotic distribution function $\sigma(\xi)$ of the values of $x(t)$ as defined by (2). It turns out that $\sigma(\xi)$ is of a more fundamental character than $\rho(\xi)$ in that ρ may be expressed explicitly in terms of σ whereas on starting with ρ the determination of σ would depend upon the treatment of a quadratic integral equation of the Runge-Pólya type.

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† VII, VIII. The Roman numbers refer to the list of papers at the end of the article.

The boundedness of $x(t)$ implied by the H-condition is not a necessary restriction and is introduced only for sake of simplicity. In fact, due to the Chebyshev appraisal of the contribution of the vicinity of $\xi = \pm\infty$ to the momenta (3), our method is valid in the case of a non-bounded $x(t)$, provided that the Hamburger momentum problem (2) is a determined one, as it is in the Gaussian case.[‡] Also, there seems to be no essential difficulty in extending the following considerations to multidimensional cases.[§]

Without reference to a given time-function $x(t)$, a function $\phi(\xi)$, $-\infty < \xi < +\infty$, is called a distribution function if it is monotone and such that $\phi(-\infty) = 0$, $\phi(+\infty) = 1$. Two distribution functions are not considered as distinct if they are identical save at the set of their discontinuity points, which is at most denumerable. For every pair ϕ_1, ϕ_2 of distribution functions there exists exactly one distribution function represented at all its continuity points by the integral

$$\int_{-\infty}^{+\infty} \phi_1(\xi - \eta)\, d\phi_2(\eta), \tag{4}$$

usually denoted by $\phi_1 * \phi_2 = \phi_1 * \phi_2(\xi)$, and one has[¶]

$$L(u; \phi_1 * \phi_2) = L(u; \phi_1) L(u; \phi_2), \text{ where } L(u; \psi) = \int_{-\infty}^{+\infty} \exp(iu\xi)\, d\psi(\xi), \tag{5}$$

[†] Cf. V where further references also are given.

[‡] Cf. VI, I.

[§] Cf. II.

[¶] Cf. III.

so that $\phi_1 * \phi_2 = \phi_2 * \phi_1$ in virtue of the Lévy uniqueness theorem of Fourier-Stieltjes transforms. It is easy to see that if ϕ_1 and ϕ_2 are such that their momenta defined by (3) exist, then the momenta of $\phi_1 * \phi_2$ also exist and

$$\mu_n(\phi_1 * \phi_2) = \sum_{m=0}^{n} \binom{n}{m} \mu_{n-m}(\phi_1)\mu_m(\phi_2) \qquad (n = 0, 1, 2, \cdots). \tag{6}$$

The formula (6) follows by the Leibniz rule if one differentiates the product (5) with respect to u at $u = 0$, the differentiation behind the integral sign being readily legalized. ϕ will be termed a damped distribution function if the integrals (3) are, in reality, not improper integrals but there exists an $R > 0$ such that $\phi(\xi) = 0$ if $-\infty < \xi < -R$ and $\phi(\xi) = 1$ if $R < \xi < +\infty$. A set of damped distribution functions will be termed uniformly damped if there exists a common R for all elements of the set. We shall need (6) only in the case where both ϕ_1, ϕ_2 are damped. Then $\phi_1 * \phi_2$ also is damped in virtue of (4), and (6) follows from (5) still more easily than in the general case.

We shall need the following theorem †: If $\phi_T(\xi)$, $0 < T < +\infty$, is a set of uniformly damped distribution functions depending upon a parameter T and if

$$\lim_{T=+\infty} \mu_n(\phi_T) \qquad (n = 0, 1, 2, \cdots) \tag{7}$$

exists, then ‡ there exists a distribution function $\phi(\xi)$ such that

$$\lim_{T=+\infty} \phi_T(\xi) = \phi(\xi) \tag{7a}$$

holds at all continuity points ξ of the limit function ϕ; furthermore,

$$\lim_{T=+\infty} \mu_n(\phi_T) = \mu_n(\phi) \qquad (n = 0, 1, 2, \cdots). \tag{7b}$$

In what follows, we have to consider together with a distribution function ϕ a "transposed" function $\bar{\phi}$ defined by

$$\bar{\phi}(\xi) = 1 - \phi(-\xi) \tag{8}$$

so that $\bar{\phi}$ also is a distribution function. $\bar{\phi}$ is obtained from ϕ by a reflection at $\xi = 0$ so that $\psi = \bar{\phi}$ implies $\phi = \bar{\psi}$ and $\phi = \bar{\phi}$ holds if and only if the distribution of the probabilities represented by ϕ depends only upon $|\xi|$. Our considerations regarding the connection between the two distribution functions σ and ρ of a time-function $x(t)$ will lead to the expression $\phi * \bar{\phi}$, which is, in virtue of the theorem mentioned in connection with (4), a distribution function. Furthermore, if $\chi = \phi * \phi$, where ϕ is arbitrary, then $\chi = \tilde{\chi}$ in virtue of $\phi_1 * \phi_2 = \phi_2 * \phi_1$, so that any distribution function representable in the form $\phi * \tilde{\phi}$ is symmetric with respect to the origin $\xi = 0$. An equivalent definition of the distribution function $\phi * \tilde{\phi}$ is $L(u; \phi * \tilde{\phi}) = | L(u; \phi) |^2$, a relation implied by (5) and (8), the numbers $L(u; \phi)$, $L(u; \tilde{\phi})$ being conjugated complex for all values of u $(-\infty < u < +\infty)$.

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† Cf. VI.

‡ And only in this case.

Let now $x(t)$, $-\infty < t < +\infty$, denote an H-function, T a positive number and Ω_T the square $| t_1 | \leqq T$, $| t_2 | \leqq T$ of area $4T^2$ in a real (t_1, t_2)-plane. For a given real number ξ, let $\{T; \xi\}$ denote the set of those points (t_1, t_2) in Ω_T for which

(9a)
$$x(t_1) - x(t_2) < \xi.$$

Put

(9b)
$$\rho_T(\xi) = \text{meas}\,\{T; \xi\} : \text{meas}\,\{T; +\infty\} \qquad (\text{meas}\,\{T; +\infty\} = 4T^2)$$

where meas { } is a two-dimensional Lebesgue measure. Thus

$$\rho_T(\xi), \qquad -\infty < \xi < +\infty,$$

is for every T a distribution function representing the repartition of the fluctuation states of $x(t)$ when t is restricted to the finite range $| t | \leqq T$. In cases where the use of differentials is permissible, $d\rho_T(\xi)$ represents the probability that a given fluctuation state $\xi = x(t_1) - x(t_2)$ of $x(t)$ changes between the dates $t = -T$ and $t = T$ by the amount $d\xi$. It may be mentioned that $\rho_T = \tilde{\rho}_T$ inasmuch as (9a) does not discriminate between "past" and "future," i. e., between $t_1 < t_2$ and $t_1 > t_2$, as is required by the viewpoint of *independent* fluctuations, for instance.

Such probability interpretations have, however, an interest only if there exists an asymptotic distribution of the fluctuations, i. e., if ρ_T approaches a limit function when $T \rightarrow +\infty$. In order to prove the existence of a limit distribution, we notice first that on placing $\phi_T = \rho_T$ our criterion (7) for the existence of (7a) is applicable. In fact, the set $\rho_T(\xi)$, $0 < T < +\infty$, of distribution functions is uniformly damped. For it is clear from (9a) and (9b) that for the number R occurring in the definition of uniform dampedness one may choose any number R satisfying the inequality $| x(t) | < R/2$, $-\infty < t < +\infty$, and there do exist numbers R satisfying this inequality inasmuch as $x(t)$ is an H-function. Hence if we prove that

(10)
$$\lim_{T=+\infty} \mu_n(\rho_T) \qquad (n = 0, 1, 2, \cdots)$$

exists, it will be proven that there exists a distribution function $\rho(\xi)$ such that

$$\text{(10a)} \qquad \lim_{T=+\infty} \rho_T(\xi) = \rho(\xi)$$

at all continuity points ξ of $\rho(\xi)$ and that

$$\text{(10b)} \qquad \lim_{T=+\infty} \mu_n(\rho_T) = \mu_n(\rho) \qquad (n = 0, 1, 2, \cdots).$$

Now from (3), (9a) and (9b)

$$\text{(11)} \quad \mu_n(\rho_T) = \int_{-T}^{T}\int_{-T}^{T} (x(t_1) - x(t_2))^n \, dt_1 dt_2/(2T)^2 \quad (n = 0, 1, 2, \cdots),$$

the approximating sums of the latter Lebesgue double integral being precisely the approximating sums of the simple Stieltjes integral $\mu_n(\rho_T)$. Moreover, the double integral may be evaluated as an iterated integral. Hence it approaches a limit when $T \to +\infty$, viz.

$$\text{(12)} \qquad \lim_{T=+\infty} \int_{-T}^{T}\int_{-T}^{T} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mathfrak{M}(x^{n-m}) \mathfrak{M}(x^m),$$

inasmuch as $x(t)$ is an H-function so that the time-averages (1) exist. It follows therefore from (11) that the limits (10) also exist. This completes the proof of the existence of the asymptotic distribution function (10a) of the fluctuations of $x(t)$.

Moreover, from (10b), (11) and (12),

$$\mu_n(\rho) = \sum_{m=0}^{n} \binom{n}{m} \mathfrak{M}(x^{n-m}) \mathfrak{M}((-x)^m),$$

where $\mathfrak{M}(x^{n-m}) = \mu_{n-m}(\sigma)$ and

$$\text{(6a)} \qquad \mathfrak{M}((-x)^m) = \mu_m(\bar{\sigma}) \qquad (m = 0, 1, 2, \cdots)$$

in virtue of (2), (3) and (8). Hence from (6)

$$\mu_n(\rho) = \mu_n(\sigma * \bar{\sigma}) \qquad (n = 0, 1, 2, \cdots).$$

Since ρ and $\sigma * \bar{\sigma}$ are damped distribution functions, the so-called Lerch theorem regarding momentum determinateness is applicable. Consequently,

$$\rho = \sigma * \bar{\sigma}.$$

This is the integral relation between ρ and σ mentioned in the introduction.

The asymptotic distribution function σ of $x(t)$ has been investigated in detail in the case where $x(t)$ is almost-periodic with linearly independent frequencies † and also in the case where x is the Fourier-Stieltjes transform of a continuous function of bounded variation.‡ The relation $\rho = \sigma * \bar{\sigma}$ permits of obtaining the same sharp results regarding ρ. Besides, $\bar{\sigma} = \sigma$ in both cases.§

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† IX.

‡ IV.

§ *Ibid.*

In particular,[†] on considering the non-differentiable function of Weierstrass as an almost-periodic function $x(t)$ with linearly independent frequencies, it follows that its $\rho(\xi)$, $-\infty < \xi < +\infty$, everywhere possesses derivatives of arbitrarily high order and that in a more precise manner

$$L(u;\rho) = (\prod_{n=1}^{\infty} J_0(a^n u))^2 \qquad (0 < a < 1).$$

This product of Bessel functions has nothing to do with the Fourier-Stieltjes transform $\alpha \exp(-\beta u^2)$, $\alpha = \alpha(\beta)$, of a Gaussian distribution function so that the infinitesimal fluctuations of $x(t)$ can hardly be considered as haphazard or as independent of each other. Thus a remark of Perrin [‡] as to the analogy between the Brownian movement and the Weierstrass function is not a very fortunate one. Correspondingly, our ρ does not concern a "general" $x(t)$, chosen in the function space at random, but it concerns, in the present case, precisely the Weierstrass $x(t)$.

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V. R. E. A. C. Paley †, N. Wiener and A. Zygmund, "Notes on random functions," *Mathematische Zeitschrift*, vol. 37 (1933), pp. 647-668.

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X. A. Wintner, "A note on the non-differentiable function of Weierstrass, *ibid.*, pp. 603-605.

[†] Cf. X.

[‡] Referred to in V.

ON THE DISTRIBUTION OF SUCCESSIVE IMAGES IN THE POINCARÉ TRANSFORMATION PROBLEM OF A CIRCLE INTO ITSELF.

By D. C. Lewis, Jr.[*] and Aurel Wintner.

We consider the points on a circle C of radius $1/(2\pi)$ and denote by x the distance, measured in a certain sense along the circumference, from a fixed point O on C to a variable point P. Since the circumference is of unit length, we agree that two values of x which differ by an integer shall correspond to the same point P on C.

An equation of the form $\bar{x} = f(x)$, where $f(x)$ is continuous, monctonic and nowhere constant and satisfies the functional relation $f(x+1) \equiv f(x) + 1$, obviously defines a one-to-one order preserving transformation T of the points of C into themselves. Let us take x_0 arbitrarily and let $x_n = f(x_{n-1})$, $(n = 1, 2, \cdots)$. Then it is well known that $x_n = \alpha n + 0(1)$ as $n \rightarrow \infty$, where α is a number independent of n or x_0. It is also well known that a necessary and sufficient condition that α be irrational is that neither T nor any of its iterates have a fixed point on C. We shall assume that such is the case. The set S of the cluster points of the sequence $\{x_n\}$ is then independent of x_0 and is either the whole circumference C or a nowhere dense perfect subset of C. If $S = C$, the transformation $\bar{x} = f(x)$ may be transformed by a one-to-one order preserving transformation $\bar{x} = G(x)$ $[\equiv G(x-1) + 1]$ of C into a rotation of C through the irrational angle $2\pi\alpha$, so that $G\{f[G^{-1}(x)]\} \equiv x + \alpha$, where G^{-1} denotes the inverse of G. A similar result holds in the case $S \neq C$ by introducing a proper convention regarding the asymptotically empty set $S - C$.[†]

Let $\phi(x)$ be an arbitrary continuous periodic function with the period 1.

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[†] Cf. H. Poincaré, *Oeuvres complètes*, t. 1, pp. 137-158; H. Kneser, "Regulärer Kurvenscharen auf den Ringflächen," *Mathematische Annalen*, Bd. 91 (1924), pp. 135-154; and J. Nielsen, "Om topologiske Afbildninger af en Jordankurve paa sig selv," *Matematisk Tidsskrift*, B (1928), pp. 39-46. Also G. D. Birkhoff implicitly treated the case $S \neq C$ in his construction of an example of a discontinuously recurrent motion. "Quelques théorèmes sur le mouvement des systèmes dynamiques," *Bulletin de la Société Mathématique de France*, t. 40 (1912), pp. 305-323.

[‡] T. Carleman, "Sur les caractéristiques du tore," *C. R. Acad. Sci.*, Paris, t. 195 (1932), pp. 478-481. His proof is based on the F. Riesz theory of linear functional transformations.

Then Carleman showed that $\lim_{n=\infty} (1/n) \sum_{k=1}^{n} \phi(x_k)$ exists.‡ Since we may take $\phi(x) = e^{\pm 2\pi mix}$, where $i = (-1)^{1/2}$ and $m = 0, 1, 2, \cdots$, it follows that the sequence of points, $P_1, P_2, P_3, \cdots$, corresponding to $x_1, x_2, x_3, \cdots$ possesses a distribution function $g(x)$. What we mean by this is clarified in the following definition:

Let $\nu(n, \bar{x})$ represent the number of the n points $P_1, P_2, \cdots, P_n$ which lie on the open interval $0 < x < \bar{x}$ $(\leqq 1)$; then the sequence of points $P_1, P_2, P_3, \cdots$ possesses the distribution function $g(\bar{x}) = \lim_{n=\infty} [\nu(n, \bar{x})/n]$, if this limit exists for all $\bar{x}$, except possibly for a denumerable set.

Concerning this definition we remark that, if $g(x+0) = g(x-0)$, then x does not belong to the above mentioned exceptional denumerable set.*

It should also be noted that, conversely, the validity of Carleman's theorem can be deduced from the existence of a distribution function. This is obvious, since one can approximate to $\phi(x)$ uniformly by means of step functions chosen in such a way that the values of x corresponding to the end points of the steps do not coincide with the discontinuity points of $g(x)$.

In this note we give a direct proof of the existence of a distribution function with the help of the Kronecker-Weyl theorem.† Besides being more elementary than Carleman's method, the present proof gives a new interpretation of the function G which turns out to be simply the distribution function and hence shows automatically that the latter is continuous throughout so that the limit of $\nu(n, \bar{x})/n$ exists everywhere. Moreover we obtain an additional result, expressing the fact that the P_n's are clustering very strongly at each of their limit points. We enunciate the precise theorem as follows:

$g(x)$ exists and is everywhere continuous on $0 < x \leqq 1$, is independent of x_0 and is not a constant on any open interval containing a point of S. Furthermore, on placing $g(x+1) \equiv g(x) + 1$, $-\infty < x < +\infty$, the function g is identical with the function designated above by G.

For the sake of definiteness take first $x_0 = 0$.

Consider another circle C' of unit circumference and denote by y the distance measured in a certain sense along the circumference from a fixed point on C' to a variable point Q. Here, again, two values of y differing by an integer correspond to the same Q. Consider the sequence of points $Q_0, Q_1, Q_2, \cdots$ corresponding respectively to $y = 0,\ \alpha,\ 2\alpha, \cdots$. We now define a point function $G(P)$ on the denumerable set $P_0, P_1, P_2, \cdots$ as follows: $G(P_n) = Q_n$ $(n = 0, 1, 2, \cdots)$.

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* $g(x+0)$ and $g(x-0)$ must both exist since $g(x)$ is obviously monotonic. Cf. also E. K. Haviland and A. Wintner, "A note on the Kronecker-Weyl theorem," *American Journal of Mathematics*, vol. 56 (1934), pp. 17-24.

† H. Weyl, *Ueber die Gleichverteilung von Zahlen mod. Eins. Math. Ann.*, Bd. 77, pp. 313-352.

Now O and an arbitrary point $\bar{P}$ of C divide the points $P_1, P_2, P_3, \cdots$ into two sets, in such a manner that every point of one set precedes all the points of the other set as, starting from O, we traverse C in the positive sense. But it is known that the geometrical order of the $Q_1, Q_2, Q_3, \cdots$ on C' is the same as that of the $P_1, P_2, P_3, \cdots$ on C.* Hence the Q_n's are also divided into two sets such that every point of one set precedes all the points of the other and such that a particular $Q_n = G(P_n)$ belongs to the first or second set of the Q's according as the corresponding P_n belongs to the first or second set of the P's. This division into sets gives us a Dedekind cut of the denumerable everywhere dense set of the Q_n's. This cut defines a point $\bar{Q}$ which we make to correspond with the point $\bar{P}$ by writing $\bar{Q} = G(\bar{P})$, thus defining the function $G(P)$ for all points of C. Or, upon writing x and y instead of P and Q, we have a mapping of C upon C' by means of the equation, $y = G(x)$, where $G(x)$ is continuous, monotonic, and such that $G(x + 1) \equiv G(x) + 1$, $G(0) = 0$.

If $\bar{y} = G(\bar{x})$, it is obvious, since geometrical order is preserved in this mapping, that the number of points $Q_1, Q_2, \cdots, Q_n$ in the interval $0 < y < \bar{y} \leqq 1$ is equal to the number of points $P_1, P_2, \cdots, P_n$ in the interval $0 < x < \bar{x} \leqq 1$, namely $\nu(n, \bar{x})$. Hence, by Weyl's theorem of equidistribution, $\lim_{n=\infty} \nu(n, \bar{x})/n$ exists and is equal to the length of the interval $0 < y < \bar{y}$, which is simply $\bar{y}$ or $G(\bar{x})$. Thus we see that the distribution function exists and is none other than $G(x)$ as just defined: $g(x) \equiv G(x)$, $0 < x \leqq 1$.

Let I be an open interval containing a point P of S. Since P is a limit point of the denumerable set $P_1, P_2, \cdots$, there must be at least two of these points in I, say P_i and P_j. But $Q_i = G(P_i)$ and $Q_j = G(P_j)$ are distinct points since α is irrational. Hence G, or g, can not be a constant in any open interval containing a point of S. On the other hand, $G(x)$ *is* a constant on any interval *not* containing a point of S, as it is easy to prove directly but is obvious *a priori* since $G(x)$ is the distribution function.

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* Cf. Arnaud Denjoy, "Sur les courbes définies par les équations différentielles à la surface du tore," *Journal de Mathématiques*, 9e série, vol. 11 (1932), pp. 333-375. Also Poincaré, Kneser, and Nielsen, *loc. cit.*

From the definition and properties of $G(x)$ above given it follows that $G[f(x)] \equiv G(x) + \alpha$. Thus the distribution function and the set S is the same for all sequences of points represented by $x_0, x_1, x_2, \cdots$ for x_0 not necessarily zero as supposed above.

If $C = S$, a single valued inverse G^{-1} of G exists so that $x = G^{-1}(y)$. Hence the above functional relation may be written in the form,

$$G\{f[G^{-1}(y)]\} \equiv y + \alpha,$$

thus showing that the transformation $\bar{x} = f(x)$ can in this case be transformed into a pure rotation through the angle $2\pi\alpha$. According to a fundamental result of Denjoy this is always the case if $f(x)$ possesses a continuous derivative of bounded variation which nowhere vanishes.

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FORMAL SOLUTIONS OF IRREGULAR LINEAR DIFFERENTIAL EQUATIONS. PART I.

By Frances Thorndike Cope.

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1. Introduction. A complete formal solution of the general homogeneous linear differential equation with rational coefficients was first obtained by Fabry[*] in 1885. He found that for an arbitrary point x_0 there exist always n distinct, that is, linearly independent, formal solutions of the general type

$$y(x) = e^{Q(x)}[s_0(x)\log^k(x-x_0) + s_1(x)\log^{k-1}(x-x_0) + \cdots + s_k(x)],$$

where $Q(x)$ is a polynomial in $(x-x_0)^{1/m}$, and the $s_i(x)$ $(i=0,1,\cdots,k)$ are formal series in ascending powers of $(x-x_0)^{1/m}$, m being a positive integer, and k a positive integer or zero. This result is of importance in the theory of equations with irregular singular points, for it serves as a basis for the study of the character of the actual, that is, analytic, solutions in the neighborhood of such a singular point. A new and simpler method of establishing it, analogous to that developed by Birkhoff[†] for the study of linear difference equations, is given below.

For the sake of simplicity in the expressions involved we shall consider solutions relative to the point ∞. This is no real restriction, for any singular point x_0 can be taken into the point ∞ by a transformation which does not alter the rational character of the coefficients. It is then convenient to let the coefficients be expanded as series in descending powers of x. In fact, instead of requiring that the coefficients be rational functions, we shall require only that they be expressible as formal series, convergent or divergent, in descending powers of $x^{1/p}$, where p is a positive integer.

The choice of the *basic integer* p in any particular case is arbitrary to a certain extent, for a series in powers of $x^{1/p}$ can equally well be regarded as a series in powers of $x^{1/mp}$, where m is any positive integer. This possibility of replacing the original basic integer by an integral multiple of it is essential to our proof, which depends upon the fact (to be proved later) that any equation of this type is reducible[*] if the basic integer is properly chosen. For the present, however, we suppose one of the admissible values of the basic integer to have been arbitrarily chosen, and also a particular determination of $x^{1/p}$.

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[*] C. E. Fabry, *Thèse* (Faculté des Sciences, Paris, 1885).

[†] G. D. Birkhoff, "Formal theory of irregular linear difference equations," *Acta Mathematica*, vol. 54 (1930), pp. 205-246.

Then our principal result is the following theorem:

THEOREM I. *A linear homogeneous differential equation of the n-th order, i. e., an equation of the form*

$$(1)\qquad a_0(x)y^{(n)}(x)+a_1(x)y^{(n-1)}(x)+\cdots+a_n(x)y(x)=0\quad (a_0(x)\not\equiv 0)$$

in which the coefficients $a_i(x)$ are formal series in descending powers of $x^{1/p}$, p being a positive integer, has always n linearly independent formal solutions of the general type

$$(2)\qquad y(x)=s_0(x)\log^k x+s_1(x)\log^{k-1}x+\cdots+s_k(x),$$

where k is a positive integer or zero, and the $s_i(x)$ have the form

$$(3)\qquad s_i(x)=e^{Q(x)}(b_{i,-r}\,x^{r/mp}+b_{i,1-r}\,x^{(r-1)/mp}+\cdots),$$

in which m is a positive integer $\leq n$ and $Q(x)$ is a polynomial in $x^{1/mp}$ which is not expressible as a polynomial in $x^{2/mp}$, $x^{3/mp}$, or any higher integral power of $x^{1/mp}$. The complete set of n solutions consists of one or more subsets of the form

$$(4)\qquad y_{j+1}(x)=\sum_{i=0}^{j}\frac{j!}{i!\,(j-i)!}\,s_i(x)\log^{j-i}x\qquad (j=0,1,\cdots,k).$$

It is understood that the same determination of $\log x$ is taken in all the solutions of any subset. Which determination is chosen is immaterial, for replacing $\log x$ by $\log x+h\pi(-1)^{1/2}$ (h an integer) in the subset (4) replaces the original solutions by a new set $\bar{y}_i(x)$ $(i=1,\cdots,k+1)$ of the same character, and such that the $\bar{y}_i(x)$ are linear combinations of $y_1(x)$, $y_2(x)$, $\cdots, y_{k+1}(x)$.

A resumé of the principal definitions and theorems of the algebra and differential calculus of these formal solutions is given in § 2. The proof of Theorem I is given in § 3, and further properties of the complete set of formal solutions, analogous to those of a complete set of analytic solutions, are given in § 4.

The second part of the paper will include the proof of the converse of

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[*] A linear homogeneous differential equation $L(y)=0$ is said to be reducible if there exist linear differential operators M and N, of lower order than that of L, and with coefficients subject to the same conditions as those of L, such that $M(N(y))=L(y)$ identically in y.

Theorem I and of the equivalence of a single equation of the form (1) to the most general system of simultaneous linear differential equations with rational coefficients, as well as some applications of Theorem I.

2. Formal solutions and linear dependence. We shall be concerned with certain formal operations on quantities of one of the following types:

(a) formal series in descending powers of $x^{1/p}$, that is, series, either convergent or divergent, of the form $a_{-r}x^{r/p} + a_{1-r}x^{(r-1)/p} + \cdots$, where the coefficients $a_{-r}, a_{1-r}, \cdots$ are constants and x is a complex variable;

(b) expressions of the more general non-logarithmic form

$$(5) \qquad a_1(x)e^{Q_1(x)} + a_2(x)e^{Q_2(x)} + \cdots + a_m(x)e^{Q_m(x)},$$

in which the $a_i(x)$ $(i = 1, 2, \cdots, m)$ are formal series (we shall henceforth mean "formal series in descending powers of $x^{1/p}$" by the term "formal series"), and the $Q_i(x)$ $(i = 1, 2, \cdots, m)$ are of the form $Q_i(x) = c_{i1}x^{s/p} + c_{i2}x^{(s-1)/p} + \cdots + c_{is}x^{1/p}$ and are all different;

(c) expressions of the still more general form

$$(6) \qquad S_0(x)\log^k x + S_1(x)\log^{k-1} x + \cdots + S_k(x),$$

in which the coefficients $S_i(x)$ are of the non-logarithmic form (5). Throughout this section we shall use the notation $a(x)$, $b(x)$, $c(x)$ to represent formal series, $R(x)$, $S(x)$, $T(x)$ to represent expressions of the general non-logarithmic type (5), and $U(x)$, $V(x)$, $W(x)$, and $Z(x)$ to represent expressions of the general type (6).

The fundamental formal properties which we assume for these expressions are given by the following definitions:

DEFINITION 1. *The two formal series*

$$a(x) = a_{-r}x^{r/p} + a_{1-r}x^{(r-1)/p} + \cdots \quad \textit{and} \quad b(x) = b_{-r}x^{r/p} + b_{1-r}x^{(r-1)/p} + \cdots$$

are said to be formally equal if and only if $a_{-j} = b_{-j}$ $(j = r, r-1, r-2, \cdots)$. *In particular,* $a(x)$ *is said to be formally equal to zero if and only if* $a_{-j} = 0$ $(j = r, r-1, r-2, \cdots)$.

The two expressions of non-logarithmic form,

$$S(x) = a_1(x)e^{Q_1(x)} + \cdots + a_m(x)e^{Q_m(x)}$$

and

$$T(x) = b_1(x)e^{Q_1(x)} + \cdots + b_m(x)e^{Q_m(x)},$$

are said to be formally equal if and only if $a_i(x) = b_i(x)$ $(i = 1, 2, \cdots, m)$.

In particular, $S(x)$ *is said to be formally equal to zero if and only if* $a_i(x)=0$ $(i=1,2,\cdots,m)$.

The two expressions of general form,

$$U(x)=S_0(x)\log^k x+\cdots+S_k(x) \text{ and } V(x)=T_0(x)\log^k x+\cdots+T_k(x),$$

are said to be formally equal if and only if $S_i(x)=T_i(x)$ $(i=0,1,\cdots,k)$, *and, in particular,* $U(x)$ *is said to be formally equal to zero if and only if* $S_i(x)=0$ $(i=0,1,\cdots,k)$.

DEFINITION 2. *The sum* $a(x)+b(x)$ *and the difference* $a(x)-b(x)$ *of the series* $a(x)$ *and* $b(x)$ *are defined as the formal series*

$$\sum_{j=r}^{-\infty}(a_{-j}+b_{-j})x^{j/p} \quad \textit{and} \quad \sum_{j=r}^{-\infty}(a_{-j}-b_{-j})x^{j/p}$$

respectively.

The sum $S(x)+T(x)$ *and the difference* $S(x)-T(x)$ *of the non-logarithmic expressions* $S(x)$ *and* $T(x)$ *are defined as the non-logarithmic expressions*

$$\sum_{i=1}^{m}(a_i(x)+b_i(x))e^{Q_i(x)} \quad \textit{and} \quad \sum_{i=1}^{m}(a_i(x)-b_i(x))e^{Q_i(x)}$$

respectively.

The sum $U(x)+V(x)$ *and the difference* $U(x)-V(x)$ *are defined as the expressions*

$$\sum_{i=0}^{k}(S_i(x)+T_i(x))\log^{k-i}x \quad \textit{and} \quad \sum_{i=0}^{k}(S_i(x)-T_i(x))\log^{k-i}x$$

respectively.

DEFINITION 3. *The product* $a(x)b(x)$ *of the formal series* $a(x)$ *and* $b(x)$ *is defined as the formal series*

$$\sum_{j=2r}^{-\infty}(a_{-r}b_{r-j}+a_{1-r}b_{r-j-1}+\cdots+a_{r-j}b_{-r})x^{j/p},$$

where it is understood that $a_{-i}=b_{-i}=0$ *for* $i>r$; *the product* $S(x)T(x)$ *of the non-logarithmic expressions* $S(x)$ *and* $T(x)$ *is defined as the non-logarithmic expression*

$$\sum_{i=1}^{m}\sum_{j=1}^{m}a_i(x)b_j(x)e^{Q_i(x)+Q_j(x)};$$

and the product $U(x)V(x)$ *of the two expressions* $U(x)$ *and* $V(x)$, *of the general form* (6), *is defined as the expression*

$$\sum_{i=0}^{k}\sum_{j=0}^{k} S_i(x)T_j(x)\log^{2k-i-j} x.$$

THEOREM A. *If $a(x)$ and $b(x)$ are formal series, and $b(x) \neq 0$, then there is a unique formal series $c(x)$, of the same type, such that $b(x)c(x) = a(x)$.*[*]

DEFINITION 4. *The series $c(x)$ of the preceding theorem is called the quotient of $a(x)$ by $b(x)$, and is denoted by $a(x)/b(x)$.*

If $T(x)$ is of the form

$$b(x)e^{Q(x)} \qquad (b(x) \neq 0), \tag{7}$$

then the quotient $S(x)/T(x)$ is defined as the non-logarithmic expression

$$\sum_{i=1}^{m} (a_i(x)/b(x))e^{Q_i(x)-Q(x)}$$

and the quotient $U(x)/T(x)$ is defined as

$$\sum_{i=0}^{k} (S_i(x)/T(x))\log^{k-i} x.$$

(For our purposes a definition of division by an expression of the general form (6) is not necessary.)

DEFINITION 5. *The derivative of the formal series $a(x)$ is defined as the series*

$$\sum_{j=r}^{-\infty} (ja_{-j}/p)x^{j/p-1};$$

the derivative of the non-logarithmic expression $S(x)$ is defined as

$$\sum_{i=1}^{m} [a'_i(x) + a_i(x)Q'_i(x)]\, e^{Q_i(x)};$$

and the derivative of the expression $U(x)$ is defined as

$$\sum_{i=0}^{k} [S'_{k-i}(x)\log x + ix^{-1} S_{k-i}(x)]\log^{i-1} x.$$

From these definitions it is easy to verify that the ordinary formal algebraic laws and formal rules of differentiation apply to these expressions, and if we generalize the ordinary definitions of determinant and of matrix by letting the elements be expressions of the form (6) then the elementary theorems on determinants and the theory of linear equations are still valid, since their proof depends only on the algebraic operations defined above. Only when division is involved will slight modifications be necessary, as in the case of the following theorems:

---

[*] Proved directly by substituting $c(x) = c_{-s}x^{s/p} + c_{1-s}x^{(s-1)/p} + \cdots$ $(c_{-s} \neq 0)$ and evaluating $s, c_{-s}, c_{1-s}, \cdots$ successively.

THEOREM B. (Cramer's Rule). *If in the equations*

$$
(8)\qquad \begin{array}{c} U_{11}y_1 + \cdots + U_{1n}y_n = V_1 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ U_{n1}y_1 + \cdots + U_{nn}y_n = V_n \end{array}
$$

the coefficients U_{ij} $(i, j = 1, 2, \cdots, n)$ and the right-hand terms V_i $(i = 1, 2, \cdots, n)$ are expressions of the form (6), and the determinant of the coefficients, $U = | U_{ij} |$, is of the non-logarithmic form (7) and is different from zero, then the system of equations has one and only one solution (of the form (6)), namely $y_i = U_i/U$ $(i = 1, 2, \cdots, n)$, where U_i is the n-rowed determinant obtained from U by replacing the elements of the i-th column by the elements $V_1, V_2, \cdots, V_n$.

THEOREM C. *A sufficient condition that the equations*

$$
\begin{array}{c} U_{11}y_1 + \cdots + U_{1n}y_n = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ U_{n1}y_1 + \cdots + U_{nn}y_n = 0 \end{array}
$$

in which the U_{ij} $(i, j = 1, 2, \cdots, n)$ are expressions of the form (6), have a solution other than $y_1 = y_2 = \cdots = y_n = 0$, is that the rank r of the matrix (U_{ij}) $(i, j = 1, 2, \cdots, n)$ be less than n, and that at least one of the non-vanishing r-rowed determinants of the matrix be of the non-logarithmic form (7).

The condition that the determinant U be zero is also necessary for the existence of a solution other than $y_1 = y_2 = \cdots = y_n = 0$.

The theory of linear dependence of expressions of this form is based on

DEFINITION 6. *If $y^{[1]}, y^{[2]}, \cdots, y^{[m]}$ are expressions of the general type (6), they are said to be linearly dependent if they satisfy formally some linear homogeneous relation,*

$$
(9)\qquad F_1y^{[1]} + F_2y^{[2]} + \cdots + F_my^{[m]} = 0,
$$

in which $F_1, F_2, \cdots, F_m$ are constants, not all zero. Otherwise the expressions are said to be linearly independent, or distinct.

Likewise the m *sets of* n *expressions each,* $y_1^{[i]}, y_2^{[i]}, \cdots, y_n^{[i]}$ $(i = 1, 2, \cdots, m)$, *are said to be linearly dependent if and only if* m *constants, not all zero, exist such that*

$$F_1 y_j^{[1]} + F_2 y_j^{[2]}, + \cdots + F_m y_j^{[m]} = 0 \qquad (j = 1, 2, \cdots, n).$$

The elementary theorems on linear dependence of sets of constants then apply also to expressions of the form (6), as can readily be verified.

The formal solutions of Theorem I, and those formal expressions which we shall be especially interested in, may be regarded as expressions of the form (6) in which $m = 1$. When it is necessary to consider solutions of the more general form (6) as well as those of this simpler type, we shall distinguish between them by calling the latter *elementary,* the former *composite,* formal solutions. In the remainder of this section, and elsewhere when no qualifying adjective is used, we shall mean elementary formal solutions.

For these formal solutions $y_i^{[j]}(x)$ certain additional theorems on linear dependence are true which have no analogues in the case in which the $y_i^{[j]}(x)$ are constants. Two of these may be stated as follows:

THEOREM D. *A necessary and sufficient condition for the linear dependence of the set* $y^{[1]}(x), \cdots, y^{[m]}(x)$, *where*

$$y^{[j]}(x) = s_0^{[j]}(x) \log^k x + s_1^{[j]}(x) \log^{k-1} x + \cdots + s_k^{[j]}(x) \quad (j = 1, 2, \cdots, m)$$

and the $s_i^{[j]}(x)$ $(i = 0, 1, \cdots, k;\ j = 1, 2, \cdots, m)$ *are non-logarithmic, is that there exist constants* $F_1, \cdots, F_m$, *not all zero, such that*

$$F_1 s_i^{[1]} + F_2 s_i^{[2]} + \cdots + F_m s_i^{[m]} = 0 \qquad (i = 0, 1, \cdots, k).$$

THEOREM E. *A necessary and sufficient condition that the set*

$$\begin{aligned} y_i(x) &= e^{Q_1(x)} \bar{y}_i(x) && (i = 1, 2, \cdots, m_1), \\ &\cdots\cdots \\ y_i(x) &= e^{Q_h} \bar{y}_i(x) && (i = m_{h-1} + 1, \cdots, m_h) \end{aligned}$$

where the $\bar{y}_i(x)$ $(i = 1, 2, \cdots, m_h)$ *are of the form* (2) *but contain no exponential factors, be linearly independent is that each of the sets* $y_{m_{j-1}+1}(x), \cdots, y_{m_j}(x)$ $(j = 1, 2, \cdots, h)$ *be linearly independent.*

From the definitions of the algebraic operations and differentiation for expressions of the form (6) it is clear that the following theorems regarding the solutions of linear differential equations are true of such formal solutions as well as of analytic solutions:

THEOREM F. *If* $y_1(x), y_2(x), \cdots, y_n(x)$ *are solutions of a linear homogeneous differential equation, then any linear combination of these solutions with constant coefficients is also a solution.*

THEOREM G. *If* $Y(x)$ *is a particular solution of a non-homogeneous linear differential equation* $L(y) = r(x)$, *and* $y(x)$ *is the general solution of the corresponding homogeneous equation, then* $y(x) + Y(x)$ *is the general solution of the non-homogeneous equation.*

Additional theorems on the linear dependence of formal solutions of the equation (1) will be given in § 4.

In the remainder of this paper only *formal* series and *formal* solutions will be dealt with, and the unqualified terms *series* and *solution* are to be taken in this sense. Thus equations involving the independent variable x are to be interpreted as *formal* equations, i. e., identities in x, and the symbol $\equiv$ will in general be reserved for those cases in which we have an identity in the dependent variable.

When no ambiguity is caused by doing so, the term *series* will occasionally be applied to the whole expression (2) or (3). It will be convenient to use the term *non-logarithmic solution* for a solution of the form (3), and we shall also use a generalization of the terms *normal*, and *anormal*, series. A solution (2) will be said to be *of normal form*, or a *normal series*, with respect to the basic integer p, if it is expressible in terms of integral powers of $x^{1/p}$, that is, if we have $m = 1$ in the expression (2). On the other hand, a solution which is expressible in the form (2) only if m is greater than unity, will be said to be *of anormal form*, or an *anormal series*, with respect to the basic integer p. The terms *normal* and *anormal* will be used in this sense, referring, unless otherwise specified, to the basic integer which has been used in expressing the coefficients of the differential equation in question. If the basic integer is unity this definition reduces to the usual one.

3. Proof of the fundamental theorem. In proving Theorem I, it is convenient to take equation (1) in the form

$$(1') \qquad y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = 0.$$

This can always be done, for, since $a_0(x) \not\equiv 0$, we may divide the original equation through by $a_0(x)$ and obtain new coefficients $\bar{a}_i(x) = a_i(x)/a_0(x)$ $(i = 1, 2, \cdots, n)$ which are subject to all the conditions imposed on the original coefficients.

The method of proof is to reduce the original question to that of the reducibility of the general equation (1′) of order $n > 1$ by establishing the following facts:

(1) *that in the case* $n = 1$ *the equation* (1′) *has a non-logarithmic solution of normal form;*

(2) *that if there is a non-logarithmic solution for every equation* (1′), *then there is a complete set of* n *linearly independent solutions of type* (2), *consisting of subsets of type* (4), *for every such equation;*

(3) *that there is at least one non-logarithmic solution* (3) *if only every such equation of order* $n > 1$ *is reducible;*

and then prove

(4) *that every equation* (1′) *of order* $n > 1$ *is reducible when the basic integer* p *is properly chosen.*

3. 1. Proof of Part 1. The proof of Part 1 is by direct computation as follows:

We have given the equation

$$y'(x) + a(x)y(x) = 0. \tag{10}$$

where

$$a(x) = x^{\sigma/p}(a_{-\sigma} + a_{1-\sigma}x^{-1/p} + a_{2-\sigma}x^{-2/p} + \cdots) \quad (\sigma, \text{ an integer; } a_{-\sigma} \neq 0).$$

Let

$$s(x) = e^{Q(x)}x^{r/p}(b_{-r} + b_{1-r}x^{-1/p} + b_{2-r}x^{-2/p} + \cdots), \tag{11}$$

where $b_{-r} \neq 0$, and

$$Q(x) = c_1x^{s/p} + c_2x^{(s-1)/p} + \cdots + c_sx^{1/p} \qquad (s, \text{ an integer}), \tag{12}$$

that is, let $s(x)$ be of type (3) with $m = 1$. Then assume that $s(x)$ is a solution of the equation (10) and obtain conditions on the constants $s, c_1, c_2, \cdots, c_s, r, b_{-r}, b_{1-r}, \cdots$.

The equation (10) becomes, after substituting $y(x) = s(x)$ and dividing through by $e^{Q(x)}$,

$$\begin{aligned}[Q'(x) + a(x)]\,[b_{-r}x^{r/p} + b_{1-r}x^{(r-1)/p} + \cdots]\\ + (rb_{-r}/p)x^{r/p-1} + [(r-1)b_{1-r}/p]x^{(r-1)/p-1} + \cdots = 0.\end{aligned}$$

Since $b_{-r} \neq 0$, this equation can be satisfied only if the leading term of the factor $Q'(x) + a(x)$ is of degree -1, or less, in x. This requirement can be met when $\sigma > -p$ by taking

$$s = \sigma + p \neq 0,\ c_1 = -(p/s)a_{-\sigma},\ c_2 = [-p/(s-1)]a_{1-\sigma}, \cdots,\ c_s = -pa_{p-1}.$$

In the case when σ is $\leq -p$ we let $Q(x) = 0$.

Then in either case the equation to be satisfied takes the form

$$(13)\qquad [a_p x^{-1} + a_{p+1} x^{-1/p-1} + \cdots]\,[b_{-r} x^{r/p} + b_{1-r} x^{(r-1)/p} + \cdots] + \frac{rb_{-r}}{p} x^{-1+r/p} + \frac{(r-1)b_{1-r}}{p} x^{-1+(r-1)/p} + \cdots = 0.$$

By multiplying out, collecting terms, and equating to zero the coefficient of each power of x, we find that b_{-r} may be chosen arbitrarily and the remaining constants $r, b_{1-r}, b_{2-r}, \cdots$ are then uniquely determined by the equations

$$r = -pa_p, \quad b_{i-r} = (p/i)\,[a_{p+1} b_{i-r-1} + \cdots + a_{p+i} b_{-r}], \quad (i = 1, 2, 3, \cdots).$$

Thus we have proved

THEOREM II. *Every differential equation of the first order of type* (1′) *has a non-logarithmic solution of normal form.*

3. 2. Proof of Part 2. Part 2 is proved indirectly as follows:

If we assume the statement (2) to be false, then there must exist equations of form (1′) which have at least one non-logarithmic solution but do not have a complete set of n linearly independent solutions of the general type (2), made up of subsets of type (4), and there must be a least order for which such equations exist.

Let

$$(14)\qquad L(y) = 0$$

be such an equation of this least order n, where $L(y)$ denotes the left-hand member of an equation (1′). Then we must have $n > 1$, since in the case of an equation of the first order the one non-logarithmic solution assumed constitutes the required complete set of solutions.

Now if we let $s(x)$ be one of the non-logarithmic solutions of equation (14) and substitute $y(x) = s(x)\bar{y}(x)$, the equation (14) is transformed into a new equation of the same type in $\bar{y}(x)$. Also the new equation must be satisfied by $\bar{y}(x) = 1$, since $y(x) = s(x)$ is a solution of the original equation. But a linear homogeneous equation,

$$(14')\qquad \bar{y}^{(n)} + \bar{a}_1 \bar{y}^{(n-1)} + \cdots + \bar{a}_n \bar{y} = 0,$$

can have the solution $\bar{y} =$ a constant only if $\bar{a}_n(x) = 0$, in which case the equation may be regarded as one of $(n-1)$-st order in $\bar{y}'$.

This equation of $(n-1)$-st order in $\bar{y}'$ will have a complete set of $n-1$ distinct solutions $\bar{s}_1(x), \bar{s}_2(x), \cdots, \bar{s}_{n-1}(x)$ of the type (2), which can be divided into sets of type (4), for this has been assumed the case for all equations (1′) of order less than n.

At this point we need the following

LEMMA. *An equation of the form*

$$y'(x) = s(x), \tag{15}$$

where $s(x)$ is a formal series of type (2), has always a formal solution of type (2), of normal form.[*]

The proof of this lemma is simple in the case when $s(x)$ is non-logarithmic. In this case we have $s(x) = e^{Q(x)}(b_{-r}x^{r/p} + b_{1-r}x^{(r-1)/p} + \cdots)$, where $b_{-r} \neq 0$, r is an integer, p is a positive integer, and $Q(x) = c_1x^{s/p} + \cdots + c_sx^{1/p}$, s being a positive integer. If, in particular, $Q(x) = 0$, then the solution

$$y(x) = [pb_{-r}/(r+p)]x^{(r+p)/p} + \cdots + pb_{p-1}x^{1/p} + b_p \log x + b_{p+1}x^{-1/p} + \cdots$$

can be obtained directly by term by term integration. Otherwise the equation (15) can be transformed by the substitution $y(x) = e^{Q(x)}\bar{y}(x)$ into

$$\bar{y}'(x) + Q'(x)\bar{y}(x) = b_{-r}x^{r/p} + b_{1-r}x^{(r-1)/p} + \cdots.$$

Then, on assuming a solution of the form $y(x) = y_{-m}x^{m/p} + y_{1-m}x^{(m-1)/p} + \cdots$ (m an integer; $y_{-m} \neq 0$) for this equation and substituting, we find that the left-hand member becomes

$$\sum_{j=m-p+s}^{-\infty} \left[\frac{p+j}{p} y_{-p-j} + \sum_{i=1}^{s} \frac{ic_{s-i+1}}{p} y_{-p-j+i}\right] x^{j/p},$$

where it is understood that $y_k = 0$ if $k < -m$. In particular, the leading term on the left is $(sc_1/p)y_{-m}x^{(m-p+s)/p}$, where s and c_1 are known and are different from zero, and y_{-m} is to be different from zero. In order that this term shall be equal to the leading term on the right we must have $m - p + s = r$ and $(sc_1/p)y_{-m} = b$. Hence $m = r + p - s$, $y_{-m} = pb_{-r}/sc_1$, and the first term of $y(x)$ is completely determined. The remaining coefficients y_{1-m}, $y_{2-m}, \cdots$ can now be found successively from the equations,

$$\frac{p+r-j}{p} y_{-p-r+j} + \sum_{i=1}^{s} \frac{ic_{s-i+1}}{p} y_{-p-r+j+i} = b_{j-r},$$

where j is taken equal to $1, 2, 3, \cdots$ successively. Thus a series $\bar{y}(x)$ can be found such that $y(x) = e^{Q(x)}\bar{y}(x)$ is a solution of the original equation $y'(x) = s(x)$.

---

[*] It is to be noted that the basic integer here is not necessarily the same as in the original equation (14).

It remains to consider the case in which $s(x)$ involves $\log x$. In this case the highest power of $\log x$ which occurs in the equation to be solved can be reduced step by step until we reach an equation of the simpler form just considered, in which $s(x)$ is non-logarithmic. In fact when $s(x)$ has the general form (2) it may be written as $s(x) = s_0(x)\log^k x + s_{k-1}(x)$, where $s_0(x)$ is free from logarithms and $s_{k-1}(x)$ involves at most the $(k-1)$-st power of $\log x$. Now consider the transformation

$$y(x) = \bar{s}_0(x)\log^k x + \bar{y}(x), \tag{16}$$

where $\bar{s}_0(x)$ is a solution of type (2) for the equation $y'(x) = s_0(x)$. Since $s_0(x)$ is non-logarithmic, such a solution is known to exist. Also it is clear from the actual computation of the preceding paragraph that this solution is non-logarithmic except when $s_0(x)$ contains a term in x^{-1}. By this transformation equation (15) becomes

$$\bar{y}'(x) = s_{k-1}(x) - kx^{-1}\bar{s}_0(x)\log^{k-1} x, \tag{17}$$

in which $\log x$ appears to the $(k-1)$-st power at most when $\bar{s}_0(x)$ is non-logarithmic. In the case when $s_0(x)$ contains a term Ax^{-1}, so that $\bar{s}_0(x)$ contains a term in $\log x$, we may let $s_0(x) = Ax^{-1} + s_{00}(x)$ and use instead of (16) the transformation

$$y(x) = \{[A/(k+1)]\log x + \bar{s}_{00}(x)\}\log x + \bar{y}(x), \tag{18}$$

where $\bar{s}_{00}(x)$ is a solution (of type (2)) of the equation $y'(x) = s_{00}(x)$ and is therefore non-logarithmic. This transforms equation (15) into

$$\bar{y}'(x) = s_{k-1}(x) - kx^{-1}s_{00}(x)\log^{k-1} x, \tag{19}$$

and, since $\bar{s}_{00}(x)$ is non-logarithmic, this equation involves at most the $(k-1)$-st power of $\log x$. Consequently we can always, by a transformation of the form (16) or (18), reduce by at least one the highest power of $\log x$ which occurs, and by repeating the process at most k times we can obtain an equation which has a non-logarithmic right-hand term and hence a solution of the desired type. From the form of (16) and (18) it is clear that this leads to a solution of the desired type for the original equation (15). This completes the proof of the lemma.

By following through the details of the transformations used in proving the lemma above it can be seen that each subset (of the type (4)) of solutions of the equation of order $n-1$ in $\bar{y}'(x)$ leads to a similar subset for the equation of order n in $\bar{y}(x)$. In some cases the solution $\bar{y}(x) = C$ must be included in one of these subsets; in some cases it will constitute a distinct additional subset. In any case the solutions of the equation in $\bar{y}(x)$ are of the desired type and the complete set of solutions is made up of subsets of the type (4). The solutions of the original equation in $y(x)$ are obtained from these by multiplying them by the non-logarithmic factor $s(x)$, and hence are also of the required type and can be grouped in subsets of the type (4). Consequently the assumption made at the beginning of this section is untenable, and the second part of our proof is complete.

3. 3. Proof of Part 3. Part 3 is also proved indirectly. If this statement is not true, then we may assume that every equation (1′) of order greater than 1 is reducible, and also that there exist equations of this form which have no non-logarithmic solutions. In this case there is a least order n for which such equations exist and, as a consequence of part (1) of this proof, n is at least 2. Then let $L(y) = 0$ be such an equation, of order n. Since n is greater than 1, it follows from the first of our two assumptions that this equation is reducible, that is, that $L(y) \equiv M_{n-r}(L_r(y))$, where the equations, $M_{n-r}(y) = 0$ and $L_r(y) = 0$ are also of form (1′) and of order less than n. Any solution of $L_r(y) = 0$ is then a solution of $L(y) = 0$ also. But the equation $L_r(y) = 0$ has at least one non-logarithmic solution, since r is less than n and n was taken as the least order for which an equation of this form could fail to have a non-logarithmic solution. This contradicts the assumption that $L(y) = 0$ has no non-logarithmic solution, and hence proves the statement (3).

3. 4. Proof of Part 4. It remains now to show that every equation of the form (1′) and of order greater than 1 is reducible if the basic integer p is properly chosen. This will be done by considering three special cases and then showing that any equation (1′) can be brought under one or another of these cases by a suitable transformation. First, however, we observe that if $a_n(x) = 0$ a factorization $L(y) \equiv M_{n-1}(L_1(y))$ with $L_1(y) \equiv y'$ is always possible, and exclude this case from the discussion which follows.

The equation to be considered then is

$$L(y) \equiv y^{(n)}(x) + a_1(x)\, y^{(n-1)}(x) + \cdots + a_n(x) y(x) = 0 \quad (a_n(x) \neq 0),$$

where $a_i(x) = a_{i,-\sigma_i} x^{\sigma_i/p} + a_{i,1-\sigma_i} x^{(\sigma_i-1)/p} + \cdots \qquad (i = 1, 2, \cdots, n)$,

σ_i being an integer, and $a_{i,-\sigma_i} \neq 0$.

The three special cases to be considered are characterized by the following conditions:

Case I. $\sigma_i/i \leqq -p$ $(i = 1, 2, \cdots, n)$,

Case II. For some positive integer r, less than n,

$$\sigma_r/r > -p, \text{ and } \sigma_i/i \begin{cases} \leqq \sigma_r/r & (i = 1, 2, \cdots, r), \\ < \sigma_r/r & (i = r+1, r+2, \cdots, n). \end{cases}$$

Case III. $\sigma_n/n > -p$, $\sigma_i/i \leqq \sigma_n/n$ $(i = 1, 2, \cdots, n)$,

and the equation $\rho^n + a_{1,-\sigma_n/n}\rho^{n-1} + \cdots + a_{n,-\sigma_n} = 0$ has at least two distinct roots.

3.41. Proof of reducibility in Case I. In this case the equation has a regular singular point at $x = \infty$, and a solution of the form

$$g(x) = x^r(g_0 + g_1 x^{-1/p} + g_2 x^{-2/p} + \cdots) \qquad (g_0 \neq 0)$$

can always be found.

In fact the equation can be written in the form

$$(20) \qquad x^n y^{(n)}(x) + x^{n-1} b_1(x) y^{(n-1)}(x) + \cdots + b_n(x) y(x) = 0,$$

where $b_i(x) = b_{i0} + b_{i1} x^{-1/p} + \cdots$ $(i = 1, 2, \cdots, n)$. On substituting the series for $g(x)$ in this equation we have

$$g_0[r(r-1) \cdots (r-n+1) + \sum_{i=1}^{n-1} b_{i0} r(r-1) \cdots (r-n+i+1) + b_{n0}]$$

as the coefficient of the leading term. This is a polynomial of degree n in r and may be denoted by $F(r)$. Then the coefficient of $x^{r-k/p}$ can be written in the form

$$g_k F(r-k/p) + \sum_{l=0}^{k-1} g_l \, [\sum_{i=1}^{n-1} b_{i,k-l}(r-l/p) \cdots (r-l/p-n+i+1) + b_{n,k-l}].$$

If we choose for r the least root of the equation $F(r) = 0$, then $F(r-k/p)$ does not vanish for any positive value of k and the condition that the coefficient of $x^{r-k/p}$ shall be zero for $k = 1, 2, 3, \cdots$ gives us equations which can be solved successively for the ratios g_1/g_0, g_2/g_0, $\cdots$. Thus it is always possible to find a solution $g(x)$ of the equation (20) or (1′).

The differential expression $y'(x) - g'(x)y(x)/g(x)$ is of the same type as $L(y)$, and if it is denoted by $L_1(y)$ then $L(y)$ can be expressed as

$$L(y(x)) \equiv M_{n-1}(L_1(y(x))) + G(x)y(x)$$

where M_{n-1} is a differential expression of the required type, of order $n-1$, and $G(x)$ is a formal series. Then since the equations $L(y)=0$ and $L_1(y)=0$ are both satisfied by $y(x)=g(x)$, we must have $G(x)=0$, and the equation $L(y)=0$ is reducible.

3.42. Proof of reducibility in Case II. In this case, for some positive integer r less than n, the following conditions are satisfied:

$$\left\{\begin{array}{ll} \text{(i)} & -p<\sigma_r/r, \\ \text{(ii)} & \sigma_i/i \leqq \sigma_r/r \text{ for } i<r, \\ \text{(iii)} & \sigma_i/i<\sigma_r/r \text{ for } i>r. \end{array}\right. \tag{21}$$

We let

$$L_r(y) \equiv y^{(r)}(x)+p_1(x)y^{(r-1)}(x)+\cdots+p_r(x)y(x) \tag{22a}$$

$$M_{n-r}(y) \equiv y^{(n-r)}(x)+\lambda_1(x)y^{n-r-1}(x)+\cdots+\lambda_{n-r}(x)y(x), \tag{22b}$$

where the coefficients $p_i(x)$ and $\lambda_i(x)$ are required to be of the same form as the original $a_i(x)$, but are otherwise arbitrary. Then we wish to show that these coefficients can be determined so as to make the symbolic product $M_{n-r}(L_r(y))$ identically equal to $L(y)$.

From equation (22a) we have, by successive differentiation,

$$L_r{}^{(k)}(y) \equiv \sum_{j=0}^{r+k}\{p_j+kp'_{j-1}+\cdots+[k!/(k-l)!\,l!]p_{j-l}^{(l)}+\cdots+p_{j-k}^{(k)}y^{(r+k-j)}\},$$

where it is understood that $p_0(x)=1$, and $p_i(x)=0$ if i is negative or is greater than r.

On substituting these values in the identity

$$M_{n-r}(L_r(y))=L_r{}^{(n-r)}(y)+\lambda_1L_{(r)}^{(n-r-1)}(y)+\cdots+\lambda_{n-r}L_r(y),$$

we reduce this to the form

$$M_{n-r}(L_r(y))=y^{(n)}+A_1y^{(n-1)}+\cdots+A_ny,$$

where

$$A_i=p_i+\text{linear terms in } p'_{i-1}, p''_{i-2},\cdots,p_{i-n+r}^{(n-r)}$$

$$+\sum_{l=1}^{i-1}\lambda_l[p_{i-l}+\text{linear terms in } p'_{i-l-1},\cdots,p_{i-n+r}^{(n-r-l)}]+\lambda_i$$

$$(i=1,2,\cdots,n),$$

where it is understood that $\lambda_0(x)=1$, and $\lambda_i(x)=0$ if i is negative or greater than $n-r$. Then, in order that $L(y)\equiv M_{n-r}(L_r(y))$, it is necessary that $a_i(x)=A_i(x)$ $(i=1,2,\cdots,n)$. This gives us n equations of the form

$$(23)\qquad \begin{aligned} a_i(x) = p_i &+ \text{linear terms in } p'_{i-1}, \cdots, p^{(n-r)}_{i-n+r} \\ &+ \sum_{l=1}^{i-1} \lambda_l [p_{i-l} + \text{linear terms in } p'_{i-l-1}, \cdots, p^{(n-r-l)}_{i-n+r}] + \lambda_i \\ &\qquad\qquad (i = 1, 2, \cdots, n) \end{aligned}$$

to be satisfied formally. This condition can always be satisfied by a proper choice of the coefficients p_{ik} and λ_{ik}, where we denote by p_{ik} the coefficient of $x^{-k/p}$ in the series $p_i(x)$, and by λ_{ik} the coefficient of $x^{-k/p}$ in the series $\lambda_i(x)$. The determination of these coefficients is as follows:

Denote by e_i the greatest integer not greater than $i\sigma_r/r$, and take

$$(24)\qquad \lambda_{ik} = 0, \text{ if } k \leqq -e_i; \qquad p_{ik} = \begin{cases} 0, & \text{if } k < -e_i, \\ a_{ik}, & \text{if } k = -e_i. \end{cases}$$

Then the leading terms in the right-hand member of the equation are

$$p_{i,-e_i} x^{e_i/p} + \sum_{l=1}^{i-1} \lambda_{l,1-e_l} p_{i-l,-e_{i-l}} x^{(e_{i-l}+e_l-1)/p} + \lambda_{i,1-e_i} x^{(e_i-1)/p},$$

but from the definition of e_i it is readily seen that $(e_{i-l} + e_l - 1)$ is less than e_i, so that the highest power of $x^{1/p}$ which occurs on the right of equation (23) is the e_i-th and its coefficient is $p_{i,-e_i} = a_{i,-e_i}$. On the other hand the conditions (21 ii) and (21 iii) ensure that the leading term of $a_i(x)$ is of at most the e_i-th degree in $x^{1/p}$, and its coefficient is $a_{i,-e_i}$. Thus conditions (24) ensure that the leading terms in the right and left members of equation (23) are the same. The requirement that the coefficient of $x^{(e_i-k)/p}$ shall be the same on both sides of equation (23) gives us equations of the form

$$(25)\qquad a_{i,k-e_i} = p_{i,k-e_i} + \sum_{l=1}^{i-1} \lambda_{l,k-e_l} p_{i-l,e_l-e_i} + \lambda_{i,k-e_i} + F_{ik} \quad (i = 1, 2, \cdots, n),$$

where F_{ik} is a known polynomial in the coefficients

$$(26)\qquad \begin{cases} \lambda_{i,1-e_i}, \lambda_{i,2-e_i}, \cdots, \lambda_{i,k-1-e_i} & (i = 1, 2, \cdots, n-r), \\ p_{i,1-e_i}, p_{i,2-e_i}, \cdots, p_{i,k-1-e_i} & (i = 1, 2, \cdots, r). \end{cases}$$

Since $e_l - e_i$ is less than $k - e_i$, the factor p_{i-l,e_l-e_i} is one of the set (26), so that for each value of k we have a set of n equations linear in the n quantities

$$(27)\qquad p_{1,k-e_1}, p_{2,k-e_2}, \cdots, p_{r,k-e_r};\ \lambda_{1,k-e_1}, \cdots, \lambda_{n-r,k-e_{n-r}},$$

and by taking $k = 1, 2, \cdots$ successively we can solve the corresponding sets of equations successively for $p_{i,k-e_i}$ and $\lambda_{i,k-e_i}$ in terms of the previously determined coefficients **(26).**

To show that the set of equations (25) actually determines the n coefficients (27) uniquely in terms of the coefficients (26) we need only inspect this set of equations somewhat more closely. They may be written as follows:

$$\begin{aligned}
\phi_{1k} &= p_{1,k-e_1}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\phi_{rk} &= p_{r,k-e_r} + \lambda_{1,k-e_1} p_{r-1,e_1-e_r} + \cdots + \lambda_{r-1,k-e_{r-1}} p_{1,e_{r-1}-e_r} + \lambda_{r,k-e_r}, \\
\phi_{r+1,k} &= \lambda_{1,k-e_1} p_{r,e_1-e_{r+1}} + \cdots + \lambda_{r,k-e_r} p_{1,e_r-e_{r+1}} + \lambda_{r+1,k-e_{r+1}}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\phi_{nk} &= \lambda_{n-r,k-e_{n-r}} p_{r,e_{n-r}-e_n},
\end{aligned}$$

where the left-hand terms are known. From the definition of e_i it follows that $e_i - e_{r+i} = -\sigma_r$ $(i = 1, 2, \cdots, n-r)$, so that $p_{r,e_i-e_{r+i}} = p_{r,-\sigma_r} = a_{r,-\sigma_r} \neq 0$, and the last $n-r$ equations can be solved successively, beginning with the last, for $\lambda_{n-r,k-e_{n-r}}, \cdots, \lambda_{1,k-e_1}$. Then $p_{1,k-e_1}, \cdots, p_{r,k-e_r}$ obviously can be determined successively from the first r equations of the set.

Thus the series $p_i(x)$ and $\lambda_i(x)$ can always be determined so that $a_i(x) = A_i(x)$, and hence $L(y) \equiv M_{n-r}(L_r(y))$, and the reducibility of equation (1′) in Case II is established.

3. 43. Proof of reducibility in Case III. In this case we have

$$(28) \qquad \left\{ \begin{array}{ll} \text{(i)} & \sigma_i/i \leq \sigma_n/n \text{ for all values of } i, \\ \text{(ii)} & -p < \sigma_n/n, \\ \text{(iii)} & \text{at least two distinct roots of the equation} \end{array} \right.$$

$$(29) \qquad \rho^n + a_{1,-\sigma_n/n}\rho^{n-1} + \cdots + a_{i,-i\sigma_n/n}\rho^{n-i} + \cdots + a_{n,-\sigma_n} = 0.$$

Here it is assumed, as indicated by the notation $a_{i,-i\sigma_n/n}$ in the above equation, that the basic integer p in the equation (1′) has been so chosen that σ_n/n is an integer. This is always possible.

We let ρ_1 be a root of equation (29), of multiplicity r. Then the equation can be put in the form

$$(29') \qquad (\rho - \rho_1)^r(\rho^{n-r} + \alpha_1\rho^{n-r-1} + \cdots + \alpha_i\rho^{n-r-i} + \cdots + \alpha_{n-r}) = 0,$$

where $\rho_1^{n-r} + \alpha_1\rho_1^{n-r-1} + \cdots + \alpha_{n-r} \neq 0$, and, since $a_{n,-\sigma_n} \neq 0$, ρ_1 and α_{n-r} are both different from zero.

Then we define $L_r(y)$ and $M_{n-r}(y)$ by equations (22) as in Case II, except that r is now the multiplicity of the root ρ_1. Then, as in Case II, we have to satisfy equations (23). This time, however, we let the leading term of $p_i(x)$ (or $\lambda_i(x)$) be of degree $i\sigma_n/n$ in $x^{1/p}$, that is, let

$$(30)\qquad p_{ik} = \lambda_{ik} = 0 \quad \text{for} \quad k < -i\sigma_n/n,$$

and take

$$(31)\qquad \begin{cases} \lambda_{i,-i\sigma_n/n} = \alpha_i, \\ p_{i,-i\sigma_n/n} = \text{the } (i+1)\text{-th coefficient in the expansion of } (\rho - \rho_1)^r. \end{cases}$$

This gives us, as the leading term in the right-hand member of equation (23), $[p_{i,-ig} + \sum_{l=1}^{i-1} \alpha_l p_{i-l,(l-i)g} + \alpha_i] x^{ig/p}$, where $g = \sigma_n/n$. The coefficient, enclosed in brackets, is precisely the coefficient of ρ^{n-i} in the left-hand member of equation (29′), which is equal to $a_{i,-i\sigma_n/n}$, since (29′) is identical with (29). Hence the leading terms of the right and left members of equation (23) have been made the same by conditions (30) and (31).

As in Case II, the remaining coefficients

$$p_{i,k-ig} \ (i = 1, 2, \cdots, r;\ k = 1, 2, \cdots),\ \lambda_{i,k-ig} \ (i = 1, 2, \cdots, n-r;\ k = 1, 2, \cdots)$$

can be determined for successive values of k from the equations obtained by requiring that the coefficient of $x^{(ig-k)/p}$ be the same in both members of equation (23). In this case these equations have the form

$$(32)\qquad a_{i,k-ig} = p_{i,k-ig} + \sum_{l=1}^{i-1} (\lambda_{l,-lg} p_{i-l,k-(i-l)g} + \lambda_{l,k-lg} p_{i-l,(l-i)g}) + \lambda_{i,k-ig} + G_{ik}$$
$$(i = 1, 2, \cdots, n),$$

where G_{ik} is a known polynomial in the coefficients

$$\begin{array}{ll} \lambda_{i,-ig}, \lambda_{i,1-ig}, \cdots, \lambda_{i,k-1-ig} & (i = 1, 2, \cdots; n-r); \\ p_{i,-ig}, p_{i,1-ig}, \cdots, p_{i,k-1-ig} & (i = 1, 2, \cdots, r). \end{array}$$

These n equations, linear in the n quantities

$$(33)\qquad p_{i,k-ig} \ (i = 1, 2, \cdots, r), \qquad \lambda_{i,k-ig} \ (i = 1, 2, \cdots, n-r)$$

are the same as the conditions imposed on these quantities by requiring that the identity in u,

$$\sum_{i=1}^{n} \psi_{ik} u^{n-i} \equiv A_r(u) C_{n-r-1}(u) + B_{n-r}(u) D_{r-1}(u),$$

exist, where $\psi_{ik} = a_{i,k-ig} - G_{ik}$, and $A_r(u)$ and $B_{n-r}(u)$ are the known polynomials

$$A_r(u) \equiv u^r + \cdots + p_{i,-ig} u^{r-i} + \cdots + p_{r,-rg},$$
$$B_{n-r}(u) \equiv u^{n-r} + \cdots + \lambda_{i,-ig} u^{n-r-i} + \cdots + \lambda_{n-r,(r-n)g},$$

and $C_{n-r-1}(u)$ and $D_{r-1}(u)$ are the polynomials

$$
\begin{aligned}
C_{n-r-1}(u) &\equiv \lambda_{1,k-g}u^{n-r-1} + \cdots + \lambda_{i,k-ig}u^{n-r-i} + \cdots + \lambda_{n-r,k-(n-r)g}\\
D_{r-1}(u) &\equiv p_{1,k-g}u^{r-1} + \cdots + p_{i,k-ig}u^{r-i} + \cdots + p_{r,k-rg},
\end{aligned}
$$

in which the coefficients are to be determined. The known polynomials $A_r(\rho)$ and $B_{n-r}(\rho)$ are, as a consequence of the choice of $p_{i,-ig}$ and $\lambda_{i,-ig}$ (equations (31)), the first and second factors respectively of the left-hand term of equation (29′), and hence A_r and B_{n-r} are relatively prime. Consequently we may apply the theorem of algebra that if P_n, A_r, B_{n-r} are known polynomials of degrees indicated by the subscripts, and A_r and B_{n-r} are relatively prime, then there exists one, and only one, pair of polynomials C_{n-r-1} and D_{r-1}, of degrees $n-r-1$ and $r-1$ respectively, which satisfy the identity

$$P_n \equiv A_r C_{n-r-1} + B_{n-r}D_{r-1}.$$

This ensures the existence of a unique set of coefficients (33) which satisfy the equations (32).

Thus by taking $k = 1, 2, 3, \cdots$ successively, it is always possible to determine the sets of coefficients,

$$
\begin{array}{ll}
\lambda_{i,1-ig} \ (i = 1, 2, \cdots, n-r), & p_{i,1-ig} \ (i = 1, 2, \cdots, r)\\
\lambda_{i,2-ig} \ (i = 1, 2, \cdots, n-r), & p_{i,2-ig} \ (i = 1, 2, \cdots, r)\\
\cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}
$$

successively, so that equations (23) are satisfied, that is, so that

$$L(y) \equiv M_{n-r}(L_r(y)),$$

and the reducibility of equation (1′) is established in Case III.

3. 44. Conclusion of proof of Part 4. Now any equation (1′) which does not belong to Case I, II, or III must satisfy the following conditions:

$$
(34) \quad \left\{
\begin{array}{lll}
\text{(i)} & \sigma_i/i \leqq \sigma_n/n & (i = 1, 2, \cdots, n),\\
\text{(ii)} & -p < \sigma_n/n, & \\
\text{(iii)} & \rho^n + a_{1,-\sigma_n/n}\rho^{n-1} + \cdots + a_{n,-\sigma_n} = (\rho - \rho_1)^n & (\rho_1 \neq 0).
\end{array}
\right.
$$

This requires that $a_{i,-i\sigma_n/n}$ $(i = 1, 2, \cdots, n)$ be different from zero, that is, that $\sigma_i/i = \sigma_n/n$ $(i = 1, 2, \cdots, n-1)$, and, in particular $\sigma_1 = \sigma_n/n$, and also that $n\rho_1 = -a_{1,-\sigma_n/n} = -a_{1,-\sigma_1}$.

In this case we substitute

$$(35) \qquad y = e^{Q(x)}\bar{y},$$

where $Q(x) = p\rho_1 x^{(p+\sigma_1)/p}/(p+\rho_1)$. The equation (1′) will then be transformed into an equation of the same type in the new variable $\bar{y}$, which we may denote by

$$\bar{L}(\bar{y}) \equiv \bar{y}^{(n)} + \bar{a}_1 \bar{y}^{(n-1)} + \cdots + \bar{a}_n \bar{y} = 0, \tag{36}$$

where $\bar{a}_i(x) = \bar{a}_{i,-\bar{\sigma}_i} x^{\bar{\sigma}_i/p} + \cdots$, and, in particular, $\bar{a}_1(x) = nQ'(x) + a_1(x)$. Moreover a sufficient condition for the reducibility of the original equation (1′) is the reducibility of the transformed equation (36).

Consequently if equation (36) comes under Case I, II, or III the equation (1′) is reducible. Otherwise equation (36) must satisfy conditions of the form (34) and another transformation, $\bar{y} = e^{\bar{Q}(x)}\bar{\bar{y}}$, leads to another equation of the same type, which is reducible only if equation (36) is reducible. The transformation (35) however, makes $\bar{\sigma}_1 \leqq \sigma_1 - 1$, since from the equations

$$\bar{a}_1(x) = nQ'(x) + a_1(x), \qquad Q'(x) = \rho_1 x^{\sigma_1/p}, \qquad n\rho_1 = -a_{1,-\sigma_1},$$

we have

$$\bar{a}_1(x) = n\rho_1 x^{\sigma_1/p} - (a_{1,-\sigma_1} x^{\sigma_1/p} + a_{1,1-\sigma_1} x^{(\sigma_1-1)/p} + \cdots) = a_{1,1-\sigma_1} x^{(\sigma_1-1)/p} + \cdots.$$

Hence if the equation (1′) does not belong to Case I, II, or III, we must have $\bar{\sigma}_n/n \leqq (\sigma_n - 1)/n$, and after a second transformation, if the new equation in $\bar{\bar{y}}$ does not belong to Case I, II, or III, we must have $\bar{\bar{\sigma}}_n/n \leqq (\sigma_n - 2)/n$. Thus as long as the equation does not fall under Case I, II, or III, it may be transformed in this way, so that the value of the ratio σ_n/n is each time decreased by at least $1/n$. This can occur only a finite number of times before we have an equation in which $\sigma_n/n < -p$, which necessarily belongs to Case I, and is therefore reducible. Consequently we can always transform the original equation into one which is reducible by a finite number of transformations of type (35), and the reducibility of the original equation follows from the reducibility of the new one. This completes the proof of

Theorem III. *Every equation of the form* (1′) *is reducible if the basic integer is properly chosen.*

This is the last step of the proof of Theorem I, for we have shown that the reducibility of all equations of the type (1′) insures the existence of a non-logarithmic solution, and hence of a complete set of solutions, for each such equation.

4. Properties of the complete set of solutions. In the first place we observe that, although the formal solutions (6) may be multiple-valued and each one may go over into another one when the point x in the complex x-plane describes a closed curve encircling the origin p times, nevertheless the linear dependence or independence of any set of solutions remains unaltered. This follows from the fact that the equation (9), by means of which linear dependence is defined, is an identity in x.

Next we shall verify, for the case of formal solutions of the type (6), the two following theorems:

THEOREM IV. *A necessary and sufficient condition for the linear dependence of the set of formal solutions* $y_1(x), \cdots, y_n(x)$ *of a linear homogeneous differential equation of the form* (1) *is that their Wronskian, the determinant* $\Delta(y_1, \cdots, y_n) \equiv | y_1, \cdots, y_n^{(n-1)} |$ *be formally equal to zero.*

THEOREM V. *Any formal solution of a linear differential equation of the form* (1) *can be expressed as a linear combination, with constant coefficients, of any set of n linearly independent formal solutions.*

We first prove the condition, $\Delta(y_1, y_2, \cdots, y_n) = 0$, to be necessary for the linear dependence of the set $y_1, y_2, \cdots, y_n$. If there exist constants $F_1, F_2, \cdots, F_n$, not all zero, such that $F_1y_1 + F_2y_2 + \cdots + F_ny_n = 0$, then we have

$$F_1y_1^{(i)} + F_2y_2^{(i)} + \cdots + F_ny_n^{(i)} = 0 \qquad (i = 0, 1, \cdots, n-1).$$

This set of equations can be satisfied by a set of constants not all zero only if the determinant of the coefficients of $F_1, F_2, \cdots, F_n$ is zero, and this determinant is precisely $\Delta(y_1, y_2, \cdots, y_n)$.

If the set $y_1, y_2, \cdots, y_{n-1}$ is of the type described in Theorem I, that is, if $y_1, y_2, \cdots, y_{n-1}$ are elementary formal solutions and the set is made up entirely of subsets of the form (4), the sufficiency of the condition can be proved by much the same method as is used for analytic solutions. In this case we denote by Δ_{ij} the cofactor of the element in the i-th row and j-th column of Δ. Then it can be shown * that Δ_{nn} is of the non-logarithmic form (7) for which division has been defined. Consequently, if $\Delta = 0$ and $\Delta_{nn} \neq 0$, we have

$$(37) \qquad y_n^{(n-1)} = -[(\Delta_{n1}/\Delta_{nn})y_1^{(n-1)} + \cdots + (\Delta_{n,n-1}/\Delta_{nn})y_{n-1}^{(n-1)}]$$

from the expanded form $\Delta \equiv y_1^{(n-1)}\Delta_{n1} + \cdots + y_{n-1}^{(n-1)}\Delta_{n,n-1} + y_n^{(n-1)}\Delta_{nn}$ of the determinant. Then if $F_1, F_2, \cdots, F_{n-1}$ are required to be expressions of the general form (6) and subject to the conditions

---

* By the method used in Part II, § 5, to prove equation (40) free from logarithms and exponential factors.

$$(38) \qquad y_n^{(i)} = -[F_1 y_1^{(i)} + \cdots + F_{n-1} y_{n-1}^{(i)}] \qquad (i = 0, 1, \cdots, n-2),$$

we have, solving the set of simultaneous equations (38) by Cramer's rule, $F_i = \Delta_{ni}/\Delta_{nn}$ $(i = 1, 2, \cdots, n-1)$, so that equation (37) becomes

$$(37') \qquad y_n^{(n-1)} = -[F_1 y_1^{(n-1)} + \cdots + F_{n-1} y_{n-1}^{(n-1)}].$$

On differentiating each equation of the set (38) and subtracting the result for the i-th equation from the $(i-1)$-st equation (considering equation (37′) as the last of the set) we have

$$0 = F'_1 y_1^{(i)} + \cdots + F'_{n-1} y_{n-1}^{(i)} \qquad (i = 0, 1, \cdots, n-2).$$

Then this set of simultaneous linear equations in the variables $F'_1, F'_2, \cdots, F'_{n-1}$ has the unique solution $F'_i = 0$ $(i = 1, 2, \cdots, n-1)$, since the determinant of the coefficients is Δ_{nn}.

We now need the

LEMMA. *If $y(x)$ is of the form (6), and $y'(x) = 0$, then $y(x)$ is a constant.*

The proof for the special case in which $y(x)$ is an elementary non-logarithmic solution is obtained from the computation of § 3.1 by taking $a(x) = 0$ in the equation (10). If $y(x)$ is of the general form $y(x) = S_1(x) + \cdots + S_q(x)$, where $S_1(x), \cdots, S_q(x)$ are of type (2) and each involve a different exponential factor, then $y'(x) = S'_1(x) + \cdots + S'_q(x)$, and $y'(x)$ can be zero only if $S'_1(x) = \cdots = S'_q(x) = 0$. But if

$$S_i(x) = s_0(x) \log^k x + s_1(x) \log^{k-1} x + \cdots + s_k(x)$$

where $s_0(x), \cdots, s_k(x)$ are of elementary non-logarithmic form, then

$$S'_i(x) = s'_0(x) \log^k x + \cdots + [s'_{j+1}(x) + (k-j) x^{-1} s_j(x)] \log^{k-j-1} x + \cdots + [s'_k(x) + x^{-1} s_{k-1}(x)].$$

Then $S'_i(x)$ can be zero only if $s'_0(x) = 0$, that is, since $s_0(x)$ is non-logarithmic, only if $s_0(x)$ is a constant c_0. In this case $s'_1(x) = -c_0 k x^{-1}$, but this equation can be satisfied by a formal series of type (3) only if $c_0 = 0$, in which case we have $s_1(x)$ equal to a constant. Thus $s_0(x), \cdots, s_{k-1}(x)$ successively can be proved equal to zero, so that $S_i(x)$ must be non-logarithmic and a constant, and the lemma is proved for the general case.

By applying this lemma we find that the coefficients $F_i(x)$ are constants, so that the first of the equations (38) establishes the linear dependence of the set of solutions $y_1, \cdots, y_n$.

If the cofactor Δ_{nn} is zero, we observe that it is precisely the Wronskian of $y_1, \cdots, y_{n-1}$ and hence, by the above argument, this set of solutions is linearly dependent unless the first principal minor of Δ_{nn} is zero. By repeating this argument we work back until we reach a set of k $(< n)$ solutions which is linearly dependent. Then, since this subset $y_1, \cdots, y_k$ is linearly dependent, so is the original set $y_1, \cdots, y_n$.

In order to establish the sufficiency of the condition when $y_1, \cdots, y_n$ is any set of solutions of the form (6), we need to use Theorem V for the special case in which the given set of distinct solutions consists (with the possible exception of one solution) of subsets, of type (4), of elementary solutions. We therefore proceed to prove Theorem V for this special case.

Let $\bar{y}_{n+1}$ be any solution of the differential equation (1) and let $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ be a set of linearly independent solutions made up of subsets, of the form (4), of elementary solutions. Then the $n+1$ equations $a_0\bar{y}_i^{(n)} + a_1\bar{y}_i^{(n-1)} + \cdots + a_n\bar{y}_i = 0$ $(i = 1, 2, \cdots, n+1)$ must be satisfied simultaneously. Since these equations are linear in $a_0, a_1, \cdots, a_n$, they can have a set of solutions other than $0, 0, \cdots, 0$ only if the determinant of the coefficients is equal to zero. This determinant is simply the Wronskian of the set $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n, \bar{y}_{n+1}$, and this set is of the special type for which Theorem IV has already been proved. Hence this set is linearly dependent, that is, there exists a relation $F_1\bar{y}_1 + \cdots + F_n\bar{y}_n + F_{n+1}\bar{y}_{n+1} = 0$, in which the $F_1, \cdots, F_{n+1}$ are constants, not all zero, and F_{n+1} is certainly not zero, since $\bar{y}_1, \cdots, \bar{y}_n$ are linearly independent. Consequently we have $\bar{y}_{n+1} = (1/F_{n+1})[F_1\bar{y}_1 + \cdots + F_n\bar{y}_n]$.

Then any set of solutions $y_1, y_2, \cdots, y_n$ of equation (1) can be expressed as $y_i = \sum_{j=1}^{n} c_{ij}\bar{y}_j$ $(i = 1, 2, \cdots, n)$, the coefficients c_{ij} being constants and $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ being a linearly independent set of elementary solutions which can be grouped in subsets of type (4). Then

$$\Delta(y_1, \cdots, y_n) = | y_i^{(j)} | = | \sum_{k=1}^{n} c_{ik}\bar{y}_k^{(j)} | = | c_{ij} | \cdot | \bar{y}_i^{(j)} | = | c_{ij} | \, \Delta(\bar{y}_1, \cdots, \bar{y}_n),$$

so that if $\Delta(y_1, \cdots, y_n) = 0$ we must have $| c_{ij} | = 0$, and hence, from the equations $y_i = \sum_{j=1}^{n} c_{ij}\bar{y}_j$ $(i = 1, 2, \cdots, n)$, the set $y_1, y_2, \cdots, y_n$ must be linearly dependent.

Now that the proof of Theorem IV has been completed, the proof given for Theorem V in the special case applies in the general case also.

APPENDIX.

Note on Formal Solutions of Linear Difference Equations.

The results of § 2 apply, with only slight modifications, to the formal solutions of the linear homogeneous *difference* equation discussed by Birkhoff,* and have been implicitly assumed by him. The formal solutions of such an equation are of the form (6), except that the exponential factor $e^{Q(x)}$ is replaced by $x^{\mu x}e^{Q(x)}$, μ being a constant. We therefore replace the non-logarithmic expressions of type (5) by expressions of the form

$$(5') \qquad a_1(x)e_1(x) + \cdots + a_m(x)e_m(x),$$

in which the factors $e_1(x), \cdots, e_m(x)$ are of the form $e_i(x) = x^{\mu_i x}e^{Q_i(x)}$ $(i = 1, 2, \cdots, m)$ and are all different, and the factors $a_1(x), \cdots, a_m(x)$ are formal series. The most general formal expressions to be considered then are of the type

$$(6') \qquad S_0(x)\log^k x + S_1(x)\log^{k-1} x + \cdots + S_k(x),$$

where $S_0(x), \cdots, S_k(x)$ are of the form (5′). The definition of linear dependence (Definition 6) is modified by allowing the coefficients $F_1, \cdots, F_m$ in equation (9) to be of the form

$$(a) \qquad F_i = c_i e^{2\pi k_i \sqrt{-1}\, x},$$

where $c_1, \cdots, c_m$ are constants, not all zero, and $k_1, \cdots, k_m$ are integers. Definition 5 is replaced by

DEFINITION 5′. *If $a(x)$ is the formal series*

$$a_{-r}x^{r/p} + a_{1-r}x^{(r-1)/p} + \cdots,$$

then $a(x+n)$, n being any integer, is the formal series obtained by expanding each term of the series

$$a_{-r}(x+n)^{r/p} + a_{1-r}(x+n)^{(r-1)/p} + \cdots$$

as a formal series by the binomial theorem and combining terms in like powers of $x^{1/p}$.

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* *Loc. cit.*

If $S(x)$ is an expression of the non-logarithmic form (5′), *then $S(x+n)$, n being an integer, is the expression, of the form* (5′), *obtained from*

$$\sum_{i=1}^{m} a_i(x+n)e_i(x+n)$$

by expanding the factors $e_i(x+n)$ in the form (5′) *by the rules which apply when these factors are regarded as analytic functions, and expanding the factors $a_i(x+n)$ as formal series according to the preceding paragraph.*

If $U(x)$ is an expression of the general form (6′), *then $U(x+n)$ is the expression, of the form* (6′), *obtained from*

$$\sum_{i=0}^{k} S_i(x+n)\log^{k-i}(x+n)$$

by expanding the coefficients $S_i(x+n)$ in the form (5′) *according to the preceding paragraph and taking*

$$\log(x+n) = \log x + n/x - \tfrac{1}{2}(n/x)^2 + \cdots.$$

With these generalizations of the definitions, the ordinary formal algebraic laws are still valid, and formal equations in expressions of the form (6′) are to be regarded as identities in x. Theorems B-E of § 2 remain unchanged and Theorems F, G, IV, and V are replaced by the following:

THEOREM F′. *If $y_1(x), y_2(x), \cdots, y_n(x)$ are solutions of a linear homogeneous difference equation, then any linear combination of them with coefficients of the form* (*a*), *i. e., any expression of the form* (6′) *which is linearly dependent on the set $y_1, \cdots, y_n$, is also a solution of the equation.*

THEOREM G′. *If $Y(x)$ is a particular solution of a non-homogeneous linear difference equation $L(y) = r(x)$, and $y(x)$ is the general solution of the corresponding homogeneous equation $L(y) = 0$, then $y(x) + Y(x)$ is the general solution of the non-homogeneous equation.*

THEOREM IV′. *A necessary and sufficient condition for the linear dependence of the n formal solutions $y_1, \cdots, y_n$ of a homogeneous linear difference equation of the n-th order is that the determinant of Casorati, $D(y_1, \cdots, y_n) \equiv | y_1(x), y_2(x+1), \cdots, y_n(x+n-1) |$, be formally equal to zero.*

THEOREM V′. *Any formal solution of a linear homogeneous difference equation of n-th order is linearly dependent on any set of n linearly independent formal solutions of the equation.*

The proof of Theorem IV′ follows the same lines as the proof of the corresponding theorem for analytic solutions. To prove the condition necessary we assume that the solutions satisfy an equation $F_1y_1 + \cdots + F_ny_n = 0$, where $F_1, \cdots, F_n$ are of the form (a). From this, since $F_i(x+1) = F_i(x)$, we have $F_1y_1(x+i) + \cdots + F_ny_n(x+i) = 0$ $(i = 0, 1, \cdots, n-1)$. This system of n simultaneous equations in the n variables $F_1, \cdots, F_n$ can have a solution other than $F_1 = \cdots = F_n = 0$ only if the determinant of the coefficients is zero, and this determinant is precisely the determinant of Casorati.

In proving the condition sufficient we restrict ourselves at first to the case in which the set $y_1, \cdots, y_{n-1}$ is of the type described in the fundamental existence theorem.* In this case, if we denote by D_{ij} the cofactor of the element in i-th row and j-th column of D, the determinant D_{nn} is of elementary non-logarithmic form. Then if we assume $D = 0$ and $D_{nn} \neq 0$ we have

$$\text{(b)} \quad -y_n(x+n-1) = \frac{D_{n1}}{D_{nn}} y_1(x+n-1) + \cdots + \frac{D_{n,n-1}}{D_{nn}} y_{n-1}(x+n-1).$$

We let $F_1, \cdots, F_{n-1}$ be expressions of the form (6′) determined by the conditions

$$\text{(c)} \quad y_n(x+i) = -[F_1(x)y_1(x+i) + \cdots + F_{n-1}(x)y_{n-1}(x+i)] \qquad (i = 0, 1, \cdots, n-2).$$

Then $F_i = D_{ni}/D_{nn}$ $(i = 1, \cdots, n-1)$, and the equation (b) becomes

$$\text{(d)} \quad y_n(x+n-1) = -[F_1(x)y_1(x+n-1) + \cdots + F_{n-1}(x)y_{n-1}(x+n-1)].$$

On substituting $x+1$ for x in the i-th equation of the set (c) and subtracting from the result the $(i+1)$-st equation of the set (considering equation (d) as the n-th of the set) we obtain the equations

$$0 = \sum_{j=1}^{n-1} [F_j(x+1) - F_j(x)] y_j(x+i) \qquad (i = 1, \cdots, n-1),$$

which can be satisfied only by $F_i(x+1) - F_i(x) = 0$ $(i = 1, \cdots, n-1)$, since the determinant of the coefficients is D_{nn}.

The next step depends on the

LEMMA. *If $y(x)$ is a formal solution of the equation $y(x+1) - y(x) = 0$, then $y(x)$ is of the form (a).*

This can be proved by considering first the case in which $y(x)$ is of the elementary non-logarithmic type, establishing the fact for this case by direct substitution and evaluation of constants, and then reducing the general case to this one, step by step.

Applying the lemma, we find that $F_i = c_ie^{2\pi m_i\sqrt{-1}\,x}$ $(i = 1, \cdots, n-1)$, so that the first equation of the set (c) establishes the linear dependence of the set of solutions $y_1, \cdots, y_n$.

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* I. e., consists of elementary formal solutions which can be arranged in sets of the form (4).

If $D_{nn} = 0$, then, since D_{nn} is the determinant of Casorati for the first $n-1$ solutions, these solutions are, by the preceding argument, linearly dependent unless the first principal minor of D_{nn} is zero. Thus we can work back step by step until a set of solutions $y_1, \cdots, y_k$ $(k < n)$ is obtained which is linearly dependent. Then the whole set $y_1, \cdots, y_n$ must be linearly dependent.

In order to prove the condition sufficient in the general case we require Theorem V′ for the case in which the given set of linearly independent solutions is of the special type just considered. To prove Theorem V′ for this special case, let $\bar{y}_1, \cdots, \bar{y}_n$ be such a set of linearly independent elementary solutions, consisting of subsets of type (4), for the equation

$$a_0(x)y(x+n) + \cdots + a_n(x)y(x) = 0 \qquad (a_0(x) \not\equiv 0,\ a_n(x) \not\equiv 0)$$

and let y_i be any solution of this equation. Then, by substituting $\bar{y}_1, \cdots, \bar{y}_n, y_i$ successively for y in this equation, we obtain a system of $n+1$ equations, linear in the $n+1$ coefficients $a_i(x)$. Since these must be satisfied simultaneously by a set of values of the $a_i(x)$ not all zero, the determinant of the coefficients must vanish. Hence, since it is the determinant of Casorati for the $n+1$ solutions, they are linearly dependent, and, since the solutions $\bar{y}_1, \cdots, \bar{y}_n$ are linearly independent, y_i must be expressible as a linear combination of $\bar{y}_1, \cdots, \bar{y}_n$.

The proof of Theorem IV′ for the general case can now be completed. Any set of solutions $y_1, \cdots, y_n$ can be expressed as

$$\text{(e)} \qquad y_i = \sum_{j=1}^{n} F_{ij}\bar{y}_j \qquad (i = 1, \cdots, n),$$

where $\bar{y}_1, \cdots, \bar{y}_n$ is a set of solutions of the special form already considered and the coefficients F_{ij} are of the form (a). Then

$$\begin{aligned} D(y_1, \cdots, y_n) &= | \bar{y}_i(x+j-1) | = \Big| \sum_{k=1}^{n} F_{ik}\bar{y}_k(x+j-1) \Big| \\ &= | F_{ij} | \cdot | \bar{y}_i(x+j-1) | = | F_{ij} | \cdot D(\bar{y}_1, \cdots, \bar{y}_n). \end{aligned}$$

Hence $D(y_1, \cdots, y_n)$ can be zero only if $| F_{ij} | = 0$, and in this case we have from the equations (e) a linear homogeneous equation of the form (9′), which establishes the linear dependence of the set of solutions $y_1, \cdots, y_n$.

Thus Theorem IV′ has been proved for the most general case. The proof of Theorem V′ given for the special case is therefore valid in the general case also.

ON THE INVARIANCE OF A GENERALIZED GRAMIAN IN A RIEMANNIAN FUNCTION SPACE.*

By HENRY P. THIELMAN.

In this paper we consider a special Riemannian function space manifold † in which the length of a *vector* $y(s)$ is defined by means of a symmetric, properly positive, definite, integral, quadratic form of the following type:

$$(1) \qquad (yy)_g = g_{\alpha\beta} y^\alpha y^\beta + g_\alpha (y^\alpha)^2 > 0 \qquad (g_{\alpha\beta} = g_{\beta\alpha},\ g_\alpha \neq 0 \text{ in } (a,b))$$

for all real continuous functions $y(s)$ which do not vanish identically in the interval (a, b).

(1) stands for the expression

$$\int_a^b \int_a^b g(\alpha, \beta) y(\alpha) y(\beta)\, d\alpha\, d\beta + \int_a^b g(\alpha) y^2(\alpha)\, d\alpha > 0.$$

These conventions

1) *of representing the arguments of functions as continuous subscripts and superscripts, and*

2) *of letting the repetition of a continuous index in the same term, once as a subscript and once as a superscript, signify Riemannian integration with respect to that index over the interval* (a, b) will be used throughout this paper. Parentheses around any continuous index denote that the integration convention 2) does *not* apply to that index. All functions occurring in this paper are assumed to be real, and continuous in their closed regions of definition.

It is easily shown that the *Fredholm determinant* $D[g_{\alpha\beta}/(g_\alpha g_\beta)^{1/2}]$ of (1) does not vanish (i. e., unity is not a characteristic value of $D[g_{\alpha\beta}/(g_\alpha g_\beta)^{1/2}]$). In fact suppose

$$D[g_{\alpha\beta}/(g_\alpha g_\beta)^{1/2}] = 0.$$

Then the homogeneous Fredholm integral equation

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* Presented to the American Mathematical Society, April 13, 1933.

† A. D. Michal, "Affinely connected function space manifolds," *American Journal of Mathematics*, vol. 50 (1928), pp. 473-517.

$$z^{\alpha} + [g_{\alpha\beta}/(g_{\alpha}g_{\beta})^{1/2}]z^{\beta} = 0$$

will possess a continuous solution z^{α} which does not vanish identically. Hence if we let $y^{\alpha} = z^{\alpha}/(g(\alpha))^{1/2}$ it follows from (4) that

$$g_{\alpha}{}^{1/2}y^{(\alpha)} + (g_{\alpha\beta}/g_{\alpha}{}^{1/2})y^{\beta} = 0.$$

Multiplying this by $g_{\alpha}{}^{1/2}y^{(\alpha)}$ and integrating with respect to α we get

$$g_{\alpha}(y^{\alpha})^2 + g_{\alpha\beta}y^{\alpha}y^{\beta} = 0$$

which contradicts (1).

If we define the inner product, the angle between vectors, and the outer product of n vectors in a way analagous to the one used in n-dimensional vector analysis, and in Euclidean function space geometry,* we are led to a geometrical interpretation of the following generalized Gramian

$$(2) \qquad |(y_iy_j)_g| = \begin{vmatrix} (y_1y_1)_g & (y_1y_2)_g & \cdots & (y_1y_n)_g \\ (y_2y_1)_g & (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (y_ny_1)_g & (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix},$$

where

$$(y_iy_j)_g = g_{\alpha\beta}y_i{}^{\alpha}y_j{}^{\beta} + g_{\alpha}y_i{}^{\alpha}y_j{}^{\beta}$$
$$(i, j = 1, 2, \cdots, n;\ g_{\alpha\beta} = g_{\beta\alpha},\ g_{\alpha} \neq 0, \text{ in } (a, b)).$$

It can be interpreted as the square of the volume of the parallelepiped generated by the vectors $y_1{}^{\alpha_1}, y_2{}^{\alpha_2}, \cdots, y^{\alpha_n}$ having the same origin. If $(yy)_g$ is of type (1) and the y_i $(i = 1, 2, \cdots, n)$ are linearly independent then the Gramian in (2) is always greater than zero. For $g_{\alpha\beta} \equiv 0$, $g_{\alpha} \equiv 1$, $n = 2$, the statement that (2) is positive becomes Schwarz's inequality.†

Since we thus see that the generalized Gramian (2) plays an important part in Riemannian function space geometry, the question naturally arises, "What are the most general transformations of a certain type that leave this Gramian invariant?"

In order to answer this question we consider the general linear functional transformation of the third kind

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* G. Kowalewski, "Über Functionenräume," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Naturwissenschaftliche Klaase,* 120 Band, Abteilung IIª, I. Mitteilung, pp. 77-109, II. Mitteilung, pp. 1435-1472 (1911).

† E. H. Moore gives a broader generalization of Schwarz's inequality in "On the fundamental functional operation of a general theory of linear integral equations," *Proceedings of the Fifth International Congress of Mathematicians,* Cambridge University Press (1913), pp. 230-255.

$$y^a = K^a\bar{y}^a + K_\beta{}^a y^\beta \tag{3}$$

where $K^a \neq 0$ in (a, b) and the Fredholm determinant $D[K_\beta{}^a/K^a] \neq 0$. It is a well known fact that under these conditions the transformations (3) form a group.* An answer to the proposed question is given by the following theorem:

THEOREM I. *The subgroups of the group of linear functional transformations of the third kind which leave the integral quadratic form* $(yy)_g$, *and hence* $(y_iy_j)_g$, *invariant are identical with the subgroups of the group of transformations of the third kind that leave the generalized Gramian* $|(y_iy_j)_g|$ *invariant.*

It is obvious that every transformation which leaves $(y_iy_j)_g$ invariant will leave the Gramian $|(y_iy_j)_g|$ invariant. We shall now prove the converse of this statement.

Let us consider a Gramian which is invariant under the transformations (3). That is to say

$$\begin{vmatrix} (\bar{y}_1\bar{y}_1)_g & (\bar{y}_1\bar{y}_2)_g & \cdots & (\bar{y}_1\bar{y}_n)_g \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_1)_g & (\bar{y}_n\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} = \begin{vmatrix} (y_1y_1)_g & (y_1y_2)_g & \cdots & (y_1y_n)_g \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (y_ny_1)_g & (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix}. \tag{4}$$

Let us expand (4) to have it in the form

$$\begin{aligned} &(\bar{y}_1\bar{y}_1)_g \begin{vmatrix} (\bar{y}_2\bar{y}_2)_g & \cdots & (\bar{y}_2\bar{y}_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} + Qg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n]\bar{y}_1{}^a\bar{y}_1{}^\beta \\ &= (y_1y_1)_g \begin{vmatrix} (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix} + Qg\alpha\beta[y_2, y_3, \cdots, y_n]y_1{}^a y_1{}^\beta, \end{aligned} \tag{5}$$

where $Qg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n]$, and $Qg\alpha\beta[y_2, y_3, \cdots, y_n]$ are the coefficients of a quadratic form in $\bar{y}_1$ and y_1 respectively. Next keep $\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n$ constant and calculate the first variation of (5). Doing this we obtain

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* A. D. Michal and T. S. Peterson, "The invariant theory of functional forms under the group of linear functional transformations of the third kind," *Annals of Mathematics*, vol. 32 (1931), pp. 431-450.

$$(6)\quad \begin{aligned} &2g_a\bar{y}_1{}^a\delta\bar{y}_1{}^a \begin{vmatrix} (\bar{y}_2\bar{y}_2)_g & \cdots & (\bar{y}_2\bar{y}_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_{n1}\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} + Rg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n]\bar{y}_1{}^a\delta\bar{y}_1{}^\beta \\ &= 2g_a y_1{}^a\delta y_1{}^a \begin{vmatrix} (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix} + Rg\alpha\beta[y_2, y_3, \cdots, y_n]y_1{}^a\delta y_1{}^\beta, \end{aligned}$$

where $Rg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, y_n]$ stands for

$$2g_{a\beta} \begin{vmatrix} (\bar{y}_2\bar{y}_2)_g & \cdots & (\bar{y}_2, \bar{y}_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} + Qg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_m] + Qg\alpha\beta[y_2, y_3, \cdots, y_n],$$

and $Rg\alpha\beta[y_2, y_3, \cdots, y_m]$ for the identical expression with $\bar{y}_i$ replaced by y_i $(i = 2, 3, \cdots, n)$.

Remembering that

$$y^a = K^a\bar{y}^a + K_\beta{}^a y^\beta, \qquad K^a \neq 0 \text{ in } (a, b)$$

and hence

$$\delta y^a = K^a\delta\bar{y}^a + K_\beta{}^a\delta y^\beta,$$

we can rewrite (6) in the form

$$(7)\quad \begin{aligned} \int_a^b \{&\Omega[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n]y_1{}^a \\ &+ \Lambda_{a\beta}[y_2, y_3, \cdots, y_n, y_2, y_3, \cdots, y_n, K^\sigma K_\mu{}^\lambda]y_1{}^\beta\}\delta y_1{}^a d\alpha = 0, \end{aligned}$$

where

$$(8)\quad \Omega = 2g_a \begin{vmatrix} (\bar{y}_2\bar{y}_2)_g & \cdots & (\bar{y}_2\bar{y}_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} - 2g_{(a)}(K^a)^2 \begin{vmatrix} (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix},$$

and

$$\begin{aligned}\Lambda_{\alpha\beta} = {} & Rg\alpha\beta[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n] - Rg\alpha\beta[y_2, y_3, \cdots, y_n]K^{(\alpha)}K^{(\beta)} + \\ & - Rg\sigma\alpha[y_2, y_3, \cdots, y_n]K_\beta{}^\sigma K^{(\alpha)} - Rg\beta\sigma[y_2, y_3, \cdots, y_n]K^{(\beta)}K_\alpha{}^\sigma + \\ & - Rg\sigma\rho[y_2, y_3, \cdots, y_n]K_\beta{}^\sigma K_\alpha{}^\rho \\ & - 2\begin{vmatrix} (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix} (g_{(\alpha)}K_\beta{}^\alpha K^\alpha + g_{(\beta)}K_\alpha{}^\beta K^\beta + g_\sigma K_\alpha{}^\sigma K_\beta{}^\sigma).\end{aligned}$$

By the fundamental theorem of the calculus of variations it follows from (7) that

$$\begin{aligned}\bar{y}_1{}^\alpha \Omega[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n] & \\ + \bar{y}_1{}^\beta \Lambda_{\alpha\beta}[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n, K^\sigma, K_\mu{}^\lambda] & = 0.\end{aligned}$$

In order that this relation may be true it is necessary * that

$$\Omega[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n] = 0,$$

and

$$\Lambda_{\alpha\beta}[\bar{y}_2, \bar{y}_3, \cdots, \bar{y}_n, y_2, y_3, \cdots, y_n, K^\sigma, K_\mu{}^\lambda] = 0.$$

From (8) it now follows that if the generalized Gramian of order n is an absolute invariant then, since $g_\alpha \neq 0$,

$$\begin{vmatrix} (\bar{y}_2\bar{y}_2)_g & \cdots & (\bar{y}_2\bar{y}_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (\bar{y}_n\bar{y}_2)_g & \cdots & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} = (K^\alpha)^2 \begin{vmatrix} (y_2y_2)_g & \cdots & (y_2y_n)_g \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (y_ny_2)_g & \cdots & (y_ny_n)_g \end{vmatrix}.$$

Applying the same method of reasoning to this equation as we did to (4), and repeating this process $(n-2)$ times we finally arrive at the equations

$$(9) \qquad \begin{vmatrix} (\bar{y}_{n-1}\bar{y}_{n-1})_g & (\bar{y}_{n-1}\bar{y}_n)_g \\ (\bar{y}_n\bar{y}_{n-1})_g & (\bar{y}_n\bar{y}_n)_g \end{vmatrix} = (K^\alpha)^{2(n-2)} \begin{vmatrix} (y_{n-1}y_{n-1})_g & (y_{n-1}y_n)_g \\ (y_ny_{n-1})_g & (y_ny_n)_g \end{vmatrix}$$

and

$$(10) \qquad (\bar{y}_n\bar{y}_n)_g = (K^\alpha)^{2(n-1)}(y_ny_n)_g.$$

Since the $y_1{}^\alpha, y_2{}^\alpha, \cdots, y_n{}^\alpha$ are cogredient variables under the transformations (3) it can be easily verified that as a consequence of (10) we have

$$(11) \qquad (\bar{y}_i\bar{y}_j)_g = (K^\alpha)^{2(n-1)}(y_iy_j)_g \qquad (i, j = 1, 2, \cdots, n).$$

Making use of (11) in (9) we obtain by an obvious calculation that

$$(K^\alpha)^2 = 1,$$

and hence that

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* G. Kowalewski, *loc. cit.*, pp. 1455, 1456.

$$(\bar{y}_i\bar{y}_j)_g = (y_iy_j)_g \qquad (i, j = 1, 2, \cdots, n). \tag{12}$$

Equations (4) to (12) now state that if the generalized Gramian of order n is an absolute invariant under a certain transformation of the third kind, then each element of the Gramian is an absolute invariant under the same transformation, as was to be proven.

The necessary and sufficient conditions that $(yy)_g$ and hence $(y_iy_j)_g$, be an absolute invariant under a subgroup of the group of linear functional transformations of the third kind are, however, known. We state here a theorem which is a direct consequence of a result proven elsewhere.*

THEOREM II. *A necessary and sufficient condition that* $(yy)_g$, *and hence* $(y_iy_j)_g$ *be an absolute invariant under a subgroup of the group of linear functional transformations of the third kind* (3) *with non-vanishing Fredholm determinant* $D[K_\beta{}^a/K^a]$ *is that the coefficient* K^a *and* $K_\beta{}^a$ *satisfy the equations*

$$\begin{aligned} &(K^a)^2 \equiv 1 \\ &g_{\sigma\beta}K^{(\beta)}K_a{}^\sigma + g_{a\sigma}K^{(a)}K_\beta{}^\sigma + g_{\sigma J}K_a{}^\sigma K_\beta{}^J \\ &\quad + g_{(a)}K^aK_\beta{}^a + g_{(\beta)}K^\beta K_a{}^\beta + g_\sigma K_a{}^\sigma K_\beta{}^\sigma \equiv 0. \end{aligned} \tag{13}$$

It can be easily verified that the transformations of the third kind (3) whose coefficients K^a and $K_\beta{}^a$ satisfy (13) form a group. In analogy with the situation in Euclidean function space,† (i. e., one for which $g_{a\beta} \equiv 0$, $g_a \equiv 1$) we will call this group the group of *orthogonal transformations*, and also speak of it as the *orthogonal group* in *our function space. The orthogonal transformations for which* $K^a \equiv 1$ *constitute a subgroup of this orthogonal group.*

Since the Fredholm determinant $D[g_{a\beta}/(g_ag_\beta)^{1/2}]$ is a relative scalar invariant of weight two under the group of linear functional transformations of the third kind,‡ it follows that the *Fredholm determinant* $D[K_\beta{}^a/K^a]$ *of an orthogonal transformation has the value of* 1 *or* -1.

Making use of the definition of an orthogonal group we can restate our Theorem I as follows:

THEOREM Ia. *A necessary and sufficient condition that the generalized Gramian* $|(y_iy_j)_g|$ *be an absolute invariant under a subgroup of the group of linear functional transformations of the third kind whose Fredholm deter-*

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* A. D. Michal and T. S. Peterson, *loc. cit.*, p. 442.

† G. Kowalewski, *loc. cit.*, p. 101.

‡ A. D. Michal and T. S. Peterson, *loc. cit.*, p. 443.

minant $D[K_\beta{}^\alpha/K^\alpha]$ *does not vanish, is that the subgroup be the orthogonal group of transformations of the third kind, or a subgroup of it.*

If our special Riemannian function space is Euclidean we obtain as a corollary to Theorem Ia, a known result.*

COROLLARY. *A necessary and sufficient condition that the Gramian* $|(y_i y_j)|$ *be an absolute invariant under the Fredholm group of transformations is that the group of transformations be orthogonal.*

Geometrically interpreted Theorem I states that every group of linear functional transformations of the third kind which leaves the volume of every parallelepiped unaltered in our function space, leaves also the distances and angles unaltered. By analogy with the situation in n-space and in Euclidean function space † such a group might be called a group of motions in our special Riemannian function space.

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* T. S. Peterson, "A class of invariant functionals of quadratic functional forms," *American Journal of Mathematics*, vol. 51 (1929), pp. 417-430.

† I. A. Barnett, "On a class of transformations in function space," *Annals of Mathematics*, vol. 25 (1924), pp. 205-220.

ON CONVERGENCE AND OSCILLATION OF TRANSFORMS OF SEQUENCES OF VECTORS.

By H. EARL SPENCER.

1. *Introduction.* A set of real functions of a real variable, $\| a_k(t) \|$, defines a transformation that transforms a sequence of numbers, $\{x_n\}$, into a function $y(t)$, if $y(t) = \sum_{k=1}^{\infty} a_k(t) x_k$. The variable t is restricted to range over a point set having a limit point t_0 not belonging to the set; the limit point t_0 may be either an ordinary point or one of the symbolic limits, $\pm \infty$. Let $D(t) = | t - t_0 |$ if t_0 is an ordinary point, but let $D(t) = | 1/t |$ if t_0 is one of the symbolic limits, $\pm \infty$.

The transformation $\| a_k(t) \|$ is called *regular* if it has the property that $\lim_{t \to t_0} y(t) = \lim_{n \to \infty} x_n$ for every convergent sequence $\{x_n\}$.

The *oscillation* of a sequence of numbers, $\{x_n\}$, is defined to be $\overline{\lim}\sup_{m,n \to \infty} | x_m - x_n |$, and will be represented by $\Omega(x)$. The oscillation of the function $y(t)$ is defined to be $\limsup_{t,u \to t_0} | y(t) - y(u) |$. The transformation $\| a_k(t) \|$ is said to be *repressive* if $\Omega(y) \leqq \Omega(x)$ for every bounded sequence $\{x_n\}$.

It is the purpose of this paper to consider transformations, whose elements are square matrices of functions, acting upon sequences whose terms are vectors to produce a vector whose components are functions; and set forth conditions under which such transformations are regular and conditions under which such transformations are repressive.

2. *Regular and null coercive transformations.* It has been shown * that for $\| a_k(t) \|$ to be regular it is necessary and sufficient that

(1)

(a) for some $\delta > 0$, $\sum_{k=1}^{\infty} | a_k(t) |$ converges for each t and is bounded for all t for which $D(t) < \delta$;

(b) for each k, $\lim_{t \to t_0} a_k(t) = 0$;

(c) $\lim_{t \to t_0} \sum_{k=1}^{\infty} a_k(t) = 1$.

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* I. Schur, *Journal für Mathematik,* vol. 151 (1920), p. 82.

The transformation $\| a_k(t) \|$ is called *null* if $\lim_{t \to t_0} y(t) = 0$ for every convergent sequence $\{x_n\}$. For $\| a_k(t) \|$ to be null it is necessary and sufficient that *

$$(2) \quad \text{(a) same as (1a); (b) same as (1b); (c) } \lim_{t \to t_0} \sum_{k=1}^{\infty} a_k(t) = 0.$$

The transformation $\| a_k(t) \|$ is called *coercive* if $\lim_{t \to t_0} y(t)$ exists for every bounded sequence $\{x_n\}$. For $\| a_k(t) \|$ to be simultaneously null and coercive it is necessary and sufficient that †

$$(3) \qquad \lim_{t \to t_0} \sum_{k=1}^{\infty} | a_k(t) | = 0.$$

We shall be concerned with a set of q-rowed square matrices, $\| A_k(t) \|$, for each of which the element in the i-th row and j-th column $(i, j = 1, 2, \cdots, q)$, is a real function of a real variable designated by $a_k{}^{(ij)}(t)$. Henceforth $\{x_n\}$ instead of representing a sequence of numbers will represent a sequence of vectors of q real components each; $x_n{}^{(i)}$ will be the i-th component of the vector x_n. The sum $A_1 + A_2$ is defined to be the matrix with elements $a_1{}^{(ij)} + a_2{}^{(ij)}$; and the sum $x_1 + x_2$ is defined to be the vector with components $x_1{}^{(i)} + x_2{}^{(i)}$. The product Ax, of matrix by vector will mean as usual that $(Ax)^{(i)} = \sum_{r=1}^{q} a^{(ir)} x^{(r)}$.

It is necessary to define the modulus $| x |$, of a vector x, and the modulus $| A |$, of a matrix A. We consider only definitions of $| x |$ which satisfy

$$(4) \quad \begin{array}{l} \text{(a) for any two vectors } x_1, x_2, \ | x_1 + x_2 | \leqq | x_1 | + | x_2 |, \\ \text{(b) } | x | \geqq | x^{(i)} | \ (i = 1, 2, \cdots, q), \qquad \text{(c) } | x | \leqq \sum_{r=1}^{q} | x^{(r)} |, \\ \text{(d) if } \lambda \text{ is any scalar, } | \lambda x | \leqq | \lambda | \cdot | x |; \end{array}$$

and definitions of $| A |$ which satisfy

$$(5) \quad \begin{array}{l} \text{(a) for any two matrices } A_1, A_2, \ | A_1 + A_2 | \leqq | A_1 | + | A_2 |, \\ \text{(b) } | A | \geqq | a^{(ij)} |, \ (i, j = 1, 2, \cdots, q), \qquad \text{(c) } | A | \leqq \sum_{r=1}^{q} \sum_{s=1}^{q} | a^{(rs)} |, \\ \text{(d) if } \lambda \text{ is any scalar } | \lambda A | \leqq | \lambda | \cdot | A |. \end{array}$$

The further demand will be made upon the definitions of $| A |$ and $| x |$, for any matrix A and vector x, that together they satisfy

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* I. Schur, *loc. cit.*

† I. Schur, *loc. cit.*

$$(6) \qquad |Ax| \leqq |A| \cdot |x|.$$

Three different definitions of $|x|$ that satisfy (4) are:

$$(7) \quad \begin{aligned} &(\text{a}) \ |x| = \sum_{r=1}^{q} |x^{(r)}|; \qquad (\text{b}) \ |x| = \{\sum_{r=1}^{q} |x^{(r)}|^p\}^{1/p} \ (p > 1); \\ &(\text{c}) \ |x| = \text{maximum } |x^{(r)}|, \quad (r = 1, 2, \cdots, q). \end{aligned}$$

Both (7a) and (7c) can be regarded as limiting cases of (7b), as $p \rightarrow 1$ and as $p \rightarrow \infty$ respectively.

Some definitions of $|A|$ which satisfy (5) are:

$$(8) \quad \begin{aligned} &(\text{a}) \quad |A| = \{\sum_{r=1}^{q} \sum_{s=1}^{q} |a^{(rs)}|^p\}^{1/p} \quad (p \geqq 1); \\ &(\text{b}) \quad |A| = \sum_{r=1}^{q} \{\sum_{s=1}^{q} |a^{(rs)}|^p\}^{1/p} \quad (p \geqq 1). \end{aligned}$$

Here also the limiting cases as $p \rightarrow \infty$ may be included.

Condition (6) is not necessarily satisfied if we select $|x|$, $|A|$ at random from (7) and (8) respectively. For example (7c) and the limiting case of (8a) as $p \rightarrow \infty$, where $|A| = \text{maximum } |a^{(ij)}|$, $(i, j = 1, 2, \cdots, q)$, do not necessarily satisfy (6). On the other hand, (7c) and (8b) do satisfy (6) whether we use a limiting case of (8b) or not.

The meanings of $\lim_{n\rightarrow\infty} x_n = x$ and $\lim_{t\rightarrow t_0} A(t) = A$ will be respectively, $\lim_{n\rightarrow\infty} |x_n - x| = 0$ and $\lim_{t\rightarrow t_0} |A(t) - A| = 0$. From (4b) and (4c) it follows that $\lim_{n\rightarrow\infty} x_n = x$ if and only if $\lim_{n\rightarrow\infty} x_n^{(i)} = x^{(i)}$ for each i; and from (5b) and (5c) it follows that $\lim_{t\rightarrow t_0} A(t) = A$ if and only if $\lim_{t\rightarrow t_0} a^{(ij)}(t) = a^{(ij)}$ for each i and j.

Henceforth $y(t)$ will represent a vector of q components, its i-th component $y^{(i)}(t)$, being a real function of the real variable t. If $y(t) = \sum_{k=1}^{\infty} A_k(t) x_k$, then the sequence of vectors $\{x_n\}$ is transformed into a vector $y(t)$ by $\| A_k(t) \|$.

The transformation $\| A_k(t) \|$ will be called *regular* if $\lim_{t\rightarrow t_0} y(t) = \lim_{n\rightarrow\infty} x_n$ for every convergent sequence $\{x_n\}$.

THEOREM 1. *For* $\| A_k(t) \|$ *to be regular it is necessary and sufficient that*

(9)

(a) *for some* $\delta > 0$, $\sum_{k=1}^{\infty} | A_k(t) |$ *converges for each* t *and is bounded for all* t *for which* $D(t) < \delta$;

(b) *for each* k, $\lim_{t \to t_0} A_k(t) = 0$; (c) $\lim_{t \to t_0} \sum_{k=1}^{\infty} A_k(t) = I$.

The symbols 0 and I are used to represent respectively the null matrix having all its elements zero, and the identity matrix having each principal diagonal element 1, each other element 0.

To prove that (9c) is necessary, let i, j be an arbitrary but fixed pair of integers, and consider the sequence $\{x_n\}$, such that for all n, $x_n^{(r)} = [1$, if $r = j$; 0, if $r \neq j]$. Then $\lim_{n \to \infty} x_n^{(r)} = [1$, if $r = j$; 0, if $r \neq j]$. Since $\| A_k(t) \|$ is regular, $\lim_{t \to t_0} y^{(r)}(t) = \lim_{n \to \infty} x_n^{(r)}$, and since $y^{(r)}(t) = \sum_{k=1}^{\infty} a_k^{(rj)}(t)$, it must be that $\lim_{t \to t_0} \sum_{k=1}^{\infty} a_k^{(ij)}(t) = \lim_{t \to t_0} y^{(i)}(t) = [1$ if $i = j$; 0 if $i \neq j]$, from which (9c) follows.

To prove that (9b) is necessary consider the sequence $\{x_n\}$ of vectors defined by $x_n^{(r)} = 1$, if $r = j$, $n = k$, but $x_n^{(r)} = 0$ otherwise. Then $\lim_{n \to \infty} x_n^{(r)} = 0$, $(r = 1, 2, \cdots, q)$. For this sequence, $y^{(i)}(t) = a_k^{(ij)}(t)$. Since $\| A_k(t) \|$ is regular, $\lim_{t \to t_0} a_k^{(ij)}(t) = \lim_{t \to t_0} y^{(i)}(t) = 0$, and (9b) follows.

To prove (9a) necessary, fix j, let $\{x_k^{(j)}\}$ be an arbitrary convergent sequence converging to $x^{(j)}$, and let $x_k^{(r)} = 0$ for all k if $r \neq j$. Here $y^{(r)}(t) = \sum_{k=1}^{\infty} a_k^{(rj)} x^{(j)}$, and $\lim_{k \to \infty} x_k^{(r)} = [0$ if $r \neq j$; $x^{(j)}$ if $r = j]$, so that since $\| A_k(t) \|$ is regular, $\lim_{t \to t_0} \sum_{k=1}^{\infty} a_k^{(ij)}(t) x_k^{(j)} = [0$ if $i \neq j$; $x^{(j)}$ if $i = j]$. Since this holds for an arbitrary convergent sequence $\{x_k^{(j)}\}$, the transformation $\| a_k^{(ij)}(t) \|$ must be regular if $i = j$ or null if $i \neq j$. Then by either (1a) or (2a) as the case may be, $\sum_{k=1}^{\infty} | a_k^{(ij)}(t) |$ is bounded for all t for which $D(t) < \delta$, for some $\delta > 0$. Let an upper bound be $K^{(ij)}$. Then on account of (5c), $\sum_{k=1}^{\infty} | A_k(t) | \leqq \sum_{i=1}^{q} \sum_{j=1}^{q} K^{(ij)}$ for all t for which $D(t)$ is sufficiently near zero, and necessity of (9a) is established.

To prove (9) sufficient, let $\lim_{n \to \infty} x_n = x$; we show that $\lim_{t \to t_0} y(t) = x$ also. On account of (9) and (5b), $\| a_k^{(rs)}(t) \|$ satisfies (1) if $s = r$, or (2) if $s \neq r$. Consequently we have $\lim_{t \to t_0} \sum_{k=1}^{\infty} a_k^{(rs)}(t) x_k^{(s)} = [x^{(s)}$ if $s = r$; 0 if $s \neq r]$; hence we may write for $r = 1, 2, \cdots, q$,

$$\lim_{t\to t_0} y^{(r)}(t) = \lim_{t\to t_0} \sum_{k=1}^{\infty} \sum_{s=1}^{q} a_k^{(rs)}(t) x_k^{(s)} = \sum_{s=1}^{q} \lim_{t\to t_0} \sum_{k=1}^{\infty} a_k^{(rs)}(t) x_k^{(s)} = x^{(r)},$$

and $\lim_{t\to t_0} y(t) = x$ follows.

The transformation $\| A_k(t) \|$ will be called *null* if $\lim_{t\to t_0} y(t) = 0$ for every convergent sequence $\{x_n\}$, and *coercive* if $\lim_{t\to t_0} y(t)$ exists for every bounded sequence $\{x_n\}$.

THEOREM 2. *For* $\| A_k(t) \|$ *to be simultaneously null and coercive it is necessary and sufficient that*

$$\lim_{t\to t_0} \sum_{k=1}^{\infty} | A_k(t) | = 0. \tag{10}$$

To prove sufficiency let $\{x_k\}$ be any bounded sequence of vectors. Let $| x_k | \leqq K$ for all k. Then by (4a) and (6),

$$| y(t) | \leqq \sum_{k=1}^{\infty} | A_k(t) | \cdot | x_k | \leqq K \cdot \sum_{k=1}^{\infty} | A_k(t) |,$$

and sufficiency follows.

To prove necessity fix a pair of integers i, j and select a sequence $\{x_k\}$ in which $x_k^{(r)} = 0$ for all k if $r \neq j$, but $\{x_k^{(j)}\}$ forms an arbitrary bounded sequence of numbers. Then $y^{(i)}(t) = \sum_{k=1}^{\infty} a_k^{(ij)}(t) x_k^{(j)}$, from which it follows that $\| a_k^{(ij)}(t) \|$ is also null and coercive. Consequently (3) holds and (10) follows from (5c).

In Theorems 1 and 2, $| x |$ and $| A |$ may have any definitions satisfying (4), (5) and (6).

3. *Repressive transformations.* It has been shown that for a regular $\| a_k(t) \|$ to be repressive it is necessary and sufficient that *

$$\lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k(t) | = 1. \tag{11}$$

For a sequence of vectors $\{x_n\}$, we define the *oscillation*, $\Omega(x)$, to be $\limsup_{m,n\to\infty} | x_m - x_n |$. The oscillation of the function $y(t)$ we define to be $\limsup_{t,u\to t_0} | y(t) - y(u) |$. The transformation $\| A_k(t) \|$ will be called *repressive* if $\Omega(y) \leqq \Omega(x)$. We shall develop conditions for repressiveness for three different definitions of $|x|$.

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* W. A. Hurwitz, *American Journal of Mathematics*, vol. 52, no. 3, July 1930.

The following is a list of conditions with which we shall be concerned:

$$(12) \quad \lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ii)}(t) | = 1 \qquad (i = 1, 2, \cdots, q);$$

$$(13) \quad \lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ij)}(t) | = 0 \qquad (i, j = 1, 2, \cdots, q \text{ but } i \neq j);$$

$$(14) \quad \lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ii)}(t) - a_k^{(jj)}(t) | = 0 \qquad (i, j = 1, 2, \cdots, q);$$

$$(15) \quad \lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ij)}(t) - a_k^{(ji)}(t) | = 0 \qquad (i, j = 1, 2 \text{ but } i \neq j);$$

$$(16) \quad \lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ii)}(t) \pm a_k^{(ji)}(t) | = 1 \qquad (i, j = 1, 2 \text{ but } i \neq j).$$

THEOREM 3. *If the definition of $|x|$ is given by* (7b), *for a regular $\| A_k(t) \|$ to be repressive,* (12), (13) *and* (14) *are necessary and sufficient.*

We prove (12) necessary for repressiveness whenever $|x|$ and $|A|$ satisfy (4), (5) and (6). Select a sequence $\{x_k\}$ in which $x_k^{(r)} = 0$ for all k if $r \neq i$, while $\{x_k^{(i)}\}$ is an arbitrary bounded sequence. Then $y^{(i)}(t) = \sum_{k=1}^{\infty} a_k^{(ii)}(t) x_k^{(i)}$. For the sequence selected we have, from (4), $\Omega(x) = \Omega(x^{(i)})$ and $\Omega(y^{(i)}) \leqq \Omega(y)$. Hence when $\| A_k(t) \|$ is repressive, $\Omega(y^{(i)}) \leqq \Omega(x^{(i)})$ from which it follows that $\| a_k^{(ii)}(t) \|$ is repressive. But $\| a_k^{(ii)}(t) \|$ is regular by Theorem 1; therefore (11) holds and (12) is proved necessary.

To prove that (13) is necessary we shall suppose that (13) fails to hold and exhibit a sequence $\{x_n\}$ for which $\Omega(y) > \Omega(x)$. When (13) fails there is at least one fixed pair of integers i, j, $i \neq j$, for which $\limsup_{t\to t_0} \sum_{k=1}^{\infty} | a_k^{(ij)}(t) | > \epsilon > 0$ for some constant $\epsilon < 1$. Choose δ such that (9a) holds, and let $n_0 = 0$. Choose t_1, $D(t_1) < \delta$, such that $\sum_{k=1}^{\infty} | a_k^{(ij)}(t_1) | > \epsilon$, and then choose n_1 such that $\sum_{k=n_1+1}^{\infty} | A_k(t_1) | < 1$. Next choose t_2, $D(t_2) < \delta/2$, such that $\sum_{k=1}^{n_1} | a_k(t_2) | < \frac{1}{2}$ and $\sum_{k=1}^{\infty} | a_k^{(ij)}(t_2) | > \epsilon$; and then choose n_2 such that $\sum_{k=n_2+1}^{\infty} | A_k(t_2) | < \frac{1}{2}$. We continue this process to obtain sequences $\{n_a\}$ and $\{t_a\}$ such that $t_a \to t_0$ and

$$\text{(17a)}\quad \sum_{k=1}^{n_{a-1}} | A_k(t_a) | < 1/\alpha ; \sum_{k=n_a+1}^{\infty} | A_k(t_a) | < 1/\alpha ; \sum_{k=n_{a-1}+1}^{n_a} | a_k^{(ij)}(t_a) | > \epsilon - 2/\alpha .$$

Using these inequalities and (4) and (6), we see that for any bounded sequence $\{x_k\}$,

$$\text{(17b)}\qquad y(t_a) = O_a + \sum_{k=n_{a-1}}^{n_a} A_k(t_a) x_k ,$$

where the vector $O_a \to 0$ as $\alpha \to \infty$.

We use the sequence $\{x_k\}$ defined by:

$$x_k^{(r)} = [0 \text{ if } r \neq i, j ;\ (-1)^a \text{ if } r = i ;\ (-1)^a \epsilon^{1/p-1} \cdot \operatorname{sgn} a_k^{(ij)}(t_a) \text{ if } r = j]$$

for $n_{a-1} < k \leqq n_a$. For this sequence, $| x_k | \leqq (1 + \epsilon^{p'})^{1/p}$ and $\Omega(x) \leqq 2(1 + \epsilon^{p'})^{1/p}$ where p' is as usual defined by the equation $1/p + 1/p' = 1$. The i-th component of the transform of this sequence is given by

$$\text{(17c)}\quad (-1)^a y^{(i)}(t_a) = o_a + \sum_{k=n_{a-1}+1}^{n_a} a_k^{(ii)}(t_a) + \epsilon^{1/p-1} \cdot \sum_{k=n_{a-1}+1}^{n_a} | a_k^{(ij)}(t_a) | .$$

Here o_a is a number which $\to 0$ as $\alpha \to \infty$. Using (9) and (17a) we find, $(-1)^a y^{(i)}(t_a) > o_a + 1 + \epsilon^{p'}$, from which it follows that

$$| y^{(i)}(t_{a+1}) - y^{(i)}(t_a) | > o_a + 2 + 2\epsilon^{p'} .$$

This enables us to write

$$\Omega(y) \geqq \limsup_{a \to \infty} | y^{(i)}(t_{a+i}) - y^{(i)}(t_a) | \geqq 2 + 2\epsilon^{p'} > 2(1 + \epsilon^{p'})^{1/p} \geqq \Omega(x),$$

and necessity of (13) is established.

We next prove (14) necessary. When (14) does not hold there is at least one fixed pair of integers i and j and an $\epsilon > 0$ such that

$$\limsup_{t \to t_0} \sum_{k=1}^{\infty} | a_k^{(ii)}(t) - a_k^{(jj)}(t) | > \epsilon .$$

Choose sequences $\{n_a\}$ and $\{t_a\}$ such that $t_a \to t_0$ and

$$\text{(18a)}\quad \begin{gathered} \sum_{k=1}^{n_{a-1}} | A_k(t_a) | < 1/\alpha , \qquad \sum_{k=n_a+1}^{\infty} | A_k(t_a) | < 1/\alpha , \\ \sum_{k=n_{a-1}+1}^{n_a} | a_k^{(ii)}(t_a) - a_k^{(jj)}(t_a) | > \epsilon - 2/\alpha . \end{gathered}$$

Since (13) has already been proved necessary, we can use (13) and (18a) to obtain for any bounded sequence $\{x_k\}$,

$$\text{(18b)} \qquad y^{(r)}(t_a) = o_a + \sum_{k=1}^{\infty} a_k^{(rr)}(t_a)x_k^{(r)} = o_a + \sum_{k=n_{a-1}+1}^{n_a} a_k^{(rr)}(t_a)x_k^{(r)}.$$

Let $\sigma_{k,a} = \operatorname{sgn}(a_k^{(jj)}(t_a) - a_k^{(ii)}(t_a))$ and consider the sequence $\{x_k\}$ defined by $x_k^{(r)} = [0 \text{ if } r \neq i, j;\ (-1)^a \cdot (1 - \sigma_{k,a} \cdot \epsilon^2) \text{ if } r = i;\ (-1)^a(1 + \sigma_{k,a} \cdot \epsilon^2) \text{ if } r = j]$ for $n_{a-1} < k \leqq n_a$. For this sequence

$$| x_k | \leqq \{(1-\epsilon^2)^p + (1+\epsilon^2)^p\}^{1/p} \text{ and } \Omega(x) \leqq 2\{(1-\epsilon^2)^p + (1+\epsilon^2)^p\}^{1/p}.$$

We obtain after substitution in (18b):

$$(-1)^a y^{(i)}(t_a) = o_a + 1 - \epsilon^2 \sum_{k=n_{a-1}+1}^{n_a} a_k^{(ii)}(t_a) \cdot \sigma_{k,a};$$

$$(-1)^a y^{(j)}(t_a) = o_a + 1 + \epsilon^2 \sum_{k=n_{a-1}+1}^{n_a} a_k^{(jj)}(t_a) \cdot \sigma_{k,a}.$$

Adding the last two equations and using the definition of $\sigma_{k,a}$ and the last of relations (18a) we obtain,

$$(-1)^a\{y^{(i)}(t_a) + y^{(j)}(t_a)\} > o_a + 2 + \epsilon^3.$$

It follows that

$$| y^{(i)}(t_{a+1}) - y^{(i)}(t_a) | + | y^{(j)}(t_{a+1}) - y^{(j)}(t_a) | > o_a + 4 + 2\epsilon^3.$$

Using the fact that if $| a | + | b | > 2c$, then $| a |^p + | b |^p > 2c^p$, we conclude that

$$\{| y^{(i)}(t_{a+1}) - y^{(i)}(t_a) |^p + | y^{(j)}(t_{a+1}) - y^{(j)}(t_a) |^p\}^{1/p} > o_a + 2^{1/p}(2 + \epsilon^3),$$

and hence that $\Omega(y) \geqq 2^{1/p}(2 + \epsilon^3)$. If now ϵ be chosen so small that $2^{1/p}(2 + \epsilon^3) > 2\{(1-\epsilon^2)^p + (1+\epsilon^2)^p\}^{1/p}$, then we shall have $\Omega(y) > \Omega(x)$ and necessity of (14) is established.

We adapt the sufficiency proof given by Hurwitz* to prove that (12), (13), (14) are sufficient. To do so choose any value of i, use $a_k^{(ii)}(t)$ as a scalar and set $A_k(t) = (a_k^{(ii)}(t)) \cdot I + O_t$. Due to (13) and (14) the vector $O_t \to 0$ as $t \to t_0$. For any chosen $\epsilon > 0$ an integer α must exist such that $| x_k - x_l | < \Omega(x) + \epsilon$ when $k > \alpha$, $l > \alpha$. We have

$$y(t) = \sum_{k=1}^{\infty} A_k(t)x_k = \sum_{k=1}^{\infty} (a_k^{(ii)}(t) \cdot I)x_k + O_t = \sum_{k=1}^{\infty} (a_k^{(ii)}(t)) \cdot x_k + O_t,$$

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* W. A. Hurwitz, *loc. cit.*

or since α is fixed (9b) makes it possible to write, $y(t) = \sum_{k=\alpha+1}^{\infty} (a_k^{(ii)}(t)) \cdot x_k + O_t$. Hence $y(t) - y(u) = \sum_{k=\alpha+1}^{\infty} (a_k^{(ii)}(t)) \cdot x_k - \sum_{l=\alpha+1}^{\infty} (a_l^{(ii)}(u)) \cdot x_l + O_t$. But

$$\sum_{k=\alpha+1}^{\infty} (a_k^{(ii)}(t)) \cdot x_k - \sum_{l=\alpha+1}^{\infty} (a_l^{(ii)}(u)) \cdot x_l$$

$$= \Big(1 - \sum_{l=\alpha+1}^{\infty} a_l^{(ii)}(u)\Big) \cdot \sum_{k=\alpha+1}^{\infty} a_k^{(ii)}(t) x_k - \Big(1 - \sum_{k=\alpha+1}^{\infty} a_k^{(ii)}(t)\Big) \cdot \sum_{l=\alpha+1}^{\infty} a_l^{(ii)}(u) x_l \tag{19}$$
$$+ \sum_{l=\alpha+1}^{\infty} \sum_{k=\alpha+1}^{\infty} a_k^{(ii)}(t) a_l^{(ii)}(u) \cdot (x_k - x_l).$$

Let an upper bound of $|x_k|$ be X. The modulus of the first vector on the right of (19) $\leqq \big|1 - \sum_{l=\alpha+1}^{\infty} a_l^{(ii)}(u)\big| \cdot X \cdot \sum_{k=\alpha+1}^{\infty} |a_k^{(ii)}(t)|$ and has the limit 0 as $u \to t_0$, $t \to t_0$ by (1a) and (1c), since $\| a_k^{(ii)}(t) \|$ is regular. Likewise for the modulus of the second vector on the right of (19). Hence

$$y(t) - y(u) = \sum_{k=\alpha+1}^{\infty} \sum_{l=\alpha+1}^{\infty} a_k^{(ii)}(t) \cdot a_l^{(ii)}(u) \cdot (x_k - x_l) + O_t$$

so that

$$\Omega(y) = \limsup_{t,u \to t_0} | y(t) - y(u) |$$
$$\leqq \limsup_{t,u \to t_0} \sum_{k=\alpha+1}^{\infty} \sum_{l=\alpha+1}^{\infty} | a_k^{(ii)}(t) | \cdot | a_l^{(ii)}(u) | \cdot | x_k - x_l | \leqq \Omega(x) + \epsilon.$$

Therefore $\Omega(y) \leqq \Omega(x)$ which proves that (12), (13), (14) are sufficient. This same proof of sufficiency, unaltered, shows that (12), (13), (14) are sufficient to make $\Omega(y) \leqq \Omega(x)$ if $|x|$ is given by either (7a), (7b) or (7c).

The next theorem to be stated contains the result that necessary and sufficient conditions that $\| A_k(t) \|$ be repressive, are different according as $q = 2$ or $q > 2$. The theorem is therefore stated in two parts.

THEOREM 4a. *If the definition of* $|x|$ *is given by* (7a) *and* $q = 2$, *for a regular* $\| A_k(t) \|$ *to be repressive,* (14), (15) *and* (16) *are necessary and sufficient.*

THEOREM 4b. *If the definition of* $|x|$ *is given by* (7a) *and* $q > 2$, *for a regular* $\| A_k(t) \|$ *to be repressive,* (12), (13) *and* (14) *are necessary and sufficient.*

For any $q \geqq 2$ represent $a_k^{(ii)}(t) \pm a_k^{(ij)}(t) \mp a_k^{(ji)}(t) - a_k^{(jj)}(t)$ by $g_k^{(ij)}(t)$ and $h_k^{(ij)}(t)$ respectively. As an aid to proving that (14) is necessary we prove that $\lim_{t\to t_0} \sum_{k=1}^{\infty} | g_k^{(ij)}(t) | = 0$ and $\lim_{t\to t_0} \sum_{k=1}^{\infty} | h_k^{(ij)}(t) | = 0$. If the first of these fails to hold there exist δ, so small that (9a) holds, and sequences $\{t_a\}$, $\{n_a\}$ such that

$$\begin{gathered} D(t_a) < \delta/\alpha; \quad \sum_{k=1}^{n_{a-1}} | A_k(t_a) | < 1/\alpha; \quad \sum_{k=n_a+1}^{\infty} | A_k(t_a) | < 1/\alpha; \\ \sum_{k=n_{a-1}+1}^{n_a} | g_k^{(ij)}(t_a) | > \epsilon - 2/\alpha. \end{gathered} \tag{20}$$

Proceeding as in the proof that (13) is necessary in Theorem 3, we obtain

$$y(t_a) = O_a + \sum_{k=n_{a-1}+1}^{n_a} A_k(t_a) x_k.$$

Let $\sigma_{k,a}$ denote $\operatorname{sgn} g_k^{(ij)}(t_a)$, and consider the sequence $\{x_k\}$ for which $x_k^{(r)} = 0$ if $r \neq i$, $r \neq j$, for all k, but $[x_k^{(i)}; x_k^{(j)}] = [2 + \sigma_{k,a}(-1)^a + (-1)^a; 2 + \sigma_{k,a}(-1)^a - (-1)^a]$ for $n_{a-1} < k \leqq n_a$. For this sequence, $| x_k | \leqq 6$ and $\Omega(x) \leqq 4$. The i-th and j-th components of the transform of this sequence are given by:

$$\begin{aligned} (-1)^a y^{(i)}(t_a) = \sum_{k=n_{a-1}+1}^{n_a} [\sigma_{k,a}\{a_k^{(ii)}(t_a) + a_k^{(ij)}(t_a)\} \\ + a_k^{(ii)}(t_a) - a_k^{(ij)}(t_a) + (-1)^a\{2a_k^{(ii)}(t_a) + 2a_k^{(ij)}(t_a)\}] + o_a; \\ (-1)^a y^{(j)}(t_a) = \sum_{k=n_{a-1}+1}^{n_a} [\sigma_{k,a}\{a_k^{(ji)}(t_a) + a_k^{(jj)}(t_a)\} \\ + a_k^{(ji)}(t_a) - a_k^{(jj)}(t_a) + (-1)^a\{2a_k^{(ji)}(t_a) + 2a_k^{(jj)}(t_a)\}] + o_a. \end{aligned}$$

Subtracting the last two equations and using the definition of $\sigma_{k,a}$ and the last of relations (20) we obtain

$$(-1)^a (y^{(i)}(t_a) - y^{(j)}(t_a)) > \epsilon + 2 + o_a.$$

It follows that $| y^{(i)}(t_{a+1}) - y^{(i)}(t_a) | + | y^{(j)}(t_{a+1}) - y^{(j)}(t_a) | > 4 + 2\epsilon + o_a$. Hence $\Omega(y) > 4 + 2\epsilon > 4 \geqq \Omega(x)$, which proves that $\lim_{t\to t_0} \sum_{k=1}^{\infty} | g_k^{(ij)}(t) | = 0$ is necessary.

If we define a sequence $\{x_k\}$ such that $x_k^{(r)} = 0$, $r \neq i$, $r \neq j$, but $[x_k^{(i)}; x_k^{(j)}] = [2 + \sigma_{k,a}(-1)^a + (-1)^a; 2 - \sigma_{k,a}(-1)^a + (-1)^a]$ for $n_{a-1} < k \leqq n_a$, it can be proved in a similar manner that $\lim_{t\to t_0} \sum_{k=1}^{\infty} | h_k^{(ij)}(t) | = 0$.

We now note that

$$2\sum_{k=1}^{\infty} |\, a_k^{(ii)}(t) - a_k^{(jj)}(t) \,| \leqq \sum_{k=1}^{\infty} |\, g_k^{(ij)}(t) \,| + \sum_{k=1}^{\infty} |\, h_k^{(ij)}(t) \,|$$

so that by taking superior limits as $t \to t_0$, $\limsup\limits_{t \to t_0} \sum\limits_{k=1}^{\infty} |\, a_k^{(ii)}(t) - a_k^{(jj)}(t) \,| = 0$, which proves that (14) is necessary for any $q \geqq 2$. Also since

$$2\sum_{k=1}^{\infty} |\, a_k^{(ij)}(t) - a_k^{(ji)}(t) \,| \leqq \sum_{k=1}^{\infty} |\, g_k^{(ij)}(t) \,| + \sum_{k=1}^{\infty} |\, h_k^{(ij)}(t) \,|,$$

it follows that (15) is necessary when $q \geqq 2$. No mention of (15) is required in Theorem 4b since it is a consequence of (13) which is to be proved necessary.

For any $q \geqq 2$ it has already been shown in the proof of Theorem 3 that (12) is a necessary condition for both Theorems 4a and 4b. In the statement of Theorem 4a, (12) is omitted since it can be obtained as a consequence of (16).

To prove that (16) is necessary when $q = 2$ define a sequence $\{x_k\}$ such that $x_k^{(r)} = 0$, $r \neq i$, but $\{x_k^{(i)}\}$ is an arbitrary bounded sequence of numbers. Since $\|\, A_k(t) \,\|$ is repressive

$$\Omega(y) = \limsup_{t,u \to t_0} \{ |\, \sum_{k=1}^{\infty} (a_k^{(ii)}(t) - a_k^{(ii)}(u)) x_k^{(i)} \,| \\ + |\, \sum_{k=1}^{\infty} (a_k^{(ji)}(t) - a_k^{(ji)}(u)) x_k^{(i)} \,| \} \leqq \Omega(x) = \Omega(x^{(i)}).$$

Then certainly,

$$\limsup_{t,u \to t_0} |\, \sum_{k=1}^{\infty} (a_k^{(ii)}(t) \pm a_k^{(ji)}(t)) x_k^{(i)} \\ - \sum_{k=1}^{\infty} (a_k^{(ii)}(u) \pm a_k^{(ji)}(u)) x_k^{(i)} \,| \leqq \Omega(x^{(i)}).$$

Due to the fact that the transformations $\|\, a_k^{(ii)}(t) \pm a_k^{(ji)}(t) \,\|$ are both regular, (11) can be used to obtain, $\lim\limits_{t \to t_0} \sum\limits_{k=1}^{\infty} |\, a_k^{(ii)}(t) \pm a_k^{(ji)}(t) \,| = 1$.

To prove that (13) is necessary when $q > 2$ suppose (13) fails. Then for some pair of integers i, j, $i \neq j$, $\limsup\limits_{t \to t_0} \sum\limits_{k=1}^{\infty} |\, a_k^{(ij)}(t) \,| > \epsilon$ for some $\epsilon > 0$. Choose sequences $\{n_\alpha\}$, $\{t_\alpha\}$ such that $t_\alpha \to t_0$, $n_\alpha > n_{\alpha-1}$ and

$$(21) \quad \sum_{k=1}^{n_{\alpha-1}} |\, A_k(t_\alpha) \,| < 1/\alpha; \quad \sum_{k=n_\alpha+1}^{\infty} |\, A_k(t_\alpha) \,| < 1/\alpha; \quad \sum_{k=n_{\alpha-1}+1}^{n_\alpha} |\, a_k^{(ij)}(t_\alpha) \,| > \epsilon - 2/\alpha.$$

Use $\sigma_{k,\alpha}$ to represent $\operatorname{sgn} a_k^{(ij)}(t_\alpha)$ and define a sequence $\{x_k\}$ such that

$$[x_k^{(1)}; x_k^{(2)}; x_k^{(3)}] = [3 + \sigma_{k,a} + \sigma_{k,a}(-1)^a - (-1)^a; \\ 2 + 2(-1)^a; 1 + \sigma_{k,a} + \sigma_{k,a}(-1)^a + (-1)^a]$$

for $n_{a-1} < k \leqq n_a$ but $x_k^{(r)} = 0$ for $r > 3$ and all k. By (7a), $|x_k| \leqq 12$ and $\Omega(x) = 8$. Since (14) and (15) have been proved necessary we can use them together with (21) to obtain

$$(-1)^a\{-y^{(1)}(t_a) + y^{(2)}(t_a) + y^{(3)}(t_a)\} > 4 + 2\epsilon + o_a.$$

It follows that

$$|y^{(1)}(t_{a+1}) - y^{(1)}(t_a)| + |y^{(2)}(t_{a+1}) - y^{(2)}(t_a)| \\ + |y^{(3)}(t_{a+1}) - y^{(3)}(t_a)| > 8 + 4\epsilon + o_a.$$

Hence $\Omega(y) > \Omega(x)$ and this contradiction that $\| A_k(t) \|$ is not repressive proves that (13) is necessary when $q > 2$.

As remarked in the proof of Theorem 3, (12), (13), (14) are sufficient to make $\Omega(y) \leqq \Omega(x)$ for any one of the definitions (7) of $|x|$ for any $q \geqq 2$. The sufficiency of (14), (15), (16) when $q = 2$ remains to be proved. To do so note that because of (14) and (15),

$$y(t) = [y^{(1)}(t); y^{(2)}(t)] = [\sum_{k=1}^{\infty} \{a_k^{(11)}(t)x_k^{(1)} + a_k^{(21)}(t)x_k^{(2)}\}; \\ \sum_{k=1}^{\infty} \{a_k^{(21)}(t)x_k^{(1)} + a_k^{(11)}(t)x_k^{(2)}\}] + O_a.$$

We have taken $i = 1$, $j = 2$, which is no restriction. Since for any two real numbers α, β, $|\alpha| + |\beta|$ is either $|\alpha + \beta|$ or $|\alpha - \beta|$ and $|\alpha + \beta| \leqq |\alpha| + |\beta|$, we have for either upper or lower signs,

$$(22) \quad |y(t) - y(u)| \leqq |\sum_{k=1}^{\infty} \{(a_k^{(11)}(t) \pm a_k^{(21)}(t)) - (a_k^{(11)}(u) \pm a_k^{(21)}(u))\}x_k^{(1)}| \\ + |\sum_{k=1}^{\infty} \{(a_k^{(11)}(t) \pm a_k^{(21)}(t)) - (a_k^{(11)}(u) \pm a_k^{(21)}(u))\}x_k^{(2)}| + o_a,$$

for any pair of values t, u which satisfy (9a). On account of (16) we see that the two transformations, $\| A_k(t) \|$, in which $A_k^{(11)}(t) = a_k^{(11)}(t) \pm a_k^{(21)}(t)$, $A_k^{(12)}(t) = 0$, $A_k^{(21)}(t) = 0$, $A_k^{(22)}(t) = a_k^{(11)}(t) \pm a_k^{(21)}(t)$, satisfy (12), (13), (14) which have been proved sufficient for repressiveness. Therefore the superior limit as $t, u \to t_0$ of the right member of (22) $\leqq \Omega(x)$. Using (22), $\Omega(y) \leqq \Omega(x)$, which proves that (14), (15), (16), are sufficient.

THEOREM 5. *If the definition of* $|x|$ *is given by* (7c), *for a regular* $\| A_k(t) \|$ *to be repressive,* (12) *and* (13) *are necessary and sufficient.*

The proof that (12) is necessary has already been given in proving Theorem 3.

To prove (13) necessary we assume (13) fails to hold and obtain (17a) and (17b) as in proving (13) necessary for Theorem 3. In this case use the sequence $\{x_k\}$ defined by, $x_k^{(r)} = 0, r \neq i, r \neq j$, for all k, but $[x_k^{(i)}; x_k^{(j)}] = [(-1)^a; (-1)^a \operatorname{sgn} a_k^{(ij)}(t_a)]$ when $n_{a-1} < k \leqq n_a$, so that $|x_k| = 1$ and $\Omega(x) = 2$. For this sequence we obtain,

$$(-1)^a y^{(i)}(t_a) = 1 + \sum_{k=n_{a-1}}^{n_a} | a_k^{(ij)}(t_a) | + o_a > 1 + \epsilon + o_a,$$

from which it follows that $| y^{(i)}(t_{a+1}) - y^{(i)}(t_a) | > 2 + 2\epsilon + o_a$. But $\Omega(y) \geqq \limsup_{a \to \infty} | y^{(i)}(t_{a+1}) - y^{(i)}(t_a) | \geqq 2 + 2\epsilon > 2 = \Omega(x)$, and this proves that (13) is necessary.

To prove (12) and (13) sufficient we note that

$$y^{(i)}(t) - y^{(i)}(u) = \sum_{r=1}^{q} \left(\sum_{k=1}^{\infty} a_k^{(ir)}(t) x_k^{(r)} - \sum_{k=1}^{\infty} a_k^{(ir)}(u) x_k^{(r)} \right),$$

or because of (13)

$$y^{(i)}(t) - y^{(i)}(u) = \sum_{k=1}^{\infty} (a_k^{(ii)}(t) - a_k^{(ii)}(u)) \cdot x_k^{(i)} + o_t.$$

Since (12) is to hold, the regular transformation $\| a_k^{(ii)}(t) \|$ satisfies (11) and is therefore a repressive transformation. For this reason

$$\limsup_{t,u \to t_0} | \sum_{k=1}^{\infty} (a_k^{(ii)}(t) - a_k^{(ii)}(u)) x_k^{(i)} | \leqq \Omega(x^{(i)})$$

and consequently

$$\Omega(y^{(i)}) = \limsup_{t,u \to t_0} | y^{(i)}(t) - y^{(i)}(u) | \leqq \Omega(x^{(i)}).$$

For the definition of $|x|$ being considered this implies $\Omega(y) \leqq \Omega(x)$ and sufficiency is established.

4. *Extensions.* An examination of our criteria which determine whether a transformation $\| A_k(t) \|$ is regular, or regular and repressive, shows the following remarks to be true. Although the matrix-vector product is non-

commutative the same criteria determine a transformation to be regular, or regular and repressive, if the product is taken in the other order, namely $y(t)=\sum_{k=1}^{\infty} x_k A_k(t)$. Otherwise stated, if the transformation $\| A_k(t) \|$ is regular, or regular and repressive, the transformation $\| A'_k(t) \|$ is also regular, or regular and repressive, if $A'_k(t)$ is the conjugate matrix of $A_k(t)$.

If (7) be replaced by the corresponding definitions of the modulus of a matrix our theorems are true without change for the type of transformation in which a sequence of square matrices $\{X_k\}$ is transformed into a matrix $Y(t)$ either by $Y(t)=\sum_{k=1}^{\infty} A_k(t) X_k$ or by $Y(t)=\sum_{k=1}^{\infty} X_k A_k(t)$. Only minor changes in the proofs are needed to establish this. The same remarks apply here as in the case of transforming a sequence of vectors.

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CREMONA'S DIOPHANTINE EQUATIONS.

By ARTHUR B. COBLE.

Introduction. The purpose of this article is to present some rather specific results, as well as some general points of view, with respect to the solutions in terms of integers, positive, negative, or zero, of the two diophantine equations connected with the determination of a *complete* and *regular* linear system, $\Sigma_{p,d}$, of plane curves of *dimension* d, the generic curve of the system having the genus p. The literature is covered in the N. R. C. Report,[5] Chaps. III, IV. Let

$$x \equiv \{x_0; x_1, \cdots, x_\rho\} \tag{1}$$

be the *characteristic* of Σ, where x_0 is the *order*, $x_1, \cdots, x_\rho$ the multiplicities of the generic curve of Σ at the prescribed base points. The diophantine equations are then

$$\begin{aligned} x_1^2 + \cdots + x_\rho^2 &= x_0^2 + 1 - d - p, \\ x_1 + \cdots + x_\rho &= 3x_0 - 1 - d + p. \end{aligned} \tag{2}$$

These equations were given by Cremona for the case, $\Sigma_{0,2}$ of *homaloidal* nets. However, in connection with such nets the cases $\Sigma_{0,0}$ and $\Sigma_{0,-1}$, of respectively P-curves and D-conditions, are important.

A particular linear system has infinitely many *conjugate* linear systems under Cremona transformation of the plane. Thus the effect of a quadratic transformation with F-points at the first three base points of the system $\Sigma_{p,d}$ with characteristic x is to transform it into a system $\Sigma'_{p,d}$ with characteristic x' where

$$A_{123}: \quad \begin{aligned} x'_0 &= 2x_0 - x_1 - x_2 - x_3, \\ x'_1 &= x_0 \phantom{{}- x_1} - x_2 - x_3, \\ x'_2 &= x_0 - x_1 \phantom{{}- x_2} - x_3, \\ x'_3 &= x_0 - x_1 - x_2, \\ x'_{3+j} &= x_{3+j} \qquad\qquad (j = 1, \cdots, \rho - 3). \end{aligned} \tag{3}$$

In order to allow for the various ways in which the Q. T. may be applied to the base points and for the various orders in which the base points themselves may be arranged, we add to the transformation A_{123} in (3) the permutation group

$$\Pi(x_1, \cdots, x_\rho) \tag{4}$$

of $x_1, \cdots, x_\rho$. Then A_{123} and Π generate a group of linear transformations of determinant ± 1 whose generic element,

$$\text{(5)} \qquad \text{C. T.}: \quad \begin{aligned} x'_0 &= nx_0 - r_1x_1 \cdots - r_\rho x_\rho, \\ x'_1 &= s_1x_0 - \alpha_{11}x_1 \cdots - \alpha_{1\rho}x_\rho, \\ &\cdots\cdots\cdots \\ x'_\rho &= s_\rho x_0 - \alpha_{\rho 1}x_1 \cdots - \alpha_{\rho\rho}x_\rho, \end{aligned}$$

gives the effect upon a characteristic x of a C. T. whose F-points (ρ or less in number) are found at the basis points of $\Sigma_{p,d}$. If some of the F-points of C. T. are not included in the set of base points of Σ, the set must be enlarged by the inclusion of points of zero multiplicity.

In this generic element, (5), the first row and column furnish the characteristics,

$$(n;\ r_1, \cdots, r_\rho);$$
$$(n;\ s_1, \cdots, s_\rho),$$

of *homaloidal* nets, the inverse and direct nets of the Cremona transformation C. T. The simplest net of this character is the net of straight lines with characteristic $\{1;\ 0\,0 \cdots\} = \{1;\ 0^\rho\}$, the exponent ρ indicating repetition ρ times. The other rows and columns furnish the characteristics,

$$\{r_i;\ \alpha_{1i} \cdots \alpha_{\rho i}\},$$
$$\{s_j;\ \alpha_{j1} \cdots \alpha_{j\rho}\},$$

of P-curves of the direct and inverse transformation. The simplest P-curve has the characteristic $\{0;\ -1\,0\,0 \cdots\} = \{0;\ -1\,0^{\rho-1}\}$, this indicating the set of directions about the first basis point, which passes by the C. T. into the P-curve with characteristic $\{r_1;\ \alpha_{11} \cdots \alpha_{\rho 1}\}$. The conditions on the coefficients n, r_i, s_j, α_{ij} of (5) are all comprised by the statement that the linear group has the absolute invariants,

$$\text{(6)} \qquad \begin{aligned} Q &\equiv x_1^2 + \cdots + x_\rho^2 - x_0^2 \\ L &\equiv x_1 + \cdots + x_\rho - 3x_0. \end{aligned}$$

The linear group generated by Π and A_{123} contains a conjugate set of generating involutions. In this set there are found the *transpositions* contained in Π as well as A_{123} itself. If the base points are subject to a D-condition, the effect of the corresponding generating involution can be realized by a collineation rather than a C. T. Thus if the first two base points coincide in some direction, there is a collineation—the identity—which interchanges these two and leaves the other base points fixed. If the three F-points of the quadratic transformation A_{123} are on a line, the transformation becomes a collineation. The characteristic of this D-condition is

$$\{1;\,1\,1\,1\,0\,0\cdots\} = \{1;\,1^3\,0^{\rho-3}\};$$

the characteristic of the coincidence condition is $\{0;\,1,\,-1,\,0^{\rho-2}\}$. Thus the aggregate of D-conditions on the ρ base points is isomorphic with the aggregate of generating involutions of the linear group [cf. [1], pp. 15-17].

The general range of problems with which we are concerned may be formulated as follows: Given integer values of d, p, and ρ, to obtain all the solutions of the two diophantine equations (2), to divide these solutions into sets conjugate under the linear group generated by Π and A_{123}, to seek for each conjugate set its simplest representative, and finally to determine arithmetical criteria that a particular solution may belong to a given conjugate set. Since the linear group is infinite when $\rho \geqq 9$, the number of solutions in a conjugate set is in general infinite. The given integer ρ is necessarily positive, but d and p may be positive, negative, or zero. For example, d is -1 in the case of a D-condition.

In the case of homaloidal nets, or linear systems $\Sigma_{0,2}$, de Jonquières has classified the solutions of (2) into *geometric, arithmetic,* or *algebraic* according as the integers $\{x_0;\,x_1,\cdots,x_\rho\}$ are (a) all positive or zero, defining a geometrically existent homaloidal net; (b) all positive or zero, but defining no existent net; or (c) partly or altogether negative. This classification is not in accord with our present purpose. For example the algebraic characteristic, $\{0;\,-1\,0^{\rho-1}\}$, the set of directions about a point, is in the same conjugate set as $\{1;\,1^2\,0^{\rho-2}\}$, a line on two points. Each has a geometric existence. We shall rather find three classes of solutions, which we shall distinguish as *proper, degenerate,* or *virtual,* according as the generic curve of the system is (a) existent and irreducible; (b) existent but reducible; or (c) non-existent.

1. A canonical form of the characteristic of a linear system. We shall take the characteristic represented in (1) in the following form, prepared for the case of a set of $8+j$ base points, say the set P^2_{8+j} in the plane:

$$(7)\quad \{3(\gamma+\delta_0)-\nu;\,\gamma+\delta_0-\delta_1,\,\gamma+\delta_0-\delta_2,\cdots,\gamma+\delta_0-\delta_8,\,\kappa_1,\,\kappa_2,\cdots,\kappa_j\}.$$

For the case $j=1$ this canonical form has appeared in theses of Dr. Taylor[2] and Dr. Barber.[3] We apply it here more extensively.

It is now easy to verify that the diophantine equations (2) are satisfied if the integer parameters introduced in (7) satisfy the following conditions:

$$(8)\qquad \begin{array}{ll} \delta_0 + \delta_1 + \cdots + \delta_8 = 3\nu \quad (\nu = -1, 0, 1), & \Sigma_j \kappa_j = \gamma - 1 + p - d, \\ \delta_0^2 + \delta_1^2 + \cdots + \delta_8^2 = \nu^2 + 2\epsilon, & \Sigma_j \kappa_j^2 = \gamma^2 - 2\epsilon + 1 - p - d. \end{array}$$

The restriction of the value of ν to $0, \pm 1$ is not material, since the canonical form (7) is not affected when each δ is increased by k and when ν is increased by $3k$. We observe also that ϵ is an integer equal to or greater than 0.

The equations in the second column (8) may be replaced by the following:

$$(8.1)\qquad \begin{array}{l} \lambda \equiv 1 - p + d, \\ \gamma = \sigma_1(\kappa_1, \cdots, \kappa_j, \lambda), \\ \epsilon = \sigma_2(\kappa_1, \cdots, \kappa_j, \lambda) + \binom{p-1}{2} - d(p-1) + \binom{d}{2}, \end{array}$$

where σ_1, σ_2 are the first and second elementary symmetric polynomials in the indicated arguments.

To apply these equations to the determination of a characteristic for a given system $\Sigma_{p,d}$, and thus for given λ, we first select a $\nu = 0, 1, -1$, select $\delta_1, \cdots, \delta_8$ at random, and then determine δ_0 and ϵ from the equations (8). For this ϵ it is then necessary to select $\kappa_1, \cdots, \kappa_j$ *subject to the last of equations* (8.1). Finally γ is obtained from the second of equations (8.1), and the characteristic (7) is fixed. As an alternative one might select $\kappa_1, \cdots, \kappa_j$ at random and thus determine ϵ. It would then be necessary to determine the sets $\delta_0, \cdots, \delta_8$ which fit this ϵ. The tentative character of this process of fitting the ϵ, either on the side of the κ's, or on the side of the δ's, may be removed in great measure by a new procedure. Let

$$(9)\qquad \begin{array}{l} \mu = \epsilon - \binom{p-1}{2} + d(p-1) - \binom{d}{2} = \epsilon - \binom{\lambda}{2} + p - 1, \\ \zeta = \sigma_1(\kappa_3, \cdots, \kappa_j, \lambda), \\ i_{12} = h_1 \cdot h_2 = \zeta^2 + \mu - \sigma_2(\kappa_3, \cdots, \kappa_j, \lambda). \end{array}$$

Then the last equation (8.1) reads:

$$(\kappa_1 + \zeta)(\kappa_2 + \zeta) = i_{12} = h_1 \cdot h_2.$$

Thus

$$(10)\qquad \kappa_1 = h_1 - \zeta, \quad \kappa_2 = h_2 - \zeta.$$

Hence

(11) *Every ordered characteristic of a system* $\Sigma_{p,d}$ *is given by the canonical form* (7) *for arbitrary integral choice of* $\delta_1, \cdots, \delta_8, \kappa_3, \cdots, \kappa_j$, *and selection of* ν *from* $-1, 0, 1$, *and for an arbitrary factorization of the integer* $i_{12} = h_1 \cdot h_2$.

Indeed, for given $\delta_1, \cdots, \delta_8$, the equations (8) define δ_0 and ϵ. For given $\kappa_3, \cdots, \kappa_j$, the equations (9) define λ, μ, ζ and $i_{12} = h_1 \cdot h_2$. Then κ_1, κ_2 are determined by equations (10), and finally γ is obtained from (8.1). Thus every ordered solution of the diophantine equations (2) is determined *explicitly except to within an arbitrary factorization of a determinate integer.*

The procedure just outlined applies specifically to cases $j \geqq 2$ but it specializes readily to sets of base points $P_8{}^2$ and $P_9{}^2$. We give two examples, the first being the determination of homaloidal nets with 8 or fewer F-points. Then $p = 0$, $d = 2$, $\lambda = 3$. Since in (10) $\kappa_1 = \kappa_2 = 0$, $h_1 = h_2 = \zeta$. Then, due to the last equation (9), $\mu = 0$; and, due to the first equation (9), $\epsilon = 4$. From (8.1) we find $\gamma = 3$. For $\epsilon = 4$, the sets of δ's which satisfy (8) are [cf. (19)]:

$\nu = 0$	$\nu = 1$	$\nu = -1$
$-2 \quad -1 \quad 0^4 \quad 1^3$	$-2 \quad 0^3 \quad 1^5$	$-3 \quad 0^8$
$(-2 \quad 0^7 \quad 2)$	$(-1^3 \quad 1^6)$	$-2^2 \quad 0^6 \quad 1$
$-1^4 \quad 0 \quad 1^4$	$-1^2 \quad 0^3 \quad 1^3 \quad 2$	$-2 \quad -1^3 \quad 0^3 \quad 1^2$
$-1^3 \quad 0^4 \quad 1 \quad 2$	$-1 \quad 0^6 \quad 2^2$	$(-1^6 \quad 1^3)$
	$0^8 \quad 3$	$-1^5 \quad 0^3 \quad 2.$

In each of these sets $\{\nu;\, \delta\}$ we select one of the integers as δ_0 and, recalling that $\gamma = 3$, write down as in (7) the resulting characteristic. If we apply this to the 11 sets δ not enclosed above in parentheses, we obtain the 35 types of proper homaloidal nets with eight or fewer F-points as usually tabulated [cf. [1], p. 17]. If however we apply it to the three sets δ in parentheses (these being characterized by $\delta_i - \delta_j \equiv 0 \bmod. 2$), we get the following sets of characteristics:

$$\begin{array}{lll} \{3;\, 1^7\, -1\}, & \{5;\, 3^2\, 1^6\}, & \{7;\, 3^5\, 1^3\}, \\ \{9;\, 5\, 3^6\, 1\}, & \{11;\, 5^3\, 3^5\}, & \{13;\, 5^6\, 3^2\}, \\ \{15;\, 7\, 5^7\}. & & \end{array} \tag{13}$$

It is easy to verify that these characteristics form a complete conjugate set under the Q. T.'s (3). We see also that they are degenerate, but still geometrically constructible. Thus they can be exhibited as

$$\begin{array}{lll} \{3;\, 1^7\, 0\}\, \{0;\, 0^7\, -1\}, & \{4;\, 2^2\, 1^6\}\, \{1;\, 1^2\, 0^6\}, & \{5;\, 2^5\, 1^3\}\, \{2;\, 1^5\, 0^3\}, \\ \{6;\, 3\, 2^6\, 1\}\, \{3;\, 2\, 1^6\, 0\}, & \{7;\, 3^3\, 2^5\}\, \{4;\, 2^3\, 1^5\}, & \{8;\, 3^6\, 2^2\}\, \{5;\, 2^6\, 1^2\}, \\ \{9;\, 4\, 3^7\}\, \{6;\, 3\, 2^7\}. & & \end{array} \tag{14}$$

Each of these Cremona nets is composed of an elliptic net taken with a fixed P-curve which is the common canonical adjoint of the elliptic curves. The P-curve has no variable points in common with the generic elliptic curve of the net. The net $\{3;\, 1^7 - 1\}$ has the same constitution, being composed of the elliptic net $\{3;\, 1^7\, 0\}$ and the P-curve $\{0;\, 0^7 - 1\}$, the directions about the point p_8. Thus this type of net, characterized by de Jonquières as *algebraic*, is in the same conjugate set as the other types which he calls *arithmetic*. We prefer to say that

(15) *For* $\rho = 8$, *the equations* (2) *have, when* $p, d = 0, 2$, 35 *solutions which yield proper homaloidal nets and* 7 *solutions which yield degenerate homaloidal nets.*

As a second example, we consider the D-conditions for $P_9{}^2$, i. e., $\kappa_2 = \kappa_3 = \cdots = 0$, $p = 0$, $d = -1$, $\lambda = 0$. Then in (9) $\zeta = 0$, in (10) $h_2 = 0$, in (9) $i_{12} = 0$, whence $\mu = 0$ and $\epsilon = 1$. In (8) $\gamma = \kappa_1$. When $\epsilon = 1$,

$$(\nu;\, \delta) = (0;\, -1\, 0^7\, 1),\quad (1;\, 0^6\, 1^3),\quad (-1;\, -1^3\, 0^6).$$

Selecting a δ_0 in each of these, we get three distinct solutions

$$(16)\qquad \{3\kappa_1;\, \kappa_1 - 1, \kappa_1{}^6, \kappa_1 + 1\},\qquad \{3\kappa_1 - 1;\, (\kappa_1 - 1)^3, \kappa_1{}^6\},$$
$$\{3\kappa_1 + 1;\, \kappa_1{}^6, (\kappa_1 + 1)^3\}.$$

the various δ_0's in a $(\nu;\, \delta)$ giving only one solution due to the various multiplicities which can be chosen as $\gamma = \kappa_1$. That these constitute a single conjugate set has already been proved [cf. [1], pp. 16-17]. We observe that these are all geometrically existent in the sense that only a single condition $(d = -1)$ on the base points of $\Sigma_{0,-1}$ must be satisfied in order to ensure the existence of the rational curve indicated. In the case of a D-condition the linear diophantine equation (2) is homogeneous, and the characteristics x and $-x$ are paired. Either member of the pair indicates the same condition, since the original D-condition, $\{0;\, 1\, -1\, 0^{\rho-2}\}$, which indicates the coincidence of p_1, p_2 is identical with $\{0;\, -1\, 1\, 0^{\rho-2}\}$.

2. Lemmas concerning the canonical form in 1. We shall say that a characteristic $x = \{x_0;\, x_1 x_2 \cdots x_\rho\}$ is in the *natural order* when, for $x_0 > 0$, $x_i \geqq x_j$ if $i < j$; or when, for $x_0 < 0$, $x_i \leqq x_j$ if $i < j$. Thus any characteristic with $x_0 \neq 0$ which is in the natural order determines, in (7), (8.1), and (9), unique values of $\nu, \delta_0, \cdots, \delta_8, \epsilon, \gamma, \kappa_1, \cdots, \kappa_j$. We determine first some properties of the solutions $(\nu;\, \delta)$ of the equations $\Sigma\delta = 3\nu$ $(\nu = 0, 1, -1)$ $\Sigma\delta^2 = \nu^2 + 2\epsilon$ for given ϵ.

(17) Lemma. *If the solutions* δ *of*

$$\text{(a)}\qquad \sum_{i=0}^{i=8} \delta_i = 3\nu \qquad (\nu = 0, 1, -1)$$

are so arranged that

(b) $$\delta_i \leqq \delta_j \quad \text{if} \quad i < j,$$

then either

(c) $$\epsilon = 0 \quad \text{and} \quad (\nu; \delta) = (0; 0^9)$$

or

(d) $$\begin{aligned} \delta_0 + \delta_2 + \delta_3 - \nu &< 0, \\ \delta_0 + \delta_1 + \delta_3 - \nu &< 0, \\ \delta_0 + \delta_1 + \delta_2 - \nu &< 0. \end{aligned}$$

The last two inequalities are immediate consequences of the first and of (b). If $\delta_0 > 0$, every $\delta > 0$, and (a) is not satisfied. If $\delta_0 = 0$, and $\nu = 0$, every δ must be zero due to (a) and (b). This is the alternative case (c). If $\delta_0 = 0$, and $\nu = -1$, (a) cannot be satisfied. If $\delta_0 = 0$, and $\nu = 1$, (a) can be satisfied only by $(\nu; \delta) = (1; 0^6 1^3)$, $(1; 0^7 1\, 2)$, $(1; 0^8 3)$, all of which satisfy (d).

We may therefore assume that $\delta_0 < 0$, and will divide the proof into four cases:

(A) $\delta_3 \leqq 0$; (B) $\begin{matrix}\delta_3 > 0 \\ \delta_2 \leqq 0\end{matrix}$; (C) $\begin{matrix}\delta_2 > 0 \\ \delta_1 \leqq 0\end{matrix}$; (D) $\delta_1 > 0$.

In case (A) also $\delta_2 \leqq 0$. If either of δ_2 and δ_3 is < 0, then (d) is satisfied since $\delta_0 < 0$ and $-\nu \leqq 1$. If $\delta_2 = \delta_3 = 0$ and $\nu = 0, 1$, (d) is satisfied; if however $\nu = -1$, we have $\delta_0 < 0, \delta_1 \leqq 0$, and $\delta_0 + \delta_1 + \delta_4 + \cdots + \delta_8 = -3$. But, since $\delta_3 = 0$, $\delta_4, \cdots, \delta_8$ are zero or positive, and $\delta_0 + \delta_1 \leqq -3$. Hence $\delta_0 \leqq -2$, and (d) is satisfied.

In case (B) we write (a) in the form

$$-\delta_0 - \delta_1 - \delta_2 = \delta_3 + \cdots + \delta_8 - 3\nu = k - 3\nu \qquad (k \geqq 6).$$

Then $$-\delta_0 \geqq k/3 - \nu, \quad -\delta_3 \geqq -k/6, \quad -\delta_2 \geqq 0$$

and $$-\delta_0 - \delta_2 - \delta_3 \geqq k/6 - \nu.$$

Hence $$\delta_0 + \delta_2 + \delta_3 - \nu \leqq -k/6, \quad \text{or} \quad \delta_0 + \delta_2 + \delta_3 - \nu < 0.$$

The cases (C) and (D) are treated in the same way as the case (B) to complete the proof of the lemma.

(18) LEMMA. *For given* ϵ, ν *the maximum value of* $|\delta_i - \delta_j|$ *is* $2(\epsilon)^{1/2}$. *This maximum is attained only in the cases* $\epsilon = \eta^2$, $(\nu; \delta) = (0; -\eta\, 0^7\, \eta)$.

Proof. Case I: $\nu = 0$. Unless the maximal case mentioned occurs, $\delta_i^2 + \delta_j^2 < 2\epsilon$. When δ_i, δ_j are of opposite sign, $\delta_i^2 + \delta_j^2 \geqq -2\delta_i\delta_j$, since $(\delta_i + \delta_j)^2 \geqq 0$. Hence $\delta_i^2 + \delta_j^2 - 2\delta_i\delta_j < 4\epsilon$, or $|\delta_i - \delta_j| < 2(\epsilon)^{1/2}$.

Case II: $\nu = 1$. Since $\Sigma \delta^2 = \nu^2 + 2\epsilon$, then $\delta_i^2 + \delta_j^2 < 2\epsilon$, except in the two cases $(\nu; \delta) = (1; \eta + 3, 0^7, -\eta)$, $(1; \eta + 2, 1, 0^6, -\eta)$, and the conclusion follows as before. In these two cases ϵ is $\eta^2 + 3\eta + 4$, $\eta^2 + 2\eta + 2$ respectively, and the maximum value of $| \delta_i - \delta_j |$ is $2\eta + 3$, $2\eta + 2$ respectively. The squares of these are respectively less than 4ϵ.

Case III: $\nu = -1$. The proof of this is the same as for Case II, since $-(\nu; \delta) = (-\nu; -\delta)$.

We shall indicate the canonical form (7) of the characteristic $\{x_0; x_1 \cdot \cdot \cdot x_\rho\}$ by the notation $\{\nu, (\delta_0)\delta_1 \cdot \cdot \cdot \delta_{8\kappa}\}$, the γ which determines the order of the characteristic, and the ϵ which with ν determines the δ's, being obtained from the formulae (8.1). The isolation of a particular δ, say δ_0, in the canonical form is indicated by the parenthesis (δ_0).

It will be convenient to have a table of values $(\nu; \delta)$ for early values of ϵ. This, for $\epsilon = 0, \cdot \cdot \cdot, 7$ is:

	ϵ	$\nu = 0$	$\nu = 1$:	ϵ	$\nu = 0$	$\nu = 1$
	0	0^9		6	$-3\ 0^5\ 1^3$	$-2\ -1\ 0^2\ 1^4\ 2$
	1	$-1\ 0^7\ 1$	$0^6\ 1^3$		$-2^2\ 0^3\ 1^4$	$-2\ 0^5\ 1\ 2^2$
	2	$-1^2\ 0^5\ 1^2$	$-1\ 0^4\ 1^4$		$-2\ -1^3\ 1^5$	$-1^3\ 0^2\ 1^2\ 2^2$
			$0^7\ 1\ 2$		$-2\ -1^2\ 0^3\ 1^2\ 2$	$-1^2\ 0^4\ 1^2\ 3$
	3	$-2\ 0^6\ 1^2$	$-1^2\ 0^2\ 1^5$		. . . .	
		$-1^3\ 0^3\ 1^3$	$-1\ 0^5\ 1^2\ 2$	7	$-3\ -1\ 0^3\ 1^4$	$-3\ 0^2\ 1^6$
		. . . .			$-3\ 0^6\ 1\ 2$	$-2^2\ 1^7$
(19)	4	$-2\ -1\ 0^4\ 1^3$	$-2\ 0^3\ 1^5$		$-2^2\ -1\ 0\ 1^5$	$-2\ -1^2\ 1^5\ 2$
		$-2\ 0^7\ 2$	$-1^3\ 1^6$		$-2^2\ 0^4\ 1^2\ 2$	$-2\ -1\ 0^3\ 1^2\ 2^2$
		$-1^4\ 0\ 1^4$	$-1^2\ 0^3\ 1^3\ 2$		$-2\ -1^3\ 0\ 1^3\ 2$	$-2\ 0^5\ 1^2\ 3$
		. . . .	$-1\ 0^6\ 2^2$		. . . .	$-1^4\ 1^3\ 2^2$
			$0^8\ 3$			$-1^3\ 0^3\ 2^3$
	5	$-2\ -1^2\ 0^2\ 1^4$	$-2\ -1\ 0\ 1^6$			$-1^3\ 0^2\ 1^3\ 3$
		$-2\ -1\ 0^5\ 1\ 2$	$-2\ 0^4\ 1^3\ 2$			$-1^2\ 0^5\ 2\ 3$
		. . . .	$-1^3\ 0\ 1^4\ 2$			
			$-1^2\ 0^4\ 1\ 2^2$			
			$-1\ 0^6\ 1\ 3$			

In this table the values δ for $\nu = -1$ are to be obtained by changing the signs of the δ's for $\nu = 1$; furthermore the values δ for $\nu = 0$ are to be supplemented by changing the signs of those values given.

Since there are only a finite number of types of C.T.'s with 8 or fewer

F-points, one would naturally inquire as to the effect upon the canonical characteristic $\{\nu;\ \delta, \kappa\}$ of a C. T. with F-points at the first 8 base points isolated by the δ's. Since the κ's are not changed, the γ and ϵ are unaltered by such a C. T. The Bertini transformation, $B_{12\ldots 8}$, defined by the first eight points, has the description,

$$x'_0 = 17x_0 - \sum_{i=1}^{i=8} 6x_i,$$

$$x'_k = \ 6x_0 - \sum_{i=1}^{i=8} 2x_i - x_k \qquad (k = 1, \cdots, 8).$$

If we apply this to (7), we find that the $(\nu;\ \delta)$ is changed in sign. The Geiser transformation, $G_{2\ldots 8}$, defined by the base points $p_2, \cdots, p_8$ has the description,

$$x'_0 = 8x_0 - \sum_{i=2}^{i=8} 3x_i$$
$$x'_1 = x_1$$
$$x'_k = 3x_0 - \sum_{i=2}^{i=8} x_i - x_k \qquad (k = 2, \cdots, 8).$$

If this is applied to (7), it appears that the $(\nu;\ \delta)$ is changed in sign, and in addition δ_0 and δ_1 are interchanged. Hence

(20) *The characteristic* $\{\nu;\ (\delta_0)\delta_1\delta_2 \cdots \delta_8\ \kappa\}$ *is transformed by the Bertini transformation* $B_{12\ldots 8}$ *into* $\{-\nu;\ -(\delta_0) - \delta_1 \cdots - \delta_8\ \kappa\}$; *by the Geiser transformation* $G_{23\ldots 8}$ *into* $\{-\nu;\ -(\delta_1) - \delta_0 - \delta_2 \cdots - \delta_8\ \kappa\}$; *and by the product* $B_{1\ldots 8}\,G_{2\ldots 8}$ *into* $\{\nu;\ (\delta_1)\delta_0\delta_2 \cdots \delta_8\ \kappa\}$.

The product BG which occurs in (20) is noteworthy as being the fourth type of C. T. which is included among the generating involutions mentioned in the introduction [also cf. [1], p. 253 (6)], the first three types being the transposition $(x_i x_j)$, the Q. T., A_{123}, and the quintic transformation with six double F-points. These four types of involutions are connected respectively with the D-conditions: $\{0;\ 1 - 1\ 0^{\rho-2}\}$, $\{1;\ 1^3\ 0^{\rho-3}\}$, $\{2;\ 1^6\ 0^{\rho-6}\}$, and $\{3;\ 2\ 1^7\ 0^{\rho-8}\}$. They are the only generating involutions with less than 9 F-points.

We examine the effect upon the characteristic $\{\nu;\ \delta\ \kappa\}$ of the other two types of generating involutions, the quintic transformation $Q_{3\ldots 8}$ with double F-points at the last six of the first eight points of the characteristic, and the quadratic transformation A_{123} with simple F-points at the first three points of the characteristic. The results are as follows:

(21) *The characteristic* $\{\nu;\ \delta\,\kappa\}$ *is transformed by* $Q_{3,\ldots,8}$ *into the characteristic* $\{\nu';\ \delta'\,\kappa\}$ *where*

$$\delta'_i = \delta_i + \delta_{012} \quad (i = 0, 1, 2), \qquad \delta'_j = \delta_j + \delta_{012} + \mu \quad (j \neq 0, 1, 2),$$

and where δ_{012} *and* ν'' *are so adjusted in*

$$2\mu = 2(\delta_0 + \delta_1 + \delta_2 - \nu) = \nu'' - 3\delta_{012}$$

that $\nu' = \nu + \nu''$ *takes values* $0,\ 1,\ -1$.

(22) *The characteristic* $\{\nu;\ \delta\,\kappa\}$ *is transformed by* A_{123} *into the characteristic* $\{\nu';\ \delta'\,\kappa\}$ *where*

$$\delta'_i = \delta_i + \delta_{123} \quad (i \neq 1, 2, 3), \qquad \delta'_j = \delta_j + \delta_{123} - \mu \quad (j = 1, 2, 3),$$

and where δ_{123} *and* ν'' *are so adjusted in* $\mu = \delta_1 + \delta_2 + \delta_3 - \nu = 3\delta_{123} - \nu''$ *that* $\nu' = \nu + \nu''$ *takes values* $0,\ 1,\ -1$.

The order of the group generated by the transpositions in a characteristic, and by the quadratic transformations, when applied to the first eight points alone, is finite, and has the value 10 ! 96 [cf. [4], p. 373]. When ϵ is given, the number of solutions $(\nu;\delta)$ increases with increasing ϵ and must eventually exceed the order of this group. Hence in general the solutions $(\nu;\delta)$ for given ϵ must divide into a number of conjugate sets under the operations (22), as well as under the operations (21) and (20), which can be expressed as products of operations (22). Some numerical properties which separate conjugate sets of $(\nu;\delta)$ are as follows:

(23) *For given* ϵ *the sets* $(\nu;\delta)$ *for which* $\delta_i - \nu \equiv 0 \bmod 2$ *constitute one or more conjugate sets.*

(24) *For given* ϵ *the sets* $(0;\delta)$ *for which* $\delta_i - \delta_j \equiv 0 \bmod 3$ *constitute one or more conjugate sets.*

Indeed it is clear that each one of these two properties is invariant under (22). An example of (23) is found in the table (19) for $\epsilon = 4$, $(\nu;\delta) = (0;-2\ 0^7\ 2)$, $(1;-1^3\ 1^6)$, $(-1;-1^6\ 1^3)$, these forming one conjugate set under (22). An example of (24) occurs when

$$\epsilon = 9, \quad (\nu;\delta) = (0;-3\ 0^7\ 3), \quad (0;-2^3\ 1^6), \quad (0;-1^6\ 2^3),$$

these also forming one conjugate set.

A criterion of a different sort is the following:

(25) *For given ϵ the sets $(\nu;\delta)$ which reduce mod 3 to one of the following types (i, j, k being any cyclic advance of $0, 1, -1$),*

$$(0; i^7 j k), \quad (1; i^6 j^3), \quad (-1; i^6 k^3)$$

constitute one or more conjugate sets; while those which reduce to

$$(0; i j^4 k^4), \quad (1; i^6 k^3), \quad (-1; i^6 j^3), \quad (\pm 1; i^3 j^3 k^3), \quad (\pm 1; i^9)$$

also constitute one or more conjugate sets.

This may be verified by applying the operation (22) reduced mod 3. An example of this is found in (19) for $\epsilon = 7$, all of the sets $(\nu;\delta)$ dividing into two conjugate sets according to the criterion (25).

The conjugate sets $(\nu;\delta)$ may be exhibited from another point of view. With the κ given in the characteristic $\{\nu;\delta\kappa\}$, γ and ϵ are known when p and d are known [cf. (8.1)]. Let $(\nu;\delta)$ be a solution attached to this ϵ with $\delta_0, \delta_1, \cdots, \delta_8$ arranged in ascending order and therefore with the first eight multiplicities $x_1, \cdots, x_8$ arranged in descending order. The order, $x_0 = 3(\gamma + \delta_0) - \nu$, of the characteristic for the given $(\nu;\delta)$ will be a minimum if δ_0 is taken to be the least δ. If δ_0 is any other δ, then the order x_0 (if positive) of the characteristic can be reduced by Q. T. at the three first base points [cf. (17)]. Even if δ_0 is the least δ, the positive order can be reduced by Q. T. if $\delta_1 + \delta_2 + \delta_3 - \nu < 0$. Hence

(26) *If the order x_0 of a characteristic $\{\nu;\delta,\kappa\}$ is positive the order $x_0 = 3(\gamma + \delta_0) - \nu$ can be reduced by Cremona transformation with F-points at the first eight points unless δ_0 is the least δ and also $\delta_1 + \delta_2 + \delta_3 - 3 \geqq 0$ $(\delta_0 \leqq \delta_1 \leqq \delta_2 \cdots \leqq \delta_8)$.*

Thus we can obtain the types of characteristics for given κ, p, d (and thereby given γ, ϵ), which are irreducible under Q. T. or C. T. at the first eight points by tabulating only those sets $(\nu;\delta)$ for which $\delta_1 + \delta_2 + \delta_3 - \nu \geqq 0$. This much restricted set of solutions $(\nu;\delta)$ for $\epsilon = 0$ up to $\epsilon = 15$ follows:

	ϵ	$\nu = 0$	$\nu = 1$	$\nu = -1$
	0	0^9		
	1	$-1\ 0^7\ 1$		
	2			$-2\ -1\ 0^7$
	3	$-2\ 0^6\ 1^2$		
	4	$-2\ 0^6\ 2$		$-3\ 0^8$
	5			$-3\ -1\ 0^6\ 1$
	6	$-3\ 0^5\ 1^3$		
(27)	7	$-3\ 0^6\ 1\ 2$	$-3\ 0^2\ 1^6$	
	8		$-3\ -1\ 1^7$	$-4\ 0^7\ 1$
	9	$-3\ 0^7\ 3$		$-4\ -1\ 0^5\ 1^2$
	10	$-4\ 0^4\ 1^4$		$-4\ -1\ 0^6\ 2$
	11	$-4\ 0^5\ 1^2\ 2$	$-4\ 0\ 1^7$	
	12	$-4\ 0^6\ 2^2$	$-4\ 0^2\ 1^5\ 2$	
	13	$-4\ 0^6\ 3\ 1$	$-4\ -1\ 1^6\ 2$	$-5\ 0^6\ 1^2$
	14			$-5\ 0^7\ 2$ $-5\ -1\ 0^4\ 1^3$.
	15	$-5\ 0^3\ 1^5$	$-5\ -1\ 0^5\ 1\ 2$	

The criterion (23) distinguishes between the two solutions for $\epsilon = 4, 8, 12$; the criterion (24) distinguishes between the two solutions for $\epsilon = 9$; and the criterion (25) distinguishes between the two solutions for $\epsilon = 7, 10$, and partially distinguishes among the three solutions for $\epsilon = 13$. None of them distinguishes between the two solutions for $\epsilon = 11$, though each of these solutions gives rise to a conjugate set of solutions $(\nu; \delta)$. As ϵ increases the number of these restricted solutions $(\nu; \delta)$ also increases. However the increase is much less rapid than that of all the solutions $(\nu; \delta)$.

By changing the sign of a characteristic we prove similarly that

(28) *If the order* x_0 *of a characteristic* $\{\nu; \delta, \kappa\}$ *is negative this order can be increased by* C. T. *with F-points at the first eight points unless* δ_0 *is the greatest* δ *and also* $\delta_1 + \delta_2 + \delta_3 - \nu \gtrless 0$ $(\delta_0 \geqq \delta_1 \geqq \delta_2 \cdots \geqq \delta_8)$.

Assuming that the given characteristic has a positive order, and positive or zero multiplicities, we would normally select $\kappa_1, \cdots, \kappa_j$ to be the least multiplicities, so as to have a minimum ϵ and a minimum γ for the characteristic [cf. (8.1)]. Then ν and δ_0 are determined from the order, and $\delta_1, \cdots, \delta_8$ from the first eight multiplicities. If we interchange the eighth and ninth points of multiplicities, $\gamma + \delta_0 - \delta_8$ and κ_1, respectively, then

ν, $\delta_1, \cdots, \delta_7$, $\kappa_2, \cdots, \kappa_j$ are unaltered while γ, δ_0, δ_8, κ_1 are replaced by γ', δ'_0, δ'_8, κ'_1 where

$$\begin{gathered}\gamma' = \gamma + C_{89},\ (-\delta'_0) = (-\delta_0) + C_{89},\ \delta'_8 = \delta_8 + C_{89},\quad \kappa'_1 = \kappa_1 + C_{89},\\ C_{89} = \gamma - (-\delta_0) - \delta_8 - \kappa_1 = (\gamma + \delta_0 - \delta_8) - \kappa_1.\end{gathered} \tag{29}$$

This is the transformation (3) on the four arguments γ, $-\delta_0$, δ_8, κ_1. The additive constant C_{89} is the difference of the orders of the two points interchanged. We find that $\epsilon' = \epsilon + (\gamma - \kappa_1)(\gamma' - \gamma)$. Thus if γ is reduced, ϵ is also reduced since $\gamma - \kappa_1$ is positive.

We shall later carry out the reduction of characteristics to those of lower order by first reducing the order as far as possible by C. T. at the first eight points, and then, if possible, lower the ϵ by the interchange just mentioned.

3. The addition of characteristics. We readily find from the equations (2) that the sum,

$$\{x_0''; x_1'' \cdots x_\rho''\} = \{x_0; x_1 \cdots x_\rho\} + \{x_0'; x_1' \cdots x'_\rho\} \tag{30}$$

of the characteristics x and x' of dimension d, d' and genus p, p' respectively is a characteristic x'' of dimension d'' and genus p'', where

$$\begin{gathered}x_i'' = x_i + x_i',\ d'' = d + d' + D,\ p'' - 1 = (p-1) + (p'-1) + D,\\ D = x_0x_0' - \sum_{i=1}^{i=\rho} x_i x_i'.\end{gathered} \tag{31}$$

Thus D is the number of intersections of a member of the one system with a member of the other outside the ρ common base points of the two systems.

Again, if the given characteristic x has dimension d and genus p, the multiple $kx \equiv \{kx_0; kx_1 \cdots kx_\rho\}$ has dimension d' and genus p', where

$$\begin{gathered}d' = \binom{k+1}{2} d + \binom{k}{2}(p-1),\\ p' - 1 = \binom{k}{2} d + \binom{k+1}{2}(p-1).\end{gathered} \tag{32}$$

In particular, the value $k = -1$ yields for the characteristic $-x$ the values,

$$d' = p - 1, \quad p' - 1 = d. \tag{33}$$

An especially notable characteristic is $L \equiv \{3; 1^\rho\}$ which is invariant under the quadratic transformation (3). If the characteristic x with given p, d be added to kL, a characteristic $x + kL$ with p', d' is obtained such that

$$\begin{gathered}d' = (k+1)d - k(p-1) - \binom{k+1}{2}(\rho - 9),\\ p' - 1 = kd - (k-1)(p-1) - \binom{k}{2}(\rho - 9).\end{gathered} \tag{34}$$

Combining this with (33), we find that p' and d' for the characteristic $-x+kL$ are

$$(35) \qquad \begin{aligned} d' &= -kd + (k+1)(p-1) - \binom{k+1}{2}(\rho-9), \\ p'-1 &= -(k-1)d + k(p-1) - \binom{k}{2}(\rho-9). \end{aligned}$$

It is evident from the derivation that the substitutions (34) on $d, p-1$ of determinant $+1$ constitute for all integer values of k an infinite cyclic group which is amplified by the substitutions (35) of determinant -1 and period two into an infinite dihedral group.

An interesting application of (35) arises from the question as to when a Cremona net $p=0$, $d=2$ is converted into a Cremona net $p'=0$, $d'=2$. After factoring out $k+1$ and $k-1$ respectively, which cannot vanish simultaneously, we are left with a single condition,

$$k(\rho-9) = -6.$$

This yields eight solutions

$$(36) \qquad \begin{array}{l} \rho = \ 10;\ \ 11;\ \ 12;\ \ 15;\ 8;\ 7;\ 6;\ 3 \\ k = -6;\ -3;\ -2;\ -1;\ 6;\ 3;\ 2;\ 1. \end{array}$$

The cases in which k is positive are those cases $\rho < 9$ for which a symmetric C. T. exists. Multiplication by this symmetric transformation pairs the Cremona nets. This pairing is also effected by a change of sign of the characteristic and the addition of kL. But we shall also find all the Cremona characteristics, whose order is negative for $\rho = 10$, by changing the sign of the characteristics of positive order and subtracting $6L$. A similar procedure is evidently possible for $\rho = 11$, 12, and 15. We find in section 10 *virtual Cremona transformations* corresponding to these symmetric cases $\rho = 10$ and $\rho = 11$.

It is clear that, with the notion of addition of characteristics in mind, any characteristic x with positive order x_0, positive multiplicities $x_1, \cdots, x_k$, and negative multiplicities $x_{k+1}, \cdots, x_\rho$ will give rise to a degenerate linear system, if there exists a linear system of curves with characteristic $\{x_0; x_1 \cdots x_k\}$. In fact this latter system, taken with the sets of directions at the base points $p_{k+1}, \cdots, p_\rho$ taken x_{k+1} times, $\cdots, x_\rho$ times respectively, represents the original system. The original characteristic can be transformed into purely positive characteristics which define linear systems which degenerate into precisely the transforms of these components. An example of this is found in the nets (13). On the other hand any characteristic with negative x_0 is necessarily not constructible, and therefore is virtual.

We proceed to apply the ideas and results above to the determination of D-conditions, P-curves, and C-nets, for $\rho = 9$ and $\rho = 10$, the D-conditions for P_9^2 having already been obtained. For these cases we shall find only a small number of conjugate sets, whereas for P_{11}^2 we shall find that even the number of conjugate sets under C. T. is infinite.

4. *P*-curves for P_9^2. In this case, $p = d = 0$, and [cf. (8.1)] $\kappa_2 = \kappa_3 = \cdots = 0$, whence

$$\lambda = 1, \quad \gamma = \kappa_1 + 1, \quad \epsilon = \kappa_1 + 1.$$

If the P-characteristic has ordered multiplicities, then $\gamma + \delta_0 - \delta_8 \geqslant \kappa_1$, and $\delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_8$. Thus

$$\delta_0 - \delta_8 \geqslant -1. \tag{37}$$

Subject to this limitation, we find for $\epsilon = 0$ and $\epsilon = 1$ the following types of P-curves:

$$\begin{array}{llll} \epsilon = 0 : (\nu;\delta) = (0;0^9): & P\{0;0^8-1\}; & \\ \epsilon = 1 : (\nu;\delta) = (0;-1\,0^7\,1): & P\{3;2\,1^6\,0^2\}, & P\{6;3\,2^7\,0^2\}; \\ \qquad\qquad\quad = (1;0^6\,1^3): & P\{2;1^5\,0^4\}, & P\{5;2^6\,1^2\,0\}; \\ \qquad\qquad\quad = (-1;-1^3\,0^6): & P\{1;1^2\,0^7\}, & P\{4;2^3\,1^5\,0\}. \end{array} \tag{38}$$

We observe first that the order of the characteristic is zero as in $P\{0;0^8-1\}$, or it is positive. This is true for $\gamma = \epsilon = 0, 1$, as in the above table. We examine therefore only the cases $\gamma \geqslant 2$. The generic order is $3(\gamma + \delta_0) - \nu$. Since in (37) δ_8 is either the greatest δ, or the greatest after δ_0, then $\delta_8 \geqslant 0$. Thus $\delta_0 \geqslant -1$. Also $-\nu \geqslant -1$. Hence

$$3(\gamma + \delta_0) - \nu \geqslant 6 - 3 - 1.$$

We observe secondly that, unless $\epsilon = 0$, $(\nu;\delta) = (0;0^9)$, the order of the characteristic can be reduced by Q. T. This follows immediately from (26) and (17) if δ_0 is not the smallest δ. Let then δ_0 be the smallest δ, whence δ_8 is the largest. Then $\delta_0 < 0$ except for $(\nu;\delta) = (0;0^9)$, $(1;0^6\,1^3)$, $(1;0^7\,1\,2)$; and $\delta_8 > 0$ except for $(\nu;\delta) = (0;0^9)$, $(-1;-1^3\,0^6)$, $(-1;-2-1\,0^7)$. Two of these cases are eliminated by (37). For two others $\delta_1 + \delta_2 + \delta_3 - \nu < 0$ and the corresponding P-characteristics are reducible. Hence the order of every P-characteristic is reducible except that formed from $\epsilon = 0$, $(\nu;\delta) = (0;0^9)$, which is $P\{0;0^8-1\}$, the set of directions about the point p_9.

When the order is reduced, the reduction can be carried out by Q. T. at the first eight points until one at least of the first eight multiplicities is less than the ninth multiplicity κ_1, unless κ_1 is already -1 and $\epsilon = 0$. When this reduced characteristic is rearranged into the natural order, κ_1 is replaced by a $\kappa' < \kappa_1$, and therefore ϵ is replaced by an $\epsilon' < \epsilon$. This reduction of ϵ can be continued until $\epsilon = 0$. Hence every P-characteristic can be reduced by Q. T. to the type $P\{0; 0^8 - 1\}$, and therefore it defines a P-curve which can be transformed by C. T. into the directions about a point. Hence

(39) *Every P-characteristic for P_9^2 has the form,*

$$P\{3(\epsilon + \delta_0) - \nu; \quad \epsilon + \delta_0 - \delta_1,\ \epsilon + \delta_0 - \delta_2, \cdots, \epsilon + \delta_0 - \delta_8,\ \epsilon - 1\}$$

where ν is chosen from $0, 1, -1$; where eight of the δ's are arbitrary, and the ninth is determined from $\Sigma\delta = 3\nu$; and where ϵ is determined from $\Sigma\delta^2 = \nu^2 + 2\epsilon$. Every such P-characteristic defines a geometrically existent P-curve. The P-curves all are in a single conjugate set being reducible by C. T. *to the unique type $P\{0; 0^8 - 1\}$ of zero order. All other types have positive order and positive or zero multiplicities.*

If the δ's are so arranged that

$$\delta_0 - \delta_8 \geqq -1, \quad \delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_8,$$

then each characteristic is arranged in the natural order, and has a unique representation.

5. *C*-nets for P_9^2. For these homaloidal nets $p = 0$, $d = 2$, and

$$\lambda = 3, \quad \gamma = \kappa_1 + 3, \quad \epsilon = 3\kappa_1 + 4.$$

Since $\epsilon \geqq 0$, the lowest multiplicity which can occur in the C-characteristic is $\kappa_1 = -1$ for which $\epsilon = 1$ and

$$(\nu; \delta) = (0; -1\ 0^7\ 1), \quad (1; 0^6\ 1^3), \quad (-1; -1^3\ 0^6).$$

The corresponding nets are

$$(40) \quad \begin{matrix} C\{9; 4\ 3^7 - 1\}, & C\{8; 3^6\ 2^2 - 1\}, & C\{7; 3^3\ 2^5 - 1\}, & C\{6; 3\ 2^6\ 1 - 1\}, \\ C\{5; 2^5\ 1^3 - 1\}, & C\{4; 2^2\ 1^6 - 1\}, & C\{3; 1^7\ 0 - 1\}. & \end{matrix}$$

These are all equivalent to the last under C. T. at the first eight points.

If the lowest multiplicity is $\kappa_1 = 0$, then $\epsilon = 4$. These, being C-characteristics for P_8^2, have been determined in (15). There are 35 *proper* (i. e. geometrically existent and non-degenerate) nets which are reducible by Q. T. to the type $C\{1; 0^9\}$, the net of lines; and 6 *degenerate* (i. e. also geometrically existent) nets which are reducible by Q. T. to the last type (40).

The order of the C-characteristic is always positive. For, $3(\gamma + \delta_0) - \nu$

is zero or negative only if $\gamma + \delta_0 \leqq 0$, or if $\gamma = \kappa_1 + 3 \leqq |\delta_0|$. But $\delta_0{}^2 < \Sigma\delta^2 = \nu^2 + 2\epsilon$. Hence $\gamma \leqq \delta_0$ only if $(\kappa_1 + 3)^2 < \nu^2 + 2\epsilon$, or if $(\kappa_1 + 3)^2 < 1 + 2(3\kappa_1 + 4)$, or if $\kappa_1{}^2 < 0$, which is impossible.

The order of the C-characteristic is always reducible except in the two cases $C\{1;0^9\}$, $C\{3;1^7\,0 - 1\}$. For, if the characteristic is arranged in the natural order, $\delta_1 \leqq \delta_2 \cdots \leqq \delta_8$; and $\gamma + \delta_0 - \delta_8 \geqq \kappa_1$, i. e., $\delta_0 - \delta_8 \geqq -3$. According to (26) and (17), the order and three of the multiplicities can be reduced unless both δ_0 is the smallest δ and $\delta_1 + \delta_2 + \delta_3 - \nu \geqq 0$. Solutions $(\nu;\delta)$ of this latter type (in which δ_8 is the largest δ) are tabulated in (27), and $\delta_0 - \delta_8 \geqq -3$ is valid only when $\epsilon \leqq 4$. But these values of ϵ have been considered above, and found to yield reducible types except in the two cases mentioned. Hence the order and three of the multiplicities can be reduced step by step until either a multiplicity 0 or -1 is reached as above, or until a multiplicity less than κ_1 is obtained. Then a rearrangement of the characteristic yields an $\epsilon' < \epsilon$. Eventually then an $\epsilon = 1, 4$ must be obtained. Hence

(41) *All C-characteristics for $P_9{}^2$ have a positive order and lie in two sets conjugate under* Q. T. *The one conjugate set consists of* PROPER *homaloidal nets reducible by* Q. T. *to the net of lines $C\{1;0^9\}$. The other conjugate set consists of* DEGENERATE *nets, each made up of a proper elliptic net and the fixed canonical adjoint of the net. These nets are reducible by* Q. T. *to the net $C\{3;1^7\,0 - 1\}$, consisting of an elliptic net on seven points and the set of directions about some eighth point.*

A criterion which determines the conjugate set to which a given C-characteristic belongs is the following:

(42) *A given C-characteristic for $P_9{}^2$, $x \equiv \{x_0;x_1,\cdots,x_9\}$, defines a proper homaloidal net if the integers in x reduce modulo* 3 *to one of the following sets:*

(a) $\{0;ij^4k^4\}$, $\{1;i^6k^3\}$, $\{-1;i^6j^3\}$, $\{\pm 1;i^3j^3k^3\}$, $\{\pm 1;i^9\}$

(i, j, k being any cyclic advance of 0, 1, -1); *it defines a degenerate net if x reduces mod* 3 *to one of*

(b) $\{0;i^7\,j\,k\}$, $\{1;i^6\,j^3\}$, $\{-1;i^6\,k^3\}$.

For, it is easily verified that the Q. T., A_{123}, in (3), when reduced mod 3, transforms these two sets of reduced characteristics each into itself. Since $C\{1;0^9\}$ is in one set, and $C\{3;1^7\,0 - 1\}$ is the other, their conjugates under Q. T. must reduce mod 3 to members of the one set, or of the other, respectively.

We now seek to determine the conditions on the $(\nu;\delta)$ which ensure that the C-characteristic derived from the δ's may yield a proper homaloidal net. We reduce the δ's, ϵ, and γ mod 3. Then $\epsilon \equiv 1$, and $\gamma = \kappa_1 + 3 \equiv \kappa_1$. However κ_1 in $\epsilon = 3\kappa_1 + 4$ remains as a parameter. The conditions on the δ's reduce to

$$\Sigma\delta \equiv 0, \quad \Sigma\delta^2 \equiv 2\,(\nu = 0), \quad \Sigma\delta^2 \equiv 0\,(\nu = \pm 1).$$

Then the canonical characteristic reduces to

$$\{-\nu;\kappa_1+\delta_0-\delta_1,\ \kappa_1+\delta_0-\delta_2,\cdots,\ \kappa_1+\delta_0-\delta_8,\ \kappa_1+\delta_0-\delta_0\}.$$

This distribution of $\{x_0;x_1\cdots x_9\}$ arises from $(\nu;\delta)$ by changing the sign of the $\{\nu;\delta\}$ and by adding the $\{0;(\kappa_1+\delta_0)^9\}$, where $\kappa_1+\delta_0 \equiv 0, 1, -1 \bmod 3$. If then from any one of the set of reduced characteristics in (42 a) we subtract $\{0;(\kappa_1+\delta_0)^9\}$, and change the sign, we get a reduced set of $\{\nu;\delta\}$, which will determine the reduced characteristics. But this subtraction and change of sign leaves the list (42 a) as a whole unaltered. Hence

(43) *Given any partition of* 3ν $(\nu = 0, 1, -1)$ *into* $\delta_0,\cdots,\delta_8$ *for which* $(\nu;\delta)$ *is congruent mod* 3 *to any of the sets in* (42 a), *then this partition defines a proper homaloidal net*

$$\{3(\gamma+\delta_0)-\nu;\ \gamma+\delta_0-\delta_1,\cdots,\gamma+\delta_0-\delta_8,\kappa_1\},$$

where ϵ, κ_1, γ *are defined by* $\Sigma\delta^2 = \nu^2 + 2\epsilon$, $\epsilon = 3\kappa_1 + 4$, $\gamma = \kappa_1 + 3$.

To obtain such a characteristic once and only once it is sufficient to ensure that the multiplicities are given in the natural descending order. For selected δ_0 this requires that $\delta_1 \overset{=}{<} \delta_2 \overset{=}{<} \cdots \overset{=}{<} \delta_8$. Furthermore $\gamma+\delta_0-\delta_8 \overset{=}{>} \kappa_1$ or $\delta_0-\delta_8 \overset{=}{>} -3$. Hence

(44) *If, in the partition* $(\nu;\delta)$ *in* (43), δ_0 *is so selected that* $\delta_0-\delta_8 \overset{=}{>} -3$, δ_8 *being the largest remaining* δ, *and if* $\delta_1,\cdots,\delta_8$ *are so selected that* $\delta_1 \overset{=}{<} \delta_2 \overset{=}{<} \cdots \overset{=}{<} \delta_8$, *then each homaloidal net is obtained once and only once in the natural order.*

According to (42 a) all proper C-nets for P_9^2 are comprised under seven classes according as they reduce mod 3 to one or another of the seven types in (42 a). These are the seven types found by Dr. Taylor.[2] In particular, the reduced types $\{1;i^9\}$, $\{-1;i^9\}$ yield Dr. Taylor's types I, II, respectively. Her remaining five types are given as products

$$\text{I}\ \ A_{123},\qquad \text{II}\ \ A_{123},\qquad \text{I}\ \ A_{123}A_{456},\qquad \text{II}\ \ A_{123}A_{456},\qquad \text{I}\ \ A_{123}A_{145}.$$

That this enumeration is complete follows almost immediately from the verification that the reduced types $(\pm 1; i^9)$ yield, under multiplication by the Q. T.'s indicated, the five remaining reduced types in the set (42 a). Some corrections of the values given for the n, r_i, s_j, α_{ij} of these seven types are given by Dr. Barber.[3]

6. **D-conditions for P_{10}^2.** With $p = 0$, $d = -1$, the equations (2), when satisfied by $\{x_0; x_i\}$, are also satisfied by $\{-x_0; -x_i\}$, so that any D-characteristic can be expressed in positive or negative form. This indeed was a property of the original D-condition $\{0; 1, -1, 0^{\rho-2}\}$, which expresses the coincidence of p_1 and p_2. The equations (8.1) and (9) yield

$$\lambda = 0, \quad \epsilon - 1 = \kappa_1 \cdot \kappa_2, \quad \gamma = \kappa_1 + \kappa_2.$$

When $\epsilon = 0$, $(\nu; \delta) = (0; 0^9)$, $\kappa_1 = 1$, $\kappa_2 = -1$, we find only one D-characteristic, $\{0; 0^8\, 1\, -1\}$. When $\epsilon = 1$, and

$$(\nu; \delta) = (0; -1\, 0^7\, 1), \quad (1; 0^6\, 1^3), \quad (-1; -1^3\, 0^6),$$

we find that $\kappa_2 = 0$, $\kappa_1 = \kappa_1$, and obtain the D-characteristics given in (16). These are positive or zero throughout if $\kappa_1 \geqq 1$; negative or zero throughout if $\kappa_1 \leqq -1$.

(45) *Every positive factorization of $\epsilon - 1 = \kappa_1 \cdot \kappa_2$ $(\epsilon > 1)$ yields a D-characteristic which is positive or zero throughout; every negative factorization one which is negative or zero throughout.*

For, if we examine first the positive factorizations, we see that the theorem is true if there is only one δ which is different from 0. These two cases indeed are $(\nu; \delta) = (1; 0^8\, 3)$, $(-1; -3\, 0^8)$, for each of which $\epsilon - 1 = 3 = 3.1$ and $\gamma = 4$. Thus the order $3(\gamma + \delta_0) - \nu$ is positive in each case, and $\gamma + \delta_0 - \delta_i$ is positive in each case. If however two or more of the δ's are not zero, then

$$(a) \qquad \delta_0^2 < \nu^2 + 2\epsilon = \nu^2 + 2(\kappa_1\kappa_2 + 1).$$

If now $3(\gamma + \delta_0) - \nu \leqq 0$, then either

$$(b_1) \qquad \gamma + \delta_0 < 0, \quad |\delta_0| > \kappa_1 + \kappa_2, \quad \delta_0^2 > \kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2,$$

or

$$(b_2) \quad \gamma + \delta_0 = 0, \quad \nu = 0, \quad |\delta_0| = \kappa_1 + \kappa_2, \quad \delta_0^2 = \kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2.$$

On comparing the values of δ_0^2 in (a) and (b), we have

$$(b_1) \quad \nu^2 + 2 - \kappa_1^2 - \kappa_2^2 \geqq 2; \qquad (b_2) \quad 2 - \kappa_1^2 - \kappa_2^2 \geqq 1.$$

Since $\nu^2 = 0, 1$, and $\kappa_1 \geqq 1$, $\kappa_2 \geqq 1$, (b_1) and (b_2) cannot subsist, whence the order $3(\gamma+\delta_0)-\nu>0$. Furthermore a negative factorization of $\epsilon-1$ yields a D-characteristic with a negative order. For, otherwise a change of sign of the characteristic would give a new characteristic in which a negative order would arise from a positive factorization. Finally we observe that, if the order is positive, both positive and negative multiplicities cannot occur. For, we might take two such opposite multiplicities to be κ_1, κ_2, and then $\epsilon-1=\kappa_1\kappa_2<0$; whence $\epsilon<1$ or $\epsilon=0$. But $\epsilon=0$ contributes only the case $\{0;0^8\,1-1\}$ of zero order.

The reducibility under Q. T. of the positive characteristics is a consequence of the usual theory. If, in the equations (2) with

$$x_1 \geqq x_2 \geqq \cdots \geqq x_\rho \geqq 0,$$

the terms in x_1, x_2 are transposed, and if the second equation multiplied by x_3 be subtracted from the first, then

$$(I_1)\qquad 0 \geqq x_0(x_0-3x_3)+x_3(x_1+x_2) \\ -x_1^2-x_2^2-(1-p)(1+x_3)-d(1-x_3).$$

This inequality is necessarily valid only for $x_i \geqq 0$. The characteristic is reducible only if $x_1+x_2+x_3>x_0$. Suppose then

$$(I_2)\qquad x_0 \geqq x_1+x_2+x_3.$$

Eliminating x_0 from (I_1), we get

$$(I_3)\qquad 2x_3^2 \geqq 2x_1x_2+(1-p+d)x_3+(1-p-d).$$

If I_3 is *not* satisfied, the characteristic is reducible by Q. T. For D-, P-, and C-characteristics, (I_3) reads:

$$(I_4)\qquad \begin{array}{ll} p=0,\ d=-1: & 2x_3^2 \geqq 2x_1x_2+2; \\ p=0,\ d=0: & 2x_3^2 \geqq 2x_1x_2+x_3+1; \\ p=0,\ d=2: & 2x_3^2 \geqq 2x_1x_2+3x_3+1. \end{array}$$

We are here interested in positive characteristics for which $x_{10}>0$ since those for which $x_{10}=0$ have been obtained. Thus $x_3>0$ and none of the inequalities (I_4) are satisfied. Hence

(46) *All D-, P-, C-characteristics for P_{10}^2 with $x_1 \geqq x_2 \geqq \cdots \geqq x_{10}>0$ are reducible by* Q. T. *until x'_{10} is either zero or negative.*

If the canonical characteristic (7) for positive κ_1, κ_2 is arranged in descending order, then $\gamma+\delta_0-\delta_8 \geqq \kappa_1$, or $\kappa_2 \geqq \delta_8-\delta_0$. Hence

(47) *If $\nu=0,1,-1$ is selected, as well as eight of the δ's in $\Sigma\delta=3\nu$, if $\epsilon>0$ is determined from $\Sigma\delta^2=\nu^2+2\epsilon$, and if $\epsilon-1=\kappa_1\cdot\kappa_2$ is any positive*

factorization; if furthermore δ_0 *is selected from the* δ's *in such wise that* $\kappa_1 \geqq \kappa_2 \geqq \delta_8 - \delta_0$ *when* $\delta_1 \geqq \delta_2 \geqq \cdots \geqq \delta_8$, *then each D-condition with positive order, and positive or zero multiplicities is obtained from* (7) *once and only once in the natural order. The negative factorizations* $(-\kappa_1)(-\kappa_2)$ *yield these same D-conditions changed in sign. All these are equivalent under* Q. T. *to the one-reduced form* $D\{0; 1\,0^8 - 1\}$ *which is the only D-condition with both positive and negative multiplicities and which arises from the factorization* $-1 = (1)(-1)$ *for* $\epsilon = 0$.

7. *P*-curves for P_{10}^2. With $p = d = 0$ the equations (8.1) and (9) yield

$$\epsilon = (\kappa_1 + 1)(\kappa_2 + 1), \qquad \gamma = \kappa_1 + \kappa_2 + 1.$$

For $\epsilon = 0, \quad (\nu; \delta) = (0; 0^9), \quad \kappa_2 = -1, \quad \kappa_1 = \zeta - 1, \quad \gamma = \zeta - 1.$

We have then an infinite number of reduced forms,

$$P\{3(\zeta - 1); (\zeta - 1)^9, -1\}. \tag{48}$$

These are not conjugate under Q. T., since each can be expressed in a particular way as $(\zeta - 1)$ times the elliptic characteristic $E\{3; 1^9\,0\}$ plus the P-charactersitic $P\{0; 0^9 - 1\}$.

Let then $\epsilon > 0$, and consider any positive factorization of ϵ into $(\kappa_1 + 1)(\kappa_2 + 1)$. For this, $\kappa_1 \geqq 0$, $\kappa_2 \geqq 0$, and $\gamma > 0$. If one of the κ_1, κ_2 is zero, we have a P-characteristic for P_9^2 with $\epsilon > 0$ which, according to (39), has a positive order and positive or zero multiplicities. If κ_1, κ_2 are both positive, then $\gamma + \delta_0 > 0$ if $\gamma^2 > \delta_0^2$. But $\delta_0^2 < 2\epsilon$ if $\epsilon > 0$. And $\gamma^2 > 2\epsilon$ if $\kappa_1^2 + \kappa_2^2 > 1$, which is true. But, if $\gamma + \delta_0 > 0$, the order $3(\gamma + \delta_0) - \nu$ is positive. On the other hand, a negative factorization of $\epsilon > 0$ yields $\kappa_1 \leqq -2$, $\kappa_2 \leqq -2$, $\gamma \leqq -3$. The order now is negative if $\gamma + \delta_0 < 0$, which again is satisfied due to $\kappa_1^2 + \kappa_2^2 > 1$.

Let any P-characteristic with positive order be arranged with descending multiplicities. Then either it arises as in (48) from $\epsilon = 0$, $\zeta - 1 > 0$, or it arises from a positive factorization of $\epsilon > 0$, since the order is positive. Hence the two smallest multiplicities are zero or positive, and all of the multiplicities are zero or positive. Hence, according to (46), it can be reduced by Q. T. until an $\epsilon' < \epsilon$ is obtained for the ordered characteristic, and finally then be reduced to a type (48) for $\zeta - 1 \geqq 0$.

Let any P-characteristic with negative order be arranged with ascending multiplicities. Then either it arises as in (48) from $\epsilon = 0$, $\zeta - 1 < 0$ or it arises from a negative factorization of $\epsilon > 0$ since its order is negative. Hence the two largest multiplicities κ_1, κ_2 are $\leqq -2$, and all the multiplicities are negative. We wish to prove that this characteristic is reducible by Q. T. to the form (48). Let the sign of the characteristic be changed, thus producing a characteristic $\{y\}$ which is positive throughout and in descending order, for which $p = 1$, $d = -1$ [cf. (33)]. Then the inequality (I_3) preceding (46) shows that $\{y\}$ is reducible unless $2y_3^2 \geqq 2y_1y_2 - y_3 + 1$. If y_1 or y_2 is greater than y_3, this cannot be satisfied. Let then $y_1 = y_2 = y_3$. Reverting to the argument preceding (I_3) we now find that

$$y_4(y_4 - y_3) + \cdots + y_{10}(y_{10} - y_3) = y_0(y_0 - 3y_3) + 1 - y_3.$$

The left member is zero or negative; the right is positive if $y_0 > 3y_3$. If $y_0 < 3y_3$, $\{y\}$ is reducible. Hence we have to consider only the possibility of an irreducible case $y_1 = y_2 = y_3$, $y_0 = 3y_3$. Then the original equations become

$$y_4 + \cdots + y_{10} = 6y_3 + 1, \qquad y_4^2 + \cdots + y_{10}^2 = 6y_3^2 + 1,$$

or

$$y_4(y_3 - y_4) + \cdots + y_{10}(y_3 - y_{10}) = y_3 - 1.$$

If $y_3 > y_4$, then the left member of this last equation is at least equal to

$$y_4 + \cdots + y_{10} = 6y_3 + 1.$$

Hence $y_4 = y_3$, and similarly

$$y_5 = y_6 = y_7 = y_8 = y_3,$$

and the original equations become

$$y_9 + y_{10} = y_3 + 1, \quad y_9^2 + y_{10}^2 = y_3^2 + 1, \quad \text{or} \quad y_9 = y_3, \quad y_{10} = 1.$$

Thus $\{y\}$ either is reducible by Q. T., or is $\{3y_3; y_3^9\, 1\}$; whence the original negative P-characteristic $\{x\}$ is either reducible by Q. T., or is $\{-3y_3; -y_3^9 - 1\}$. Hence

(49) *A P-characteristic for* P_{10}^2 *with negative order is negative throughout. Such a characteristic is* VIRTUAL, *and is reducible by* Q. T. *to one of the forms* $P\{3(\zeta - 1); (\zeta - 1)^9, -1\}$ $(\zeta - 1 < 0)$. *A P-characteristic with positive order, which is positive or zero throughout, either is* DEGENERATE, *and reducible by* Q. T. *to one of the forms* $P\{3(\zeta - 1); (\zeta - 1)^9, -1\}$ $(\zeta - 1 > 0)$; *or is* PROPER *and reducible by* Q. T. *to the form* $P\{0; 0^9 - 1\}$. *The characteristics* (48) *for* $\zeta - 1 > 0$ *are the only ones which have both positive and negative integers.*

There is a very simple criterion which determines to which of the infinite number of conjugate classes (48) a particular P-characteristic belongs.

(50) *Let* $\{x\}$ *be a given P-characteristic and let* ζ *be the positive* G. C. D. *of the integers* $x_0 + 3, x_1 + 1, x_2 + 1, \cdots, x_{10} + 1$. *If* x_0 *is positive or zero, and* $\zeta = 1$, P *is a* PROPER *characteristic which defines a P-curve. If* x_0 *is positive and* $\zeta > 1$, P *is a* DEGENERATE *characteristic which defines an elliptic curve taken* $\zeta - 1$ *times and a P-curve which is the canonical adjoint of the elliptic curve. If* x_0 *is negative and* $\zeta > 1$, P *is a* VIRTUAL *characteristic.* P *is reducible by* Q. T. *to the form* $\{3(\zeta - 1)\epsilon; [(\zeta - 1)\epsilon]^9, -1\}$, *where* ϵ *is* $+1$ *or* -1 *according as* x_0 *is positive or negative.*

For, this criterion is satisfied by the reduced forms and is invariant under the Q. T. as given in (3).

8. *C*-nets for P_{10}^2. With $p = 0$, $d = 2$ the equations (8.1), (9) yield

$$\epsilon + 5 = (\kappa_1 + 3)(\kappa_2 + 3), \qquad \gamma = \kappa_1 + \kappa_2 + 3.$$

We prove first that

(51) *The negative factorizations of* $\epsilon + 5 = (\kappa_1 + 3)(\kappa_2 + 3)$ *yield C-characteristics which are negative throughout.*

For, if $\epsilon + 5 = (-j_1)(-j_2)$, where $j_1 \geqq 1$, $j_2 \geqq 1$, $j_1 j_2 \geqq 5$, then $\kappa_1 = -j_1 - 3 < 0$ and $\kappa_2 = -j_2 - 3 < 0$. Also $\gamma = -j_1 - j_2 - 3$, $\epsilon = j_1 j_2 - 5$. The order, $3(\gamma + \delta_0) - \nu$, of the net is negative if $\gamma + \delta_0 < 0$. Suppose that $\gamma + \delta_0 \geqq 0$. Then $\delta_0 \geqq j_1 + j_2 + 3$, and $\delta_0^2 \geqq (j_1 + j_2 + 3)^2$. But $2\epsilon \geqq \delta_0^2 - 1$, since $\Sigma\delta^2 = \nu^2 + 2\epsilon$. Hence $2\epsilon \geqq (j_1 + j_2 + 3)^2 - 1$, or $0 \geqq j_1^2 + j_2^2 + 6(j_1 + j_2) + 18$, which is impossible. Hence $\gamma + \delta_0 < 0$, and the order is negative.

To prove all the multiplicities negative, let δ_8 be the largest positive δ, δ_0 the smallest negative δ, so that the multiplicity $\gamma + \delta_8 - \delta_0$ has the largest possible value. To prove that $\gamma + \delta_8 - \delta_0 < 0$ suppose that $\gamma + \delta_8 - \delta_0 \geqq 0$, or $(\delta_8 - \delta_0)^2 \geqq (j_1 + j_2 + 3)^2$. Since $(j_1 - j_2)^2 \geqq 0$, $j_1^2 + j_2^2 \geqq 2j_1 j_2$. Similarly $\delta_8^2 + \delta_0^2 \geqq -2\delta_0\delta_8$. Hence

$$2(\delta_8^2 + \delta_0^2) \geqq 4j_1 j_2 + 6(j_1 + j_2) + 9 = 4\epsilon + 6(j_1 + j_2) + 29.$$

But, from $\Sigma\delta^2 = \nu^2 + 2\epsilon$, $2\epsilon \geqq \delta_0^2 + \delta_8^2 - 1$, whence $0 \geqq 6(j_1 + j_2) + 27$, which is impossible. Hence $\gamma + \delta_8 - \delta_0 < 0$, and the proof of (51) is complete.

(52) *The positive factorizations of* $\epsilon + 5 = (\kappa_1 + 3)(\kappa_2 + 3)$ *yield C-characteristics which have a positive order.*

Let

$$\epsilon + 5 = j_1 j_2, \quad j_1 \geqq 1, \quad j_2 \geqq 1, \quad j_1 j_2 \geqq 5, \quad \gamma = j_1 + j_2 - 3.$$

The order is positive if either $\gamma + \delta_0 > 0$, or $\gamma + \delta_0 = 0$, $\nu = -1$. Suppose that $\gamma+\delta_0 \leqq 0$. Then $\delta_0 \leqq -j_1-j_2+3$, or $\delta_0^2 \geqq (j_1+j_2+3)^2$. As before $2\epsilon \geqq \delta_0^2-1$, or $2j_1j_2-9 \geqq \delta_0^2$. Hence

$$2j_1j_2-9 \geqq (j_1+j_2-3)^2, \quad \text{or} \quad 0 \geqq (j_1-3)^2+(j_2-3)^2.$$

This inequality cannot be satisfied, but the equality can be satisfied, along with earlier equalities, by the values

$$j_1=j_2=3, \quad \epsilon=4, \quad \delta_1=\delta_2=\cdots=\delta_8=0, \quad \delta_0=-3, \quad \nu=-1, \quad \gamma=3,$$

yielding the net of lines $C\{1;0^{10}\}$, which itself has a positive order.

Any C-characteristic which is partially negative has precisely one negative multiplicity which is -2 *or* -1.

For, according to (51) and (52), the C-characteristic has a positive order, and arises from a positive factorization of $\epsilon+5$. Let the multiplicities be arranged in descending order. Then κ_1, κ_2 are the smallest multiplicities. They determine a positive or zero ϵ. The factors of $\epsilon+5=(\kappa_1+3)(\kappa_2+3)$ must be positive, whence, if $\kappa_2<0$, $\kappa_2=-2$, or $\kappa_2=-1$. For each of these two cases $\kappa_1 \geqq 0$, and each preceding multiplicity is zero or positive.

We list these exceptional characteristics:

$$\text{(54)} \quad \begin{array}{ll} (\alpha) & C\{3(\epsilon+3+\delta_0)-\nu;\ \epsilon+3+\delta_0-\delta_1,\cdots,\epsilon+3+\delta_0-\delta_8,\epsilon+2,- \\ (\beta) & C\{3(e+2+\delta_0)-\nu;\ e+2+\delta_0-\delta_1,\cdots,e+2+\delta_0-\delta_8,e,-1\}, \\ & \qquad e \geqq 0, \quad \epsilon=2e+1. \end{array}$$

For $\epsilon=0$ and $e=0$, they yield two reduced cases, $C\{9;3^8\,2-2\}$ and $C\{3;1^7\,0^2-1\}$.

(55) *All the C-characteristics,* (54 α), *with one negative multiplicity* -2 *can be reduced by* Q. T. *to the type* $C\{9;3^8\,2-2\}$.

Let the characteristic be arranged with decreasing multiplicities. Then $\kappa_1=\epsilon+2$ determines ϵ, and the order determines ν and δ_0. For the remaining δ's, $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_8$. The first three multiplicities are the greatest, and a Q. T. applied to these changes the order and the three highest multiplicities by the addition of $\mu=\delta_1+\delta_2+\delta_3-\nu$. If $\epsilon=0$, $\mu=0$, and the reduced type above appears. If $\epsilon>0$, μ is negative *provided that* δ_0 *is not the smallest* δ [cf. (17)]. From the decreasing multiplicities and the value $\epsilon+2$ of κ_1, we have

$$\delta_0-\delta_1 \geqq -1, \quad \delta_0-\delta_2 \geqq -1, \cdots, \delta_0-\delta_8 \geqq -1, \quad \delta_0-\delta_0=0,$$

or

$$9\delta_0-3\nu \geqq -8.$$

Hence δ_0 is zero or positive, unless $\nu=-1$ in which case δ_0 may be -1.

If δ_0 is positive, it cannot be the smallest δ. If δ_0 is zero and is the smallest δ, then either $\epsilon = 0$, or $\epsilon = 1$ and $(\nu; \delta) = (1; 1^3\, 0^6)$, in which case $\mu = -1$. If $\delta_0 = -1$ and is the lowest δ, then δ_8 must be zero else $\epsilon + 3 + \delta_0 - \delta_8 < \epsilon + 2$. Thus the only possibility is $\epsilon = 1$, $(\nu; \delta) = (-1; 0^6 - 1^3)$ which for $\delta_0 = -1$ yields $\mu = -1$. Hence the order and three highest multiplicities can be reduced by Q. T. unless $\epsilon = 0$. We reduce them until one at least is less than $\epsilon + 2$. A rearrangement of the characteristic leads to one with $\epsilon' < \epsilon$. This process can be continued until $\epsilon' = 0$ and $C\{9; 3^8\, 2 - 2\}$ appears.

In an entirely similar fashion one may prove that

(56) *All the C-characteristics,* (54 β), *with one negative multiplicity* -1 *can be reduced by* Q. T. *to the type* $C\{3; 1^7\, 0^2 - 1\}$.

We have at once from (41) for C-characteristics for $P_9{}^2$ that

(57) *All C-characteristics with positive order and one or more zero multiplicities can be reduced by* Q. T. *to the type* $C\{3; 1^7\, 0^2 - 1\}$ *or to the type* $C\{1; 0^{10}\}$.

Finally we prove that

(58) *All C-characteristics with positive order can be reduced by* Q. T. *to one of the three types:*

$$C\{9; 3^8\, 2 - 2\}, \quad C\{3; 1^7\, 0^2 - 1\}, \quad C\{1; 0^{10}\},$$

of which only the last defines a homaloidal net. The other two characteristics are DEGENERATE, *the first being the sum of a system* ($p = 3$, $d = 3$) *and twice a P-curve which has no variable intersections with the first system, and the second being the sum of an elliptic net and a similarly situated P-curve.*

For, with the help of (53), $\cdots$, (57) we need consider only positive multiplicities. Then, according to (I_4) preceding (46), the order and three of the multiplicities can be reduced by Q. T. until zero or negative multiplicities occur, and thus the theorem is proved.

We supplement this theorem by the following:

(59) *All C-characteristics with negative order are* VIRTUAL *and are negative throughout. They can be reduced by* Q. T. *to one of three types:*

$$C\{-19; -6^{10}\}, \quad C\{-21; -7^7 - 6^2 - 5\}, \quad C\{-27; -9^8 - 8 - 4\}.$$

For, we have observed in (36) that, in the case of $P_{10}{}^2$, a C-characteristic is converted into a C-characteristic by a change of sign of the characteristic and subtraction of $6L$. This changes the characteristics of positive order into characteristics of negative order, and changes the three irreducible types in

(58) into the three irreducible types of (59). We observe that the first type in (59) is a *symmetric virtual* C-characteristic. It is utilized in **10** to obtain linear transformations of Q, L with integer coefficients which have hitherto escaped notice.

9. The infinite number of conjugate sets of C-characteristics for P_{11}^2. For $p=0$, $d=2$ and P_{11}^2, the last three multiplicities being κ_1, κ_2, κ_3, the equations of condition on the canonical characteristic (7) become

$$(60)\qquad \begin{array}{ll} i_{12}=h_1\cdot h_2=\epsilon+5+3\kappa_3+\kappa_3^2, & \\ \kappa_1=h_1-3-\kappa_3, & \gamma=\kappa_1+\kappa_2+\kappa_3+3, \\ \kappa_2=h_2-3-\kappa_3, & \gamma=h_1+h_2-\kappa_3-3. \end{array}$$

Here i_{12} is always positive, its minimum value being $\epsilon+3$.

We observe first that

(61) *A positive factorization of $i_{12}=h_1\cdot h_2$ yields a C-characteristic with a positive order.*

For, the order, $3(\gamma+\delta_0)-\nu$, is positive if $\gamma+\delta_0>0$, or if $\gamma>-\delta_0$, or if $\gamma>\sqrt{\epsilon}$. If now $h_1h_2=k$ the minimum value of the sum h_1+h_2 of the positive factors h_1, h_2 is $2\sqrt{k}$. Hence $\gamma\geqq 2\sqrt{\epsilon+5+3\kappa_3+\kappa_3^2}-\kappa_3-3$. It is easy to show by clearing the radicals that the right member is $>\sqrt{\epsilon}$, i. e., $\gamma>\sqrt{\epsilon}$.

The method of **3**, in particular (36), applied to the canonical characteristic, yields the theorem:

(62) *For every characteristic C which arises from a positive factorization of $i_{12}=h_1\cdot h_2$, there is a characteristic C', obtained by changing the sign of C and subtracting $3L$, which arises from the negative factorization $i_{12}=(-h_1)(-h_2)$. In this change from C to C', the ϵ, $(\nu;\delta)$, κ_3, γ of C are replaced respectively by ϵ, $-(\nu;\delta)$, $-\kappa_3-3$, $-\gamma-3$.*

From (62) and (61) we find that

(61) *A negative factorization of $i_{12}=(-h_1)\cdot(-h_2)$ yields a C-characteristic with a negative order.*

According to (62) it will be sufficient to determine all the C-characteristics which arise from positive factorizations of $i_{12}\geqq 3$. We shall suppose the multiplicities arranged in descending order so that

$$(64)\qquad \delta_1\leqq\delta_2\leqq\cdots\leqq\delta_8,\quad \gamma+\delta_0-\delta_3\geqq\kappa_1\geqq\kappa_2\geqq\kappa_3,\quad h_1\geqq h_2.$$

According to the inequality (I_4) preceding (46), if the multiplicities are all positive, the characteristic can be reduced by Q. T. until a zero or negative multiplicity appears. If a multiplicity 0 occurs, the type has already been obtained for P_{10}^2. We are therefore interested only in the irreducible types for which $\kappa_3 \leqq -1$. Furthermore for a given ϵ we need only take those sets $(\nu;\delta)$ in the table (27), and for these take δ_0 to be the smallest δ, since other choices of $(\nu;\delta)$ and δ_0 are reducible to these. Again the descending multiplicities yield $\gamma + \delta_0 - \delta_8 \geqq \kappa_1$, and this according to (60) is equivalent to $h_2 \geqq \delta_8 - \delta_0$. Then

(65) *For every* $\epsilon \geqq 0$ *and every set* $(\nu;\delta)$ *in the table* (27) [δ_0 *being the smallest* δ *and* δ_8 *the largest*]; *for every value of* $\kappa_3 \leqq -1$; *and for every positive factorization of* $i_{12} = \epsilon + 5 + 3\kappa_3 + \kappa_3^2$ *into factors* h_1, h_2 $(h_1 \geqq h_2 \geqq \delta_8 - \delta_0)$, *there exists an irreducible type of C-characteristic:*

$$C\{3(h_1 + h_2 - \kappa_3 - 3 + \delta_0) - \nu;\ h_1 + h_2 - \kappa_3 - 3 + \delta_0 - \delta_1, \cdots,\\ h_1 + h_2 - \kappa_3 - 3 + \delta_0 - \delta_8,\ h_1 - \kappa_3 - 3,\ h_2 - \kappa_3 - 3,\ \kappa_3\}.$$

We append a short table of illustrative irreducible C-characteristics:

ϵ	$(\nu;\delta)$	κ_3	i_{12}	$h_1, h_2 \geqq$	$\delta_8 - \delta_0$	$C\{\quad\}$
0	$(0;0^9)$	-1	3	3, 1	0	$\{6;2^8\,1\,-1^2\}$
		-2	3	3, 1		$\{9;3^8\,2\,0\,-2\}$
		-3	5	5, 1		$\{18;6^8\,5\,1\,-3\}$
		-4	9	3, 3		$\{21;7^8\,4^2\,-4\}$
				9, 1		$\{33;11^8, 10, 2, -4\}$
1	$(0;-1\,0^7\,1)$	-1	4	2, 2	2	$\{3;1^7\,0^3\,-1\}$
		-2	4	2, 2		$\{6;2^7\,1^3\,-2\}$
		-3	6	3, 2		$\{12;4^7\,3^2\,2\,-3\}$
		-4	10	5, 2		$\{21;7^7\,6^2\,3\,-4\}$
		-5	16	4, 4		$\{27;9^7\,8\,6^2\,-5\}$
				8, 2		$\{33;11^7, 10^2, 4, -5\}$
2	$(-1;-2\,-1\,0^7)$	-1	5		2	
		-2	5			
		-3	7			
		-4	11			
		-5	17			
		-6	25	5, 5		$\{34;12, 11^7, 8^2, -6\}$
		-7	35	7, 5		$\{43;15, 14^7, 11, 9, -7\}$
		-8	47			
		-9	61			
		-10	77	11, 7		$\{70;24, 23^7, 18, 14, -10\}$
3	$(0;-2\,0^6\,1^2)$	-1	6		3	
		-2	6			
		-3	8			

ϵ	$(\nu;\delta)$	κ_3	i_{12}	$h_1, h_2 \geqq$	$\delta_8 - \delta_0$	$C\{\ \}$
		-4	12	4, 3		$\{18; 6^6 5^3 4 - 4\}$
		-5	18	6, 3		$\{27; 9^6 8^3 5 - 5\}$
		-6	26			
		-7	36	6, 6		$\{42; 14^6, 13^2, 10^2, -7\}$
				9, 4		$\{45; 15^6, 14^2, 13, 8, -7\}$
				12, 3		$\{51; 17^6, 16^3, 7, -7\}$
4	$(-1; -3\ 0^8)$	-1	7		3	
		-2	7			
		-3	9	3, 3		$\{10; 3^{10} - 3\}$
		-4	13			
		-5	19			
		-6	27	9, 3		$\{37; 12^9, 6, -6\}$
		-7	37			
		-8	49	7, 7		$\{49; 16^8, 12^2, -8\}$
		-9	63	9, 7		$\{58; 19^8, 15, 13, -9\}$
				21, 3		$\{82; 27^9, 9, -9\}$
	$(0; -2\ 0^7\ 2)$	-8	49	7, 7	4	$\{51; 17^7, 15, 12^2, -8\}$
		-9	63	9, 7		$\{60; 20^7, 18, 15, 13, -9\}$
		-10	79			
		-11	97			
		-12	117	13, 9		$\{87; 29^7; 27, 22, 18; -12\}$

It is clear that the two sequences of values of κ_3 and ϵ will yield an infinite number of irreducible types, all of which are degenerate. By using (62) we obtain from these an infinite number of irreducible virtual types.

10. Virtual Cremona transformations. The linear group $g_{\rho,2}$, generated by Π and A_{123} in (3), (4), whose generic element is given in (5), has the invariant quadratic and linear forms, Q, L in (6). For every element of this group, except the elements in Π, the integers n, r_i, s_j, α_{ij} are positive or zero. This element represents the effect upon a characteristic $\{x\}$ of a geometrically existent Cremona transformation. If the integers n, r_i of the element are given, the integers s_j, α_{ij} are determined to within a permutation of the subscripts j.

Thus far no linear transformations with integral coefficients and invariant forms Q, L other than those contained in $g_{\rho,2}$, have been observed. We propose to show that such transformations exist when $\rho \geqq 10$, and will say that they give the effect upon a characteristic $\{x\}$ of a *virtual Cremona transformation.*

It is to be observed that linear transformations with invariant Q, L and rational coefficients occur very early. Thus for $\rho = 1$ we have one such transformation other than the identity, namely:

$$4x'_0 = 5x_0 - 3x_1, \qquad 4x'_1 = 3x_0 - 5x_1.$$

These, and new elements which arise when $\rho > 1$, generate the group of Q, L with rational coefficients. Degenerate C-characteristics, such as $\{3; 1^7 - 1\}$ for $\rho = 8$, also determine elements with rational coefficients. If, for example, we take in (5) the $\{n; r_i\}$ to be $\{3; 1^7 - 1\}$, then the P-characteristics which take the place of the $\{s_j; \alpha_{ij}\}$ are respectively $\{1; -\frac{1}{2}(\frac{1}{2})^6 - \frac{1}{2}\}$ and $\{-1; (-\frac{1}{2})^8\}$. Thus we normally expect C-characteristics $\{n; r_i\}$ which are degenerate or virtual to yield elements with invariant Q, L which have *rational* coefficients and it is quite interesting to find that, in particular instances, such characteristics yield elements with *integer* coefficients, as do the geometric C-characteristics found in $g_{\rho,2}$.

The generator A_{123} of $g_{\rho,2}$ arises from the first symmetric element (other than the identity) which occurs when $\rho = 3$. It is natural therefore to examine the symmetric virtual C-characteristics which have been uncovered in **3** (36). There are four of these, namely

$$(67) \qquad \{-19; -6^{10}\}, \quad \{-10; -3^{11}\}, \quad \{-7; -2^{12}\}, \quad \{-4; -1^{15}\}.$$

Taking the first of these as a virtual Cremona net we seek the ten P-curves. On account of the symmetry we expect a P-curve of the form $\{-6; \alpha\beta^9\}$. The conditions (2) that this be a P-curve are $\alpha^2 + 9\beta^2 = 37$, $\alpha + 9\beta = -19$. The last of these ensures that all of the intersections of the P-curve and the net are at the base points. The similar conditions for two such P-curves lead to $2\alpha\beta + 8\beta^2 = 36$. These equations, which are sufficient to ensure the invariance of Q, L have two solutions:

$$\alpha, \beta = -1, -2; \qquad \alpha, \beta = -14/5, -9/5.$$

The corresponding two transformations are:

$$(68) \quad P_{10}{}^2 : T_{10} : \begin{array}{ll} x'_0 = -x_0 + 6L, & x'_0 = x_0 + 6M, \\ x'_i = -x_i + 2L, & x'_i = x_i + (9/5)M, \\ (i = 1, \cdots, 10) & (i = 1, \cdots, 10) \\ L \equiv -3x_0 + x_1 + \cdots + x_{10}; & M \equiv -(10/3)x_0 + x_1 + \cdots + x_{10}. \end{array}$$

T_{10}, the first of these two, has integral coefficients. The similar procedure applied to the remaining three virtual C-characteristics in (67) yields the following transformations, the first of which has integral coefficients:

$$(69) \quad P_{11}{}^2 : T_{11} : \begin{array}{ll} x'_0 = -x_0 + 3L, & x'_0 = x_0 + 3M, \\ x'_i = -x_i + L, & x'_i = x_i + (9/11)M, \\ (i = 1, \cdots, 11) & (i = 1, \cdots, 11) \\ L \equiv -3x_0 + x_1 + \cdots + x_{11}; & M \equiv -(11/3)x_0 + x_1 + \cdots + x_{11}. \end{array}$$

$$(70)\quad P_{12}^2:\ \begin{array}{ll} x'_0 = -x_0 + 2L, & x'_0 = x_0 + 2M, \\ x'_i = -x_i + \frac{2}{3}L, & x'_i = x_i + \frac{1}{2}M, \\ (i = 1, \cdots, 12), & (i = 1, \cdots, 12), \\ L \equiv -3x_0 + x_1 + \cdots + x_{12}; & M \equiv -4x_0 + x_1 + \cdots + x_{12}. \end{array}$$

$$(71)\quad P_{15}^2:\ \begin{array}{ll} x'_0 = -x_0 + L, & x'_0 = x_0 + M, \\ x'_i = -x_i + \frac{1}{3}L, & x'_i = x_i + \frac{1}{5}M, \\ (i = 1, \cdots, 15), & (i = 1, \cdots, 15), \\ L \equiv -3x_0 + x_1 + \cdots + x_{15}; & M \equiv -5x_0 + x_1 + \cdots + x_{15}. \end{array}$$

Thus each of the four symmetric cases furnishes two transformations with rational coefficients, and, of the eight transformations, two have integral coefficients, namely: T_{10} and T_{11}.

It may be observed that each of the four symmetric Cremona nets which are geometric also furnish two transformations, one of which is integral and is associated with the corresponding Cremona transformation, and the other of which has rational coefficients. It may be of interest to tabulate these rational cases. The nets being

$$\{2; 1^3\},\quad \{5; 2^6\},\quad \{8; 3^7\},\quad \{17; 6^8\}$$

the transformations with rational coefficients are:

$$(72)\quad \begin{array}{llll} P_3^2: & \begin{array}{l} x'_0 = -x_0 - L, \\ x'_i = -x_i - \frac{1}{3}L, \\ (i = 1, 2, 3), \\ L \equiv -3x_0 + x_1 + x_2 + x_3; \end{array} & P_6^2: & \begin{array}{l} x'_0 = -x_0 - 2L, \\ x'_i = -x_i - \frac{2}{3}L, \\ (i = 1, \cdots, 6), \\ L \equiv -3x_0 + x_1 + \cdots + x_6; \end{array} \\ P_7^2: & \begin{array}{l} x'_0 = x_0 - 3M, \\ x'_i = x_i - \frac{9}{7}M, \\ (i = 1, \cdots, 7), \\ M \equiv -\frac{7}{3}x_0 + x_1 + \cdots + x_7; \end{array} & P_8^2: & \begin{array}{l} x'_0 = x_0 - 6M, \\ x'_i = x_i - \frac{9}{4}M, \\ (i = 1, \cdots, 8), \\ M \equiv -\frac{8}{3}x_0 + x_1 + \cdots + x_8. \end{array} \end{array}$$

Let us consider the effect of adding to the $g_{10,2}$, generated by Π and A_{123}, the element T_{10} in (68). A set of generators of $g_{10,2}$ is the set of transpositions $(x_i x_j)$ in Π and the element A_{123} itself. These, as collineations, are determined by their spaces of fixed points, namely: $x_i - x_j$ and $x_0 - x_1 - x_2 - x_3$. But each of these is invariant to within sign under T_{10}. Thus T_{10} is interchangeable with every element of $g_{10,2}$. Also, T_{10} itself is of period two and determinant 1. Furthermore T_{10} converts the three reduced C-characteristics of positive order in (58) into the three reduced C-characteristics of negative order in (59). Hence

(73) *The group, $\bar{g}_{10,2}$, generated by adding T_{10} to $g_{10,2}$ contains an invariant*

$g_2 = 1, T_{10}$. *The elements of* $\bar{g}_{10,2}$ *not contained in* $g_{10,2}$ *transform the C-characteristics with positive orders into those with negative orders and vice-versa.*

In the group $\bar{g}_{11,2}$, generated by $g_{11,2}$ and T_{11}, the element T_{11} plays a rôle quite similar to that of T_{10} in $\bar{g}_{10,2}$.

It may be remarked that the invariant forms Q, L define a number of discontinuous groups of considerable interest. The most extensive of these is the group, $G(R)_{\rho,2}$, of linear transformations with rational coefficients. Included in this group is the aggregate, $A(RI)_{\rho,2}$, of linear transformations in which the coefficients n, r_i are integers, positive, negative, or zero; and the similar aggregate, $B(RI)_{\rho,2}$, with respect to the coefficients n, s_j. These aggregates are associated with the virtual C-characteristics. One would wish to know whether the aggregates A and B coincide, and whether these aggregates form a group. It is likely that both inquiries have an affirmative answer. Next in order is the group, $G(I)_{\rho,2}$, of linear transformations with integer coefficients. The existence of T_{10} and T_{11} show that this group is larger than the group $G(C)_{\rho,2} = g_{\rho,2}$, generated by Π and A_{123}, associated with the geometric Cremona transformations. One would wish to know whether Π, A_{123}, T_{10}, T_{11} generate this group $G(I)_{\rho,2}$. Finally there remains the long outstanding question as to whether the conditions, $n, r_i, s_j, \alpha_{ij} \overline{>} 0$, are sufficient to ensure that an element of $G(I)_{\rho,2}$ belong to $G(C)_{\rho,2}$. Mr. Gerald B. Huff[6] has recently constructed an example ($\rho = 11$) which shows that this question must be answered in the negative.

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ON THE METABELIAN GROUPS WHICH CONTAIN A GIVEN GROUP *H* AS A MAXIMAL INVARIANT ABELIAN SUBGROUP.

By H. R. Brahana.

Introduction. In a recent paper [*] we showed that the operators of order p^m in the group of isomorphisms of the abelian group H of order p^n and type $1, 1, \cdots$ belonged to conjugate sets each of which could be characterized by a partition of n. Each such operator determines a group of order p^{n+i} in the holomorph of H with certain characteristics, class, central, commutator subgroup, etc., expressible in a simple manner in the terms of the partition determined by the operator. This paper, which is a continuation of the other and whose sections are numbered consecutively with those of the other, carries on the investigation of the group of isomorphisms and the holomorph of H. It is a start on the classification of non-abelian groups of prime-power order. It is more precisely a start on the classification of abelian subgroups of order p^m and type $1, 1, \cdots$ of the group of isomorphisms of H.

Two subgroups of order p^m and type $1, 1, \cdots$ of the group of isomorphisms of H can be conjugate only if they contain the same numbers of operators of the same types. If each of these subgroups determines a metabelian group in the holomorph of H every operator in each of them corresponds to a partition of n in which the largest term is 2. There are $n/2$ or $(n-1)/2$ partitions of n with a largest term equal to 2 according as n is even or odd. Hence it is obvious that there are many distinct types of metabelian group of order p^{n+m} in the holomorph of H. In section 4 we separate the subgroups of the group of isomorphisms into classes according to the types of operator that appear in them. In section 5 we consider and classify all subgroups of the holomorph of H which contain H as a maximal invariant abelian subgroup and have quotient groups with respect to H which belong to the first class as defined in section 4. We also touch upon the question of more general groups having the same relation to H. In section 6 we consider subgroups of the second class defined in section 4.

It is unnecessary to enlarge upon the difficulties of the subject. The fact that the specializations we are compelled to make are connected in a close and simple manner with characteristics of the corresponding groups in the holomorph argues that the method of attack cannot be greatly improved. As was to be expected a characterization of the *operators* of the group of isomorphisms of H in a manageable form opens the way for a classification of its *subgroups*.

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[*] "On the isomorphisms of an abelian group of type $1, 1, \cdots$," *American Journal of Mathematics*, vol. 56 (1934), p. 53.

4. *Metabelian subgroups of the holomorph of H.* Let G be a metabelian subgroup of the holomorph of H whose quotient group with respect to H is a subgroup of I_p. Since I_p contains no operator except identity which is permutable with every operator of H it follows that H is a maximal invariant abelian subgroup of G. Every operator of G not in H transforms H according to an operator U which corresponds to a partition of n in which $n_1 = 2$. A classification of the groups G will involve a classification of the groups G/H and this must concern itself with the numbers of operators of the various types for which $n_1 = 2$. As we have seen, there are $n/2$ or $(n-1)/2$ types according as n is even or odd, for each of which $n_1 = 2$. We shall say that U is of type μ if $n_\mu = 2$ and $n_{\mu+1} = 1$.

The quotient group G/H is abelian because the commutators of G are all in H.

It will be convenient for our purposes to write I_p in another form. Consider the operators

$$(4.1) \qquad U = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & 1 & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

where the a_{ij}'s are residues mod p. It is obvious and well-known that the U's of this type form a group. The characteristic determinant is $(1-\lambda)^n$. The number of such U's is $p^{n(n-1)/2}$, the order of I_p. Hence the operators of type (4.1) constitute a Sylow subgroup of I.*

Now the operators of (4.1) for which

$$(4.2) \qquad a_{ij} = 0, \quad \text{for all } j \text{ with } i > 1,$$

constitute a group. Its order is p^{n-1} for there are $n-1$ a's in the first row and all the rest are zero. The group is abelian as may be verified by multiplication. Every operator of (4.2) is of type 1, i.e., corresponds to the partition $n = 2 + 1 + 1 + \cdots + 1$. This follows from the fact that the commutator $U^{-1}s_1Us_1^{-1} = s_2^{a_{12}}s_3^{a_{13}} \cdots s_n^{a_{1n}}$ generates the commutator subgroup of $\{H, U\}$ and is invariant under U. The group (4.2) is maximal abelian in I_p, since an operator of the form (4.1) which is permutable with every operator of

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* Cf. Dickson, "On the subgroups of order a power of p etc.," *Bulletin of the American Mathematical Society*, vol. 10 (1903-04), p. 385.

(4. 2) is itself in (4. 2). This group is generated by $U_1, U_2, \cdots, U_{n-1}$ where U_i is the operator of (4. 2) for which

$$a_{1i} = 1, \ a_{1j} = 0, \qquad j \neq i. \tag{4.3}$$

The group $\{U_1, U_2, \cdots, U_{n-1}\}$ is a subgroup of I_p and therefore none of its operators is permutable with every operator of H. Any set of α independent operators from $\{U_1, U_2, \cdots, U_{n-1}\}$ will generate a group of order p^α whose operators are all of type 1. This group will determine a group $G_\alpha = \{H, U_1, U_2, \cdots, U_\alpha\}$ which is in the holomorph of H.

The groups G_α all have centrals of order p^{n-1}. Conversely, if G is in the holomorph of H, contains H as a maximal invariant abelian subgroup, and has a central of order p^{n-1}, it is metabelian and is simply isomorphic with one of the groups G_α defined above. For generators of H may be chosen so that $n-1$ of them are in the central and the quotient group G/H will be a subgroup of (4. 2).

This set of groups $\{U_1, U_2, \cdots, U_\alpha\}$ contains only operators of type 1. There is another set of groups whose operators are all of type 1; they are characterized by the fact that the commutator subgroup arising from transformation of H by all the operators of such a group is of order p. Let U_1 and U_2 be two operators from such a group. Generators of H may be chosen so that

$$U_1^{-1} s_1 U_1 = s_1 s_2$$

and each of the other generators is invariant under U_1. Then s_2 is invariant under U_2, for otherwise $\{H, U_1, U_2\}$ would not be metabelian and not every operator of $\{U_1, U_2\}$ could correspond to a partition of n in which $n_1 = 2$. H contains some operator invariant under U_1 and not invariant under U_2, otherwise U_1 and U_2 would generate a subgroup of (4. 2) and this would be true of every pair of operators from the group in question. This operator not being s_2 may be taken to be s_3. Then let

$$U_2^{-1} s_3 U_2 = s_3 s_k$$

and let U_2 be permutable with all the other generators of H. The commutator s_k is permutable with U_1 for the same reason that s_2 is permutable with U_2. If s_k is not s_2 the operator $U_1 U_2$ transforms s_1 into $s_1 s_2$ and s_3 into $s_3 s_k$ thus giving rise to a commutator subgroup of order p^2. $U_1 U_2$ could not then be of type 1. Hence any two operators of the group must give rise to the same commutator subgroup of order p. Therefore,

(4. 4) *If G/H contains only operators of type 1, then either the central of G is of order p^{n-1} or the commutator subgroup of G is of order p.*

Groups of the second set are characterized by the fact that in (4.1) all the a's except those of one column are zero. The maximum order of a group of this set is p^{n-1}, and any two of the same order are simply isomorphic.

A group $\{U_1, U_2, \cdots, U_a\}$ which gives a G_a with central of order p^{n-1} cannot belong to the second set unless $\alpha = 1$. For if U_i and U_j are both permutable with the generators $s_2, s_3, \cdots, s_n$ and U_i and s_1 determine the same commutator as U_j and s_1, then $U_i U_j^{-1}$ is permutable with s_1 and H is not maximal abelian in G_a. From this argument we have the further result that the commutator subgroup of G_a is of order p^a.

We shall classify the metabelian subgroups of the holomorph of H according to the orders of their centrals. We have determined all of those whose centrals are of order p^{n-1} and have seen that the operators of the quotient group G/H are all of type 1. Moreover, a group of this type is determined by the order of H, its order, and the order of its central.

We shall suppose next that the order of the central of G is p^{n-2}. The quotient group G/H can contain operators of type 1 and operators of type 2 only, for an operator of type μ transforms at least μ of the independent generators of H into operators other than themselves. Let us suppose for the moment that the order of the commutator subgroup of G is greater than p. Then G/H must contain at least one operator of type 2. Generators of H may be chosen so that this operator is

$$(4.5)\qquad \begin{bmatrix} 1 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

The quotient group G/H must be contained in the subgroup of (4.1) whose operators are permutable with (4.5) and each operator of it is of type 1 or 2 leaving $s_3, s_4, \cdots, s_n$ separately invariant. It is readily verified that G/H is a subgroup of

$$(4.6)\qquad \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & \cdot & \cdot & \cdot & a_{1n} \\ 0 & 1 & a_{23} & a_{24} & \cdot & \cdot & \cdot & a_{2n} \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

If, as above, we remove the restriction on the order of the central of G, but require the operators of G/H all to be of types 1 and 2, we shall find that

G/H is either a subgroup of (4.6) or of the group obtained by interchanging rows and columns of (4.6) leaving fixed each element on the secondary diagonal. The group (4.6) is abelian and its order is p^{2n-4}. The group with the a's in two columns instead of in two rows is also abelian and of order p^{2n-4}, and its commutator subgroup is of order p^2. Any subgroup of I_φ whose operators are all of types 1 and 2 is in one or the other of these groups. If such a subgroup is in both its order is not greater than p^4.

It is necessary to note that though (4.6) contains operators of type 2, the same need not be true of its subgroups, for (4.6) contains subgroups which are also subgroups of (4.2).

In general if the order of the central of G is $p^{n-\mu}$ the quotient group G/H is a subgroup of

$$(4.7)\qquad \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & a_{1\mu+1} & a_{1\mu+2} & \cdots & a_{1n} \\ 0 & 1 & 0 & \cdots & 0 & a_{2\mu+1} & a_{2\mu+2} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 & a_{\mu\mu+1} & a_{\mu\mu+2} & \cdots & a_{\mu n} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The group (4.7) itself contains operators of type μ and no operators of type ν, where $\nu > \mu$. Since the order of the commutator subgroup determined by it is $p^{n-\mu}$, then it contains no operators of type ν, where $\nu > n - \mu$. The order of (4.7) is $p^{\mu(n-\mu)}$. We shall say that (4.7) *is of type* μ. In terms of this definition we may say that a group of type μ may contain operators of type μ or of type $n - \mu$, whichever is smaller but can contain no operator of type ν, ν greater than the smaller of the two.

We have supposed, in order to simplify the discussion, that G was a subgroup of the holomorph of H. We have been interested primarily in the quotient group G/H and if we require simply that G be metabelian and that H be maximal invariant abelian our conclusions concerning G/H are the same. We shall state the result of these considerations in the theorem:

(4.8) *If G is a metabelian group of order p^m which contains H as a maximal invariant abelian subgroup and if the order of the central of G is $p^{n-\mu}$, then G/H is a subgroup of* (4.7). *Conversely, for every subgroup of* (4.7) *there exists at least one metabelian group G which contains H as a maximal invariant abelian subgroup and whose central is of order at least $p^{n-\mu}$.*

5. *Groups for which G/H is of type* 1. Let us consider first subgroups of the holomorph of H. Such a subgroup is completely determined by the quotient group G/H which must be a subgroup of (4.2). It has been noted that such a subgroup is determined by its order. Since every operator of G/H is of type 1 it is of order p and every operator of G is therefore of order p. Moreover, no operator outside of H is permutable with every operator of H, and therefore the order of the commutator subgroup of G is p^{α}, where α is the number of independent generators of G/H.

(5.1) *For every number α, $1 \leq \alpha \leq n-1$, there exists one and only one subgroup G_{α} of the holomorph of H which contains H as a maximal invariant abelian subgroup and has a central of order p^{n-1}. The operators of G are all of order p and its commutator subgroup of order p^{α}.*

Now let us consider other groups G whose operators are all of order p. Let G_{α} be of order $p^{n+\alpha}$ and be generated by $H, U_1, U_2, \cdots, U_{\alpha}$, where the U's are of order p and each operator of $\{U_1, U_2, \cdots, U_{\alpha}\}$ transforms H according to an operator of type 1. Then if G_{α} is not in the holomorph of H the U's do not generate an abelian group. Since the group $\{U_1, U_2, \cdots, U_{\alpha}\}$ is isomorphic with an abelian group in I_p its commutators are all in H, and since G_{α} is metabelian the commutators are invariant in G_{α}. The commutator subgroup K_{α} of G_{α} must then be of order greater than or equal to p^{α}. We shall show that G_{α} is completely determined by α and k_{α}, where $p^{k_{\alpha}}$ is the order of K_{α}. It is obvious that k_{α} is not less than α and not greater than the smaller of the two numbers $n-1$ and $\alpha(\alpha+1)/2$. The second upper limit is attained when the commutators of the U's are independent of each other and of the commutators arising from transformation of H by the U's.

If $\alpha = 1$, there is one group G_1 and it is in the holomorph of H. If $\alpha = 2$, there are two groups. It is obvious that there are at least two, for the commutator $c_{12} = U_1^{-1}U_2U_1U_2^{-1}$ may be in the group $\bar{K}_2$ generated by commutators of U_1 and H and U_2 and H, or it may not. In the one case the commutator subgroup of G_2 is of order p^2 and in the other p^3. We shall show that if c_{12} is in the commutator subgroup $\bar{K}_2$, then generators U'_1 and U'_2 may be chosen so that c'_{12} is identity. Let

$$U_1^{-1}s_1U_1 = s_1s_2, \quad U_2^{-1}s_1U_2 = s_1s_3, \text{ and } c_{12} = s_2^{a}s_3^{b}.$$

Then if we take $U_1' = s_1^{-b}U_1$ and $U'_2 = s_1^{a}U_2$, we have G_2 given by H, U'_1, and U'_2 in which $c'_{12} = 1$. Thus G_2 is determined by α and k_{α}.

We shall carry out in detail the determination of the groups G_3, for there we meet and resolve all the difficulties of the general case. If $n \geq 7$ it is obvious that there are groups G_3 with K_3 of orders p^3, p^4, p^5, and p^6. Let the commutator $U_i^{-1}U_jU_iU_j^{-1}$ be denoted by c_{ij}. If c_{13} and c_{23} are in the same cyclic group we may make c_{13} to be identity by a proper choice of U'_1 to replace U_1. Let $s^a = c_{13}$ and $s = c_{23}$. Then the operator $U'_1 = U_1U_2^{-a}$ is transformed into itself by U_3. Now suppose $U_i^{-1}s_1U_i = s_1s_{i+1}$, $i = 1, 2, 3$, and $c_{13} = s_2^a s_3^b s_4^c$. G_3 is generated by $H, U_2, U'_1 = s_1^{-c}U_1U_2^{b/a}$, and $U'_3 = s_1^aU_3$. U'_3 transforms H in the same way as U_3 does and it transforms U'_1 into $c_{23}^{b/a}U'_1$, thus giving c'_{13} and c'_{23} in the same cyclic group, the situation considered above. Moreover, if c_{13} is in the group $\{s_2, s_3, s_4, c_{12}\}$ the above operators U'_1 and U'_3 give c'_{13} in the group $\{c'_{12}, c'_{23}\}$. Then, if $c_{13} = c^a{}_{12}c^b{}_{23}$, the operator $U'_3 = U_2^{-a}U_3$ gives with U_1 a commutator $c'_{13} = c^b{}_{23}$, which is again the case first considered. The argument proves that unless K_3 is of order p^6 generators U_1, U_2, and U_3 may be chosen so that two of them are permutable. The two permutable ones may be taken to be U_1 and U_2 and the argument proves that U_1, U_2, and U_3 may then be chosen so that U_1 and U_2 are still permutable and U_1 and U_3 are also permutable unless the order of K_3 is p^5. If the order of K_3 is p^4 U_2 and U_3 cannot be permutable if the other two pairs are. Hence it is obvious that the group G_3 is completely determined when the order of K_3 is known.

By means of the foregoing transformations it is possible to select α operators $U_1, U_2, \cdots, U_\alpha$ in the group G_α which with H generate G_α and which have the further properties: (1) all but $k_\alpha - \alpha$ of the operators c_{ij} are identity, where p^{k_α} is the order of K_α, and (2) if $c_{i_1j_1}$ is identity, then every operator c_{ij_1} for $i < i_1$ and every operator c_{ij} for $j < j_1$ is identity. The first property says that the c_{ij}'s which are not identity constitute with the operators $U_i^{-1}s_1U_is_1^{-1}$ a set of independent generators of K_α, and the second says that the non-identity operators c_{ij} may be taken to be the last $k_\alpha - \alpha$ when they are ordered as follows:

$$c_{12}, c_{13}, c_{23}, c_{14}, \cdots, c_{34}, c_{15}, \cdots, c_{1\alpha}, \cdots, c_{\alpha-1\alpha}.$$

It is clear that the number k_α is not greater than $n-1$, is not greater than $\alpha + \alpha(\alpha-1)/2$, and is at least α. We may summarize the results as follows:

(5.2) *A metabelian group G_α whose operators are all of order p, which contains H as a maximal invariant abelian subgroup, and whose central is of order p^{n-1} is completely determined by the numbers n, α, and k_α, and there exists such a group for every set of numbers n, α, and k_α which satisfy the inequalities: $\alpha \leq n-1$, $\alpha \leq k_\alpha \leq \alpha(\alpha+1)/2$, and $k_\alpha \leq n-1$.*

If we remove the restriction that the operators of G_α be of order p, we arrive at the class of groups which come under the title of this section. We shall not attempt the determination of all such groups, but we shall point out some of the facts that must be taken into consideration in such a determination. Let G_α be $\{H, U_1, U_2, \cdots, U_\alpha\}$, and suppose that some of the operators $U_1, U_2, \cdots, U_\alpha$ are of order p^2. Then, as we have seen in section 3, if U_i is of order p^2 every operator of the co-set HU_i is of order p^2, and the p-th powers of the operators of a co-set are the same. Hence the order of the group of p-th powers in G_α cannot be greater than p^i, where i is the number of the generators $U_1, U_2, \cdots, U_\alpha$ which are of order p^2. It is then evident that there are at least α distinct groups $G_\alpha^{(1)}, G_\alpha^{(2)}, \cdots, G_\alpha^{(\alpha)}$ with the same quotient group G_α/H. Moreover, two groups G'_α and G''_α may have the same quotient group with respect to H and have groups of p-th powers of the same order and not be simply isomorphic, for the number of p-th powers which are commutators may be different for the two groups. The cross-cut of the group of p-th powers and the commutator subgroup may be of order p^j where j is any number from 0 to i, provided of course that n is large enough so that $k_\alpha + i \leqq n - 1$. Nor does this exhaust the possibilities. Two groups G_α may have the same centrals, the same commutator subgroups, the same group of p-th powers, and the same cross-cut of the group of p-th powers and commutator subgroup, and still not be simply isomorphic. For suppose U_1 and U_2 are both of order p^2 and that they have the same p-th power. The group $\{U_1, U_2\}$ is generated by U_1 and $U_1^{-1}U_2$, the second of which is of order p. In general it is obvious that the generators of $\{U_1, U_2, \cdots, U_\alpha\}$ may be chosen so that exactly i of them are of order p^2, where p^i is the order of the group of p-th powers. Now though we may select the generators of $\{U_1, U_2, \cdots, U_\alpha\}$ so that we have certain commutators $U_i^{-1}U_jU_iU_j^{-1}$, depending only on the number k_α, equal to identity and we may select a set of generators of the same group so that the first i of them are of order p^2, it is not in general possible to make the two selections simultaneously.

We shall end this section by anticipating and answering an objection that may be raised to this method of treatment of groups of order p^m. We started with a classification of groups of order p^m and have repeatedly subdivided those classes, continuing with more and more special types. This can not be avoided from the nature of the subject. The subdivisions and subdivisions of subdivisions are made on the basis of characteristic subgroups and any classification of the groups of order p^m must take them into account.

6. *Metabelian subgroups of type 2 of the holomorph of H.* We shall study the subgroups of type 2 of I_p by considering the corresponding subgroups of the holomorph of H. Here also we shall have to content ourselves with

exhibiting the essential complexity of the problem and showing how the difficulties may be overcome, at least in the simpler cases.

Let us consider $G_2 = \{H, U_1, U_2\}$, where G_2 is metabelian and has a central of order p^{n-2}. There is first the possibility that all the operators of $\{U_1, U_2\}$ are of type 1 in which case the order of the commutator subgroup K_2 is p. (Cf. section 4.) It is easy to see that if this group is in the holomorph of H it is completely determined by the given conditions.

Every other group G_2 whose central is of order p^{n-2} is such that $\{U_1, U_2\}$ contains at least one operator of type 2. This requires that the order of the commutator subgroup K_2 be at least p^2. Since the central is of order p^{n-2} no operator of $\{U_1, U_2\}$ can give rise to more than two independent commutators and the order of K_2 cannot be greater than p^4. It is obvious that groups exist with K_2 of order p^2, p^3, and p^4. They may be defined as follows: G_2 is generated by H, U_1 and U_2, where U_1 and U_2 are permutable with all the operators of H except for the implications of the relations

$$(6.1)\qquad \begin{array}{ll} U_1^{-1}s_1U_1 = s_1s_3 & U_2^{-1}s_1U_2 = s_1s_k \\ U_1^{-1}s_2U_1 = s_2s_4 & U_2^{-1}s_2U_2 = s_2s_m. \end{array}$$

If $s_k = s_3$ and $s_m = 1$, the order of K_2 is p^2. If $s_m = 1$ and s_3, s_4, and s_k are independent, the order of K_2 is p^3. If s_3, s_4, s_k, and s_m are independent, the order of K_2 is p^4. These subdivisions correspond respectively to the cases where $\{U_1, U_2\}$ contains 2, 1, and 0 subgroups composed of operators of type 1. In the first case the subgroups composed of operators of type 1 are generated by U_2 and $U_1U_2^{-1}$; in the second case the single such subgroup is generated by U_2.

We have not shown that there are not other groups G_2 satisfying the above conditions; there are such groups. We shall examine them in order according to the order of K_2.

Let K_2 be of order p^4. Then $\{U_1, U_2\}$ contains no operator of type 1, for otherwise that operator could be used in place of U_2 and the resulting G_2 would have K_2 of order p^3.

Next suppose the order of K_2 is p^3. Then in (6.1) one of the operators s_k and s_m is not in the group $\{s_3, s_4\}$. Let this operator be $s_k = s_5$. We may then suppose that

$$(6.2)\qquad s_m = s_3^{\alpha}s_4^{\beta}s_5^{\gamma}.$$

If $\{U_1, U_2\}$ contained two subgroups composed of operators of type 1, operators from these groups could be taken for generators and K_2 would be of order p^2. Hence $\{U_1, U_2\}$ has one or no subgroups composed of operators of type 1. If it contains one such subgroup we may suppose it to be generated by U_2.

In that case we should have $\alpha = \beta = 0$ in (6.2). If we replace s_2 by $s'_2 = s_1^{-\gamma}s_2$, we have $s'_m = 1$, with s_3, s'_4, and s_5 independent. This group is then simply isomorphic with the example above for which K_2 is of order p^3. We have then to determine whether α, β, and γ in (6.2) may be selected so that $\{U_1, U_2\}$ contains no operator of type 1, and if so whether it may be done in more than one way.

If $\{U_1, U_2\}$ contains an operator of type 1 that operator is permutable with some operator of $\{s_1, s_2\}$, and all the powers of the first are permutable with all the powers of the second. Hence we may assume that the two operators in question are $U_1U_2^y$ and $s_1s_2^a$. We seek conditions on α, β, and γ in order that this may be possible.

$$U_2^{-y}U_1^{-1}s_1s_2^aU_1U_2^y = s_1s_2^a \cdot s_3s_4^as_5^y(s_3^\alpha s_4^\beta s_5^\gamma)^{ay}.$$

Setting the commutator in the above equal to identity, we have

$$\begin{aligned} 1 + \alpha ay &\equiv 0, \\ a + \beta ay &\equiv 0, \\ y(1 + \gamma a) &= 0, \mod p. \end{aligned}$$

It is obvious that neither a nor y can be zero. In order that the system have a solution it is necessary that $\alpha + \beta\gamma \equiv 0$. Consequently, if $\alpha + \beta\gamma \not\equiv 0$, the group G_2 will be distinct from the group given above whose commutator subgroup is of order p^3, for $\{U_1, U_2\}$ will contain no operator of type 1.

We may assume that β in (6.2) is 0, for if we replace U_2 by $U'_2 = U_1^{-\beta}U_2$ we obtain $s'_5 = s_3^{-\beta}s_5$ and $s'_m = s_3^\alpha s_5^\gamma$. This last commutator may be written $s_3^{\alpha+\beta\gamma}(s_3^{-\beta}s_5)^\gamma$. If then we assume that $\beta = 0$ in (6.2) and replace s_2 by $s'_2 = s_1^{-\gamma}s_2$, we have $s'_m = s_3^\alpha$. Assuming $\beta = \gamma = 0$ in (6.2) and replacing s_2 by $s'_2 = s_2^{1/\alpha}$ we have $s'_m = s_3$. Hence, if $\{U_1, U_2\}$ contains no operator of type 1, U_1, U_2, s_1, and s_2 may be selected so that U_1 and U_2 satisfy the relations (6.2) in which $\alpha = 1$ and $\beta = \gamma = 0$. The group G_2 is completely determined by the fact that K_2 is of order p^3 and that $\{U_1, U_2\}$ contains no operator of type 1.

When the order of K_2 is p^2 our example shows that $\{U_1, U_2\}$ may contain two subgroups composed of operators of type 1. The group is determined by those conditions, for if they hold U_2 and $U_1U_2^{-1}$ may be taken for generators of the group $\{U_1, U_2\}$. Both are of type 1, they are permutable with different operators of $\{s_1, s_2\}$ and give rise to commutators which are independent. Moreover, no product of their powers is permutable with any operator of $\{s_1, s_2\}$ unless one of the two generators is missing from the product. Consequently $\{U_1, U_2\}$ can contain no more than two subgroups composed of operators of type 1. We have then to consider the possibility of $\{U_1, U_2\}$ containing fewer than two subgroups composed of operators of type 1.

Since the order of K_2 is p^2 and since $\{U_1, U_2\}$ contains at least one operator of type 2, we may assume that U_1 and U_2 satisfy (6.1) where $s_k = s_3^\alpha s_4^\beta$ and $s_m = s_3^\gamma s_4^\delta$. The condition that $U_1U_2^y$ is permutable with $s_1s_2^a$ is

$$s_3s_4^a(s_3^\alpha s_4^\beta)^y(s_3^\gamma s_4^\delta)^{ay} = 1.$$

From this we obtain the congruences

$$\begin{aligned} 1 + \alpha y + \gamma a y &\equiv 0, \\ a + \beta y + \delta a y &\equiv 0, \quad \bmod p. \end{aligned}$$

Eliminating y we obtain

$$\gamma a^2 + (\alpha - \delta)a - \beta \equiv 0.$$

In order that this quadratic in a have a solution it is necessary that $(\alpha - \delta)^2 + 4\beta\gamma$ be a square, mod p.

The operator U_2 may not be written in the form $U_1U_2^y$. However, the condition that U_2 be of type 1 is simply that the determinant $\alpha\delta - \beta\gamma$ be zero. Hence, necessary and sufficient conditions that $\{U_1, U_2\}$ contain no operators of type 1 are that $\alpha\delta - \beta\gamma \not\equiv 0$ and that $(\alpha - \delta)^2 + 4\beta\gamma$ be not a square. Also, necessary and sufficient conditions that $\{U_1, U_2\}$ contain one subgroup whose operators are of type 1 are that $\alpha\delta - \beta\gamma \not\equiv 0$ and that $(\alpha - \delta)^2 + 4\beta\gamma$ be a square. In general it is possible to find a set of numbers to satisfy either of these sets of conditions.

It is necessary to discover whether G_2 is determined by the number of subgroups composed of operators of type 1 when K_2 is of order p^2. Suppose first the $\{U_1, U_2\}$ contains no operator of type 1. Then $\alpha\delta - \beta\gamma \not\equiv 0$. We may suppose that $\alpha = 0$, for if U_2 is replaced by $U'_2 = U_1^{-\alpha}U_2$, we have $s'_k = s_4^\beta$. We may suppose further that $\beta = 1$, for this may be obtained by replacing s_1 by $s'_1 = s_1^{1/\beta}$. The matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ is now in the form $\begin{pmatrix}0 & 1 \\ \gamma & \delta\end{pmatrix}$. If we replace s_2 by $s'_2 = s_1^{-\delta/2}s_2$ and U_2 by $U'_2 = U_1^{-\delta/2}U_2$, the matrix takes the form $\begin{pmatrix}0 & 1 \\ \gamma & 0\end{pmatrix}$. Having reduced the matrix to this form the transformation $s'_1 = s_1^r$, $U'_2 = U_2^{1/r}$ changes it into the form $\begin{pmatrix}0 & 1 \\ \gamma r^2 & 0\end{pmatrix}$. As r is allowed to take on the values $1, 2, \cdots, p-1$ the number γr^2 takes on the values which are squares if γ is a square, and takes on the values which are not squares if γ is not a square. The number $(\alpha - \delta)^2 + 4\beta\gamma$ for this reduced form is 4γ, which is a square if γ is a square. Hence $\{U_1, U_2\}$ contains one or no subgroups composed of operators of type 1 according as γ is or is not a square. In each case there is just one group.

The groups G_2 are therefore completely determined by the orders of their commutator subgroups and the number of subgroups composed of operators of type 1 in $\{U_1, U_2\}$. Every subgroup $\{U\}$ of $\{U_1, U_2\}$ determines a subgroup of index p of G_2. If U is of type 1, the subgroup has a central of order p^{n-1}, and if U is of type 2, the subgroup has a central of order p^{n-2}. This classification according to the number of subgroups of $\{U_1, U_2\}$ composed of operators of type 1 coincides with the classification according to the kinds of subgroups of index p that are contained in G_2. We shall summarize the preceding considerations in the theorem:

(6. 3) *The holomorph of H contains six conjugate sets of metabelian subgroups of order p^{n+2} with centrals of order p^{n-2}, and each set is determined by the order of the commutator subgroup and the number of subgroups of order p^{n+1} whose centrals are of order p^{n-1}.*

The groups $G_3 = \{H, U_1, U_2, U_3\}$. We consider next the groups generated by H and three operators U_1, U_2, and U_3, whose centrals are of order p^{n-2}. We shall use the symbol U, to denote the group $\{U_1, U_2, U_3\}$. It follows from (4. 4) that not all the operators of U are of type 1. We may therefore assume that U_1 is of type 2 and that the order of K_3 is at least p^2. If U_1, U_2, and U_3 are all of type 2 and the commutators that arise from transformation of H by them are all independent, the order of K_3 is p^6, which is as large as that order can be. We shall give examples to show that the order of K_3 may be p^i where i is any of the numbers $2, 3, \cdots, 6$.

Let U_1, U_2, and U_3 be permutable with each other and with the generators of H except for the implications of the relations:

$$(6.4)\qquad \begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3 & U_2^{-1}s_2U_2 = s_2s_k & U_3^{-1}s_1U_3 = s_1s_l \\ U_1^{-1}s_2U_1 = s_2s_4 & U_2^{-1}s_2U_2 = s_2s_k & U_3^{-1}s_2U_3 = s_2s_m. \end{array}$$

(a) If $s_k = s_m = 1$, $s_j = s_3$, and $s_l = s_4$, then the order of K_3 is p^2.

(b) If $s_k = s_m = 1$, $s_l = s_4$, and s_j is independent of s_3 and s_4, then the order of K_3 is p^3.

(c) If $s_k = s_m = 1$ and s_3, s_4, s_j, and s_l are independent, then K_3 is of order p^4.

(d) If $s_m = 1$ and the rest are independent, then K_3 is of order p^5.

(e) If all the commutators are independent, the K_3 is of order p^6.

The results for the groups G_2 suggest that we interest ourselves in the number of subgroups composed of operators of type 1 in U. In (a) there are $2p+1$ such subgroups, $p+1$ of them in $\{U_2, U_3\}$, and $p+1$ in $\{U_1U_2^{-1}, U_3\}$, which counts the group $\{U_3\}$ twice. In (b) there are $p+1$, the subgroups of $\{U_2, U_3\}$. In (c) there are $p+1$, all in $\{U_2, U_3\}$. In (d) there is one. In (e) there are none.

We investigate the possibility of the existence of other groups G_3 separating them into classes according to the order of K_3 and then subdividing those classes according to the number of subgroups of U composed of operators of type 1.

When the order of K_3 is p^6, U can contain no operator of type 1, for otherwise we could take that operator to be U_3 and the order of K_3 could not then be greater than p^5. Any two groups G_3 with K_3 of order p^6 are obviously simply isomorphic.

When the order of K_3 is p^5, U cannot contain two subgroups composed of operators of type 1, for otherwise we could suppose those subgroups to be generated by U_2 and U_3 each of which would give rise to a commutator subgroup of order p in which case the order of K_3 could be p^4 at most. In (d) U has one subgroup of the type in question; the only other possibility is that U contains none. If in the conditions which define (d) we replace $s_m = 1$ by $s_m = s_3$, we have a group U which realizes that possibility. To see that U contains no operator of type 1 we need only to examine $\{H, U_1, U_3\}$ in the light of the discussion of G_2 with K_2 of order p^3 in section 5.

It remains to be seen that the two groups described are the only ones with K_3 of order p^5. We may suppose that the order of the commutator subgroup of $\{H, U_1, U_2\}$ is p^4, for if it were p^3 then the order of the commutator subgroup of $\{H, U_1, U_3\}$ would have to be p^4 and the rôles of U_2 and U_3 could be interchanged. Then if U contains an operator of type 1 it may be taken for U_3 and one of the operators of H invariant under U_3 and not invariant under U_1 and U_2 may be taken for s_2. The generators of G_3 will then satisfy (d) above. If U contains no operator of type 1, we may assume that the generators of G_3 satisfy (d) excepting for the fact that s_m is an operator different from identity in $s_3, \cdots, s_7$, where j, k, l in (6.4) are respectively 5, 6, 7. We shall write

$$s_m = s_3^{\alpha} s_4^{\beta} s_5^{\gamma} s_6^{\delta} s_7^{\epsilon}.$$

We note first that both β and δ may be taken to be 0, for if U_3 is replaced by $U'_3 = U_1^{-\beta} U_2^{-\delta} U_3$ the new commutator s'_m is in the group $\{s_3, s_5, s'_7\}$ and s'_7 is not in the commutator subgroup of $\{H, U_1, U_2\}$. If now s_2 is replaced by $s'_2 = s_1^{-\epsilon} s_2$ then s'_m is expressible in terms of s_3 and s_5, and so we may assume that ϵ is also 0. The operator $s_m = s_3^{\alpha} s_5^{\gamma}$ is in the commutator subgroup obtained by transformation of s_1 by U_1 and U_2, and hence by a proper choice of generators of $\{U_1, U_2\}$ we may make $\alpha = 1$ and $\gamma = 0$. Therefore if U contains no operator of type 1, generators of G_3 may be chosen to satisfy (6.4) in which $s_m = s_3$ and $s_3, \cdots, s_l$ are independent.

When the order of K_3 is p^4, the number of subgroups in U composed of operators of type 1 may be $p+1$, as in (c), or it may be 2, 1, or 0. If the number were greater than $p+1$ U could be generated by three operators of type 1 and the order of K_3 could not be greater than p^3. If it were greater than 2 and not $p+1$, three operators of type 1 could be chosen for generators of U unless all the operators of type 1 were in the same group of order p^2. Let this group of order p^2 be $\{U_2, U_3\}$. Then $\{H, U_2, U_3\}$ has a central of order p^{n-1} and $\{U_2, U_3\}$ contains $p+1$ subgroups composed of operators of type 1, or $\{H, U_2, U_3\}$ has a commutator subgroup of order p and the order of K_3 is not greater than p^3.

There exists a group for each of the remaining possibilities. If in (6.4) $s_k = s_l = 1$, and s_3, s_4, s_j, and s_m are independent, we have such a group with U containing 2 subgroups composed of operators of type 1. If in (6.4) $s_m = 1$, s_3, s_4, s_j, and s_l are independent and $s_k = s_3$, we have such a group with U containing one subgroup composed of operators of type 1. If in (6.4) s_3, s_4, s_j, and s_l are independent, $s_k = s_3$, and $s_m = s_j$, we have such a group with U containing no subgroup of type 1.

The above four groups are the only groups G_3 with K_3 of order p^4. For suppose U contains either 2 or $p+1$ subgroups of order p composed of operators of type 1 and let U_2 and U_3 generate such groups. U is generated by U_2, U_3 and one other operator which must be of type 2. The centrals of $\{H, U_2\}$ and $\{H, U_3\}$ may or may not be distinct. If they are the same, generators of that central may be taken to be $s_2, s_3, \cdots, s_n$. The group G_3 is then obviously the one defined by (c). If the centrals of $\{H, U_2\}$ and $\{H, U_3\}$ are distinct, generators of H may be chosen so that U_2 is permutable with s_2, U_3 with s_1, and both with all the rest. The group G_3 is then obviously the one given above where U has 2 subgroups composed of operators of type 1.

Let U contain one subgroup composed of operators of type 1 and let it be generated by U_3. Let generators of H be chosen so that U_3 is permutable with all of them except s_2. The commutator subgroup K_2 of $G_2 = \{H, U_1, U_2\}$ will be of order p^3 or p^4. U_1 and U_2 may be chosen so that K_2 is of order p^4. If the order of K_2 is p^3 we may suppose that in (6.4) we have $s_k = s_3$ and $s_j = s_5$, for the transformations on G_2 in section 5 to put it in that form did not disturb s_1. If then U_2 is replaced by $U'_2 = U_2U_3$, the group $G'_2 = \{H, U_1, U'_2\}$ has K'_2 of order p^4. We may then suppose that $s_j = s_5$, $s_k = s_6$, $s_l = 1$, and s_m is in the group $\{s_3, s_4, s_5, s_6\}$. From considerations similar to those in section 5 it follows that we may assume $s_m = s_3^\alpha s_5^\beta$. The commutator $s_3^\alpha s_5^\beta$ is a commutator in $\{H, U_1, U_2\}$ arising from transformation of s_1 by $U_1^\alpha U_2^\beta$. If this last operator is used for U_1, the group is generated by operators which satisfy the relations given above in the case where U contains but one subgroup composed of operators of type 1.

Suppose U contains no operators of type 1. Then neither s_1 nor s_2 is permutable with any of the operators U_1, U_2, and U_3. From this it follows that the operators s_3, s_j, and s_l of (6.4) are independent as are s_4, s_k, and s_m, no matter how U_1, U_2, and U_3 are chosen. If the order of K_2 of $G_2 = \{H, U_1, U_2\}$ is p^2 then K_2 is $\{s_3, s_4\}$ and s_3, s_4, s_l, and s_m are independent so that the order of the commutator subgroup of $\{H, U_1, U_3\}$ is p^4. If the order of K_2 is p^3, we may suppose that $s_j = s_5$ and $s_k = s_3$. Then one of the operators s_l and s_m is outside the group $\{s_3, s_4, s_5\}$. Let $s_m = s_6$ be this operator. Since s_l cannot be in $\{s_3, s_5\}$ it must be in $\{s_3, s_4, s_6\}$. Replacing U_3 by U'_3 equal to the proper combination of U_1 and U_3, we have s'_l and s'_m both in the group $\{s_4, s_6\}$, and since U'_3 cannot be of type 1 the commutator subgroup of $\{H, U_2, U_3\}$ will be $\{s_3, s_4, s_5, s_6\}$. Hence, if U contains no operator of type 1, U_1 and U_2 may be chosen so that $\{H, U_1, U_2\}$ has a commutator subgroup of order p^4.

We may then assume that $s_j = s_5$ and $s_k = s_6$, where s_3, s_4, s_5, and s_6 are independent. We may assume further that s_l is expressible in terms of s_4 and s_6, for by replacing U_3 by a proper combination of U_1, U_2, and U_3 powers of s_3 and s_5 may be removed. Then a proper choice of U'_1 in the group $\{U_1, U_2\}$ will give $s'_4 = s_l$. Consequently G_3 contains a subgroup G_2 for which K_2 is of order p^3.

We shall take $\{H, U_1, U_2\}$ to be the group G_2 with K_2 of order p^3, and let $s_j = s_5$ and $s_k = s_3$. Then either s_l or s_m is a new commutator s_6, and on account of the symmetry in s_1 and s_2 of the relations defining G_2 we may suppose $s_l = s_6$. The commutator s_m is in $\{s_3, s_4, s_5, s_6\}$, but a proper choice of U_3 in the group $\{U_1, U_3\}$ removes the powers of s_4 from the expression for it. This transformation changes s_6 to some operator of the group $\{s_3, s_6\}$ which may be taken for a new s_6. We may thus assume that $s_m = s_3^\alpha s_5^\beta s_6^\gamma$. If now U_3 is replaced by $U'_3 = U_2^{-\alpha} U_3$, we have $s'_l = s_5^{-\alpha} s_6$ and $s'_m = s_5^\beta s_6^\gamma$. If s_2 is now replaced by $s'_2 = s_1^{-\gamma} s_2$, the operator s_m becomes a power of s_5. This last transformation replaces the commutator of U_2 and s_2 by $s_3 s_5^{-\gamma}$, which is not the same as the commutator of U_1 and s_1. However, if U_1 is replaced by $U_1 U_2^{-\gamma}$ the situation is restored. The commutator of U_1 and s_2 is also changed, but since it is still independent of s_3, s_5, and s_6 and appears nowhere else it may be designated by s_4. We may therefore assume that in (6.4) $s_j = s_5$, $s_k = s_3$, $s_l = s_6$, and $s_m = s_5^\beta$. Since $s_6^{1/\beta}$ will serve for s_6, we may assume that $\beta = 1$. Therefore, if G_3 has a commutator subgroup of order p^4 and U contains no operator of type 1, generators of G_3 may be chosen to satisfy (6.4) with $s_j = s_m$, $s_k = s_3$, and s_3, s_4, s_j, and s_l independent.

We now consider the case where the order of K_3 is p^3. In the example (b) above, U contains $p + 1$ subgroups composed of operators of type 1. If in (6.4) we let $s_k = s_m = 1$, $s_j = s_3$, and let s_3, s_4, and s_l be independent, then U will contain $p + 2$ subgroups composed of operators of type 1. For all the operators of $\{U_2, U_3\}$ will be of type 1 and $U_1^{-1}U_2$ will also be of type 1. If in (6.4) we let $s_j = s_3$, $s_k = 1$, and $s_l = s_m$ then U will contain three subgroups composed of operators of type 1. They are generated by U_2, U_3, and $U_1^{-1}U_2$. If in (6.4) we let $s_k = s_l = 1$, $s_j = s_5$, and $s_m = s_3$, then U will contain two subgroups composed of operators of type 1, generated by U_2 and U_3. If in (6.4) we let $s_j = s_5$, $s_k = s_3$, $s_l = s_4$, and $s_m = 1$, then U will contain but one subgroup composed of operators of type 1. In general there is also a group G_3 with K_3 of order p^3 and U containing no operators of type 1, but it is not so obvious and in order that we may not start an argument prematurely we shall not attempt to describe it here.

The above examples exhaust all the possibilities for the number of subgroups composed of operators of type 1 in U. If there are as many as 3 and not $p + 1$ such subgroups in U, we may suppose U_1, U_2, and U_3 to generate three of them. Then either $\{H, U_1\}$ and $\{H, U_2\}$ have the same central or generators of H may be chosen so that

$$\begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & & U_3^{-1}s_1U_3 = s_1s_5 \\ & U_2^{-1}s_2U_2 = s_2s_4, & U_3^{-1}s_2U_3 = s_2s_m, \end{array}$$

where $s_m = 1$ or $s_m = s_5$. In case $s_m = 1$, U contains $p + 2$ and no more subgroups of the given type. In case $s_m = s_5$, U contains 3 subgroups of the given type. If $\{H, U_1\}$ and $\{H, U_2\}$ have the same central, then $\{U_1, U_2\}$ contains only operators of type 1. We may suppose that

$$\begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & U_2^{-1}s_1U_2 = s_1s_4, & U_3^{-1}s_1U_3 = s_1s_5 \\ & & U_3^{-1}s_2U_3 = s_1s_5. \end{array}$$

The only operators of the type in question are operators of $\{U_1\}$, $\{U_2\}$ and $\{U_3\}$. We have thus shown that the number of such subgroups must be 0, 1, 2, 3, $p + 1$, or $p + 2$.

In each of the cases where U contains respectively 3 and $p + 2$ subgroups composed of operators of type 1, the argument in the last paragraph shows, by setting up a normal form, that there exists just one group.

When U contains $p + 1$ subgroups composed of operators of type 1, the same argument shows that they all belong to a subgroup of order p^2. We may choose generators of H so that the two operators U_2 and U_3 which generate this group of order p^2 are both permutable with s_2. Then U_1 must be of type 2. The commutator subgroup arising from transformation of s_1 must be of order p^3, for otherwise there would exist in $\{U_1, U_2, U_3\}$ an operator permutable with s_1 and hence of type 1 and not in $\{U_2, U_3\}$. The commutator determined by U_1 and s_2 is in the group of commutators arising from s_1. Hence we may suppose generators of G_3 to satisfy (6.4) in which $s_j = s_4$, $s_l = s_5$, $s_k = s_m = 1$, and s_4 is replaced by $s_3^{\alpha}s_4^{\beta}s_5^{\gamma}$. The exponent α must be 0, for otherwise $s'_1 = s_1^{\alpha}s_2^{-1}$ would be permutable with $U_1U_2^{\beta}U_3^{\gamma}$. If then U_2 is replaced by $U_2^{\beta}U_3^{\gamma}$, the generators of G_3 satisfy the relations defining the group for which U contains $p+1$ subgroups composed of operators of type 1 given above. Hence there is just one such group.

We suppose next that U contains two subgroups composed of operators of type 1, and let them be generated by U_2 and U_3. We may select generators of H so that in (6.4) $s_k = s_l = 1$. Then s_j and s_m will be independent for otherwise $\{U_2, U_3\}$ would contain only operators of type 1. Let $s_j = s_4$ and $s_m = s_5$. U_1 is necessarily of type 2 and one of the commutators it determines is not in the group $\{s_4, s_5\}$. On account of the symmetry in s_1 and s_2 we may suppose it to be s_3. Then the commutator arising from U_1 and s_2 is in $\{s_3, s_4, s_5\}$. Let us denote it by $s_3^{\alpha}s_4^{\beta}s_5^{\gamma}$. We may suppose $\gamma = 0$, for that result would be obtained by replacing U_1 by $U_1U_3^{-\gamma}$. Also, α must be 0, for otherwise the operator $U_1U_2^{\beta/\alpha}$ would give two commutators in the same cyclic group and would therefore be of type 1. If then U_2 is replaced by U_2^{β}, the generators of G_3 satisfy (6.4) in which $s_j = s_4$, $s_k = s_l = 1$, $s_m = s_5$.

When U contains just one subgroup composed of operators of type 1, we may suppose it to be generated by U_3 and generators of H to be chosen so that s_m in (6.4) is identity. Let $s_l = s_5$. We may suppose K_2 of $G_2 = \{H, U_1, U_2\}$ to be of order p^3, for if it is of order p^2 the groups $\{s_3, s_j\}$ and $\{s_4, s_k\}$ would be the same, $\{U_1, U_2\}$ containing no operator of type 1, and each would be K_2, in which case the order K_2 of $\{H, U_1, U_2, U_3\}$ would be p^3. The group $\{s_3, s_j, s_5\}$ is of order p^3. We may suppose

$$(6.5)\qquad \begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & U_2^{-1}s_1U_2 = s_1s_4, & U_3^{-1}s_1U_3 = s_1s_5, \\ U_1^{-1}s_2U_1 = s_2s_j, & U_2^{-1}s_2U_2 = s_2s_k. & \end{array}$$

The operators s_j and s_k are in $\{s_3, s_4, s_5\}$.

There are now two possibilities. Either s_5 is in $\{s_j, s_k\}$ or it is not. Let us suppose that s_5 is in $\{s_j, s_k\}$. We may select U_2 so that $s_k = s_5$. Then s_j may be assumed to be in $\{s_3, s_4\}$, for if the expression for it contained s_5^{γ} the operator $U'_1 = U_1U_2^{-\gamma}$ would give s'_j in $\{s_3, s_4\}$. Let $s_j = s_3^{\alpha}s_4^{\beta}$.

Since U_1 is not of type 1, β cannot be zero. α must be zero, for otherwise we could replace s_1 by $s'_1 = s_1 s_2^{-1/\alpha}$, in which case the commutator subgroup arising from transformation of s'_1 by U would be generated by s_4 and s_5 and U would contain operators of type 1 which were not powers of U_3. If now we replace U_1 by $U_1^{1/\beta}$, we have the generators of G_3 satisfying the relations (6. 5) in which $s_j = s_4$ and $s_k = s_5$.

We suppose now that s_5 is not in the group generated by commutators of s_2. We may select two independent commutators from that group and denote them by s_3 and s_4. Let

$$(6.6) \qquad \begin{aligned} &U_1^{-1} s_1 U_1 = s_1 s_j, \quad && U_2^{-1} s_1 U_2 = s_1 s_k, \quad && U_3^{-1} s_1 U_3 = s_1 s_5, \\ &U_1^{-1} s_2 U_1 = s_2 s_3, && U_2^{-1} s_2 U_2 = s_2 s_4. && \end{aligned}$$

Both s_j and s_k may be assumed to be in $\{s_3, s_4\}$. The group $\{U_1, U\}$ contains an operator which with s_1 gives s_3 for a commutator. We may therefore assume $s_k = s_3$ and $s_j = s_3^{\alpha} s_4^{\beta}$. The transformation resulting from replacing s_1 by $s'_1 = s_1 s_2^{\alpha}$ and U_1 by $U'_1 = U_1 U_2^{\alpha}$ changes s_3 into $s'_3 = s_3 s_4^{\alpha}$ and s_j into $s'_j s_3^{\alpha+2\alpha} s_4^{\beta+\alpha^2}$. This expressed in terms of the new s'_3 and s_4 is $(s'_3)^{\alpha+2\alpha} s_4^{\beta-c\alpha-\alpha^2}$. Therefore U_1 and s_1 may be chosen so that generators of G_3 satisfy (6. 6) with $s_k = s_3$ and $s_j = s_4^r$, where, as we have seen in section 5, r is not a square.

We have obtained two normal forms for the case where U contains one subgroup composed of operators of type 1. It is necessary to see that they describe groups which are not simply isomorphic. In both cases the subgroup $\{s_5\}$ is characteristic, being determined by the characteristic subgroup $\{U_3\}$ of U. The central of $\{H, U_3\}$ is characteristic, and the subgroup arising by transformation of the operators in the central of $\{H, U_3\}$ and not in the central of G_3 by operators of U is also characteristic. In one case this last group contains s_5 and in the other it does not. Consequently the two groups are distinct.

Finally we suppose that U contains no operator of type 1. Then no matter how generators of G_3 are chosen, commutators arising from transformation of s_1 by U constitute a group of order p^3, as do those arising from transformation of s_2. If we select U_1 at random from U we determine two commutators s_3 and s_4. We may write $U_1^{-1} s_1 U_1 = s_1 s_3$, $U_1^{-1} s_2 U_1 = s_2 s_4$. There exists an operator U_2 which with s_2 gives the commutator s_3. If the commutator subgroup of $\{H, U_1, U_2\}$ is of order p^2, s_1, s_2, U_1, U_2 may be chosen to satisfy the above conditions and the condition $U_2^{-1} s_1 U_2 = s_1 s_4^r$, as was shown in section 5. Then the commutator subgroup of $\{H, U_1, U_3\}$, however U_3 may be chosen outside of $\{U_1, U_2\}$, is of order p^3. We may therefore assume that the order of K_2 of $\{H, U_1, U_2\}$ is p^3. Therefore the commutator of s_1 and U_2 may be taken to be s_5. Then U contains an operator

U_3 such that $U_3^{-1}s_2U_3 = s_2s_5$. The commutator of U_3 and s_1 is in $\{s_3, s_4, s_5\}$. Hence generators of G_3 may be chosen to satisfy

$$(6.7)\qquad \begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & U_2^{-1}s_1U_2 = s_1s_5, & U_3^{-1}s_1U_3 = s_1s_k, \\ U_1^{-1}s_2U_1 = s_2s_4, & U_2^{-1}s_2U_2 = s_2s_3, & U_3^{-1}s_2U_3 = s_2s_5, \end{array}$$

where $s_k = s_3^{\alpha}s_4^{\beta}s_5^{\gamma}$.

Since U contains no operator of type 1 it is obvious that the numbers α, β, and γ are not arbitrary. For example, if $\alpha = \gamma = 0$ and $\beta = 1$, the commutator subgroup determined by $U_1U_2U_3$ is of order p generated by $s_3s_4s_5$. We seek the condition on α, β, γ that U may have the desired property.

No operator of the group $\{s_1, s_2\}$ is permutable with any operator of U. If there exists an operator of $\{s_1, s_2\}$ which is permutable with any operator of U any power of it is also permutable with the same operator of U. Such an operator could not be s_2 and hence may be written $s_1s_2^{a}$. The condition that $s_1s_2^{a}$ be permutable with $U_1^{k}U_2^{l}U_3^{m}$ is

$$s_3^{k+al+\alpha m}s_4^{ak+\beta m}s_5^{l+(\gamma+a)m} = 1.$$

From this we get the system of congruences

$$\begin{aligned} k + al + \alpha m &\equiv 0, \\ ak + \phantom{al +{}} \beta m &\equiv 0, \\ l + (\gamma + a)m &\equiv 0, \mod p. \end{aligned}$$

This system must have a solution k, l, m different from 0, 0, 0 and hence the determinant of the coefficients must be zero. This requires that

$$(6.8)\qquad a^3 + \gamma a^2 - \alpha a + \beta \equiv 0, \mod p.$$

Therefore the condition that U contains no operator of type 1 is that (6.8) be irreducible. When $p > 2$ such irreducible congruences exist and consequently numbers α, β, and γ can be found, thus determining a G_3 whose generators satisfy (6.7). Now it can be shown * that if G_3 is any group satisfying the above conditions, it contains a set of generators

$$s'_1, s'_2, \cdots, s'_n, U_1, U'_2, U'_3$$

which satisfy (6.7) and have for α', β', γ' the coefficients of any irreducible cubic of the form (6.8). Hence there is but one group with K_3 of order p^3 and U containing no operator of type 1.

There remains to consider the groups G_3 for which K_3 has the order p^2. Then U contains an operator permutable with s_1 and an operator permutable with s_2. Let these operators be U_1 and U_2 respectively. Two possibilities

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* Cf. "On Cubic Congruences." *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 962.

arise: (a) the group $\{U_1, U_2\}$ contains two subgroups composed of operators of type 1, or (b) it contains $p+1$ subgroups composed of such operators. In the former case the commutator subgroup of $\{H, U_1, U_2\}$ is of order p^2 and in the latter of order p.

Taking up case (a), we may assume

$$\begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & & U_3^{-1}s_1U_3 = s_1s_k, \\ & U_2^{-1}s_2U_2 = s_2s_4, & U_3^{-1}s_2U_3 = s_2s_m, \end{array}$$

where $s_k = s_3^a s_4^\beta$ and $s_m = s_3^\gamma s_4^\delta$. If now U_3 is replaced by $U'_3 = U_1^{-a}U_2^{-\delta}U_3$, s_k and s_m become respectively s_4^β and s_3^γ. Let us then suppose U_3 in the above is $(U'_3)^{1/\beta}$, so that $s_k = s_4$ and $s_m = s_3^\gamma$. The operator $U_1^\gamma U_2 U_3$ is of type 1 since the commutators arising from it and s_1 and s_2 are the same. Hence $\{U_1, U_2, U_3\}$ contains at least three subgroups composed of operators of type 1. The condition that $U' = U_1^k U_2^l U_3^m$ be of type 1 is that $U'^{-1}sU' = s$, where $s = s_1 s_2^a$. This gives

$$\begin{aligned} k + \gamma am &\equiv 0, \\ al + \quad m &\equiv 0, \bmod p. \end{aligned}$$

Hence, $m = -al$ and $k = \gamma a^2 l$. If $\gamma = 0$, then $k = 0$ and every operator of $\{U_2, U_3\}$ is of type 1. This comes under (b) above. We may therefore suppose for case (a) that $\gamma \not\equiv 0$. Then the above congruences have the solutions given which says that the group generated by $U_1^{a^2\gamma}U_2U_3^{-a}$ is permutable with $s_1s_2^a$. There are p choices of a; hence there are p subgroups $\{U_1^{a^2\gamma}U_2U_3^{-a}\}$. There is one subgroup composed of operators of type 1 which does not come in this form, namely $\{U_1\}$. Hence U contains at least $p+1$ subgroups composed of operators of type 1. Since every operator of U can be written in the form $U_1^k U_2^l U_3^m$ there are only $p+1$ such subgroups.

Let us consider case (b). Since the operators of $\{U_1, U_2\}$ are all of type 1, we may suppose

$$\begin{array}{lll} U_1^{-1}s_1U_1 = s_1s_3, & & U_3^{-1}s_1U_3 = s_1s_4, \\ & U_2^{-1}s_2U_2 = s_2s_3, & U_3^{-1}s_2U_3 = s_2s_m, \end{array}$$

for the commutators arising from U_3 are not both in $\{s_3\}$ and hence one may be taken to be s_4. s_m is in $\{s_3, s_4\}$ and may be assumed to be s_4 or identity, for if $s_m = s_3^a s_4^\beta$ then U_3 may be replaced by $U'_3 = U_2^{-a}U_3$ which changes s_m to s_4^β. If then $\beta \not\equiv 0$ and s_2 is replaced by $s_2^{1/\beta}$ and U_1 by $U_1^{1/\beta}$ we have the generators of G_3 satisfying the above relations with $s_m = s_4$.

If $s_m = 1$, U has at least $p+2$ subgroups composed of operators of type 1. Every operator not in $\{U_1, U_2\}$ and not in $\{U_3\}$ is of type 2 since it gives rise to a commutator subgroup of order p^2. Hence there just $p+2$ subgroups composed of operators of type 1.

If $s_m = s_4$, U_3 is also of type 1, as is $U_1U_2U_3$. The condition that $U_1{}^kU_2{}^lU_3$ be permutable with $s_1s_2{}^a$ gives

$$a + 1 \equiv 0, \text{ and } k + al \equiv 0, \text{ mod } p.$$

Then for every k there exists a group generated by $U_1{}^kU_2{}^kU_3$ whose operators are of type 1. There are p such groups and they are all outside of $\{U_1, U_2\}$. Therefore U contains $2p + 1$ subgroups composed of operators of type 1.

We have thus shown that there are three groups $\{H, U_1, U_2, U_3\}$ with commutator subgroups of order p^2 distinguished by the fact that $\{U_1, U_2, U_3\}$ contains $p + 1$, $p + 2$, or $2p + 1$ subgroups of order p with operators all of type 1. Moreover, there is just one group in each case, for we have determined a canonical form for the set of relations among the generators.

It will be worth while to tabulate the results of this section. The subgroups of the holomorph of H determined in section 5 all have centrals of order p^{n-1}; those in section 6 have centrals of order p^{n-2}. We give in order the orders of G and K and the number of subgroups composed of operators of type 1 in G/H.

G	K	no. of subgroups with operators of type 1.
p^{n+2}	p^4	0
	p^3	0, 1
	p^2	0, 1, 2
p^{n+3}	p^6	0
	p^5	0, 1
	p^4	0, 1, 2, $p + 1$
	p^3	0, 1, 1,* 2, 3, $p + 1$, $p + 2$
	p^2	$p + 1$, $p + 2$, $2p + 1$.

It is perhaps permissible to emphasize the fact that each of the cases above represents a doubly infinite system of groups, since n and p are arbitrary. Also, every one of the groups contains only operators of order p, identity excepted. Though it is convenient to study subgroups of the holomorph of H, we are primarily interested in the Sylow subgroup of order $p^{n(n-1)/2}$ in the group of isomorphisms of H. In the preceding pages we have determined all the subgroups of order p^2 and type 1, 1 and all the abelian subgroups of order p^3 and type 1, 1, 1 which contain only operators of types 1 and 2.

URBANA, ILLINOIS.

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[*] The two groups with K of order p^3 and with G/H having the same number of operators of type 1 are discussed following (6. 6).

NOTE ON A UNIQUE REPRESENTATION FOR EVERY SOLVABLE GROUP.

By A. C. LUNN and J. K. SENIOR.

It has been shown * that every solvable group G of order $g = \Pi x_i$, where $x_i = p_i^{a_i}$ $(i = 1 \cdots n)$, and the p_i's are distinct primes has an intransitive representation as a permutation group P with transitive constituents $T_1 \cdots T_n$, where T_i is of degree x_i. The purpose of this note is to point out the following features in which this representation appears as an analogue or generalization of the Cayley regular representation. The comparatively moderate degree Σx_i of the intransitive representation is a convenience for detailed computations. The proofs depend in the main on familiar theorems concerning representations together with the theorems quoted from Hall † according to which every solvable group of degree g as above has subgroups of each order prime to its index, and all those of any particular such order are conjugate.

(1) The representation P exists and is simply isomorphic with G. This was proved in the paper cited. The simple isomorphism also follows from the property (4) proved below.

(2) The representation P is uniquely determined by the abstract group G, not only as a permutation group per se, but as a representation of the abstract group, in the usual sense of "unique" as understood in connection with the meaning of "equivalence" of permutation representations. In the present instance the distinction between the two kinds of equivalence disappears, since no new conjugacies of the subgroups in question are here introduced by an external isomorphism.

Proof. In any two representations of the specified type for a given solvable group, the respective transitive constituents of the same degree are equivalent under a suitable transformation of notation, and these transformations taken together to form a single change of notation on the entire set of Σx_i letters would exhibit the equivalence of the two complete intransitive representations.‡

(3) Each transitive constituent T_i contains as a subgroup the regular representation of the corresponding Sylow subgroup S_i of G.

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* Lunn and Senior, *American Journal of Mathematics*, this volume, pp. 319, 320.

† *Loc. cit.*

‡ We are indebted to Mr. Garrett Birkhoff for independently pointing out an equivalent proof, stated in terms of correspondence of cosets.

The Sylow subgroup of order x_i in T_i must be regular since no operator in it except identity can leave any of the x_i letters invariant. The isomorphism with S_i appears from the theorem that if R is any representation of order $p^a l$ of a group G of order $p^a k$, where k is prime to p, then the Sylow subgroups of order p^a in R and G are simply isomorphic.

(4) The representation P contains implicitly in the form of induced groups of permutations on the various sets of r-tuples of letters, $r = 1 \cdots n$, every transitive representation of G of degree prime to the order of its generating subgroup. In particular, the Cayley regular representation occurs as the group on the n-tuples. Here "r-tuple of letters" means one formed by taking one letter from each of r of the sets of transitivity involved in P. By "set of r-tuples of letters" is meant the entire aggregate of r-tuples formed in this way from any one r-tuple of sets of transitivity. In particular, from the transitivity on the g n-tuples follows the simple isomorphism of G and its representation P.

The proof follows from Hall's theorems supplemented by an induction proof through increasing values of r, using the following lemma which is stated separately because it admits of certain applications to insolvable groups as well.

LEMMA. *If G is a group of order mnk where these factors are all relatively prime in pairs and m, n are not unity, and G has transitive representations of degrees m, n respectively, say on the letters $a_1 \cdots a_m$, $b_1 \cdots b_n$, then the induced group on the entire set of pairs $a_i b_j$ is transitive.*

Proof. Let the subgroups leaving a_i, b_j respectively invariant be denoted by A_i, B_j, and their intersection by K_{ij}. The index of K_{ij} under B_j is divisible by m, the index of A_i under G, and is not greater than m,* hence is m, and the order of K_{ij} is k. Hence A_i consists of n cosets of K_{ij} no two of which permute b_j in the same way, and similarly B_j consists of m cosets of K_{ij} no two of which permute a_i in the same way. Thus A_i is transitive on the pairs $a_i b_j$ for the single i and all j's, and B_j is transitive on the pairs with all i's and the single j, so that G is transitive on the entire set of all pairs $a_i b_j$.

As an instance of this lemma for the case of an insolvable group may be noted the simple group of order 168, which has two representations of degree 7 and one of degree 8. Hence there are two intransitive representations of degree 7 + 8, each of which yields as induced group on the pairs the unique transitive representation of degree 56.

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* See Speiser, *Theorie der Gruppen von endlicher Ordnung*, 2nd ed. (1927), p. 64; generalized from Miller, *Annals of Mathematics*, vol. 14 (1912-13), p. 95.

TWO-FOLD GENERALIZATIONS OF CAUCHY'S LEMMA.

By L. E. Dickson.

1. *Introduction and Summary.* We investigate the conditions on a and b under which there exist integral solutions[*] $> -k$ of the pair of equations

$$(1) \qquad a = \sum_{i=1}^{s} c_i x_i^2, \qquad b = \sum_{i=1}^{s} c_i x_i,$$

where k is an integer, while each c_i is a given positive integer. Denote the sum of the c_i by t. Necessary conditions are

$$(2) \qquad a, b \text{ integers}, \qquad b \geqq t(1-k), \qquad a \equiv b \pmod 2.$$

Our theorems on solutions of (1) apply at once to the problem to express a given integer A in the form (3_1), where $f(x) = ux^2 + vx$. By (1) we see that

$$(3) \qquad A = \sum_{i=1}^{s} c_i f(x_i), \qquad A = ua + bv$$

are equivalent equations. Hence if the latter is solvable, also the former is.

In particular, when $f(x)$ is the x-th polygonal number, or extended, or generalized polygonal number, Miss Griffiths[†] treated (3) at length. Only in the third of her problems was the investigation incomplete owing to the lack of a generalized Cauchy lemma when $c_1 = c_2 = 1$, $c_3 = 2$, $c_4 = 3$ or 4. We here cover these cases, and in fact all cases in which $t = c_1 + c_2 + c_3 + c_4 \leqq 8$.

Theorem 1. *Let $(c_1, \cdots, c_4)$ be one of the eleven sets below. Let* (2) *hold and*

$$(4) \qquad ta \geqq b^2, \ (t-1)a < b^2 + 2bk + tk^2.$$

Then there exist integral solutions $> -k$ of (1) *with $s = 4$ if the following further conditions hold.*

$(1, 1, 1, 1)$, $4a - b^2 \neq 4^m(8n + 7)$.

$(1, 1, 1, 2)$, $4(5a - b^2) \neq 25^m(5n \pm 2)$; *true if either a or b is prime to* 5.

$(1, 1, 1, 3)$, $6a - b^2 \neq 9^m(9n + 6)$.

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[*] Hence $x_i \geqq -k + 1$. For solutions $\geqq 0$ take $k = 1$.

[†] *American Journal of Mathematics*, vol. 55 (1933), pp. 102-110; *Annals of Mathematics*, vol. 31 (1930), pp. 1-12. Cited as I, II.

$(1, 1, 1, 4)$, $7a - b^2 \neq 4^m(8n + 7)$.
$(1, 1, 1, 5)$, *b odd,* $3(8a - b^2) \neq 25^m(25n \pm 10)$.
$(1, 1, 2, 2)$, $6a - b^2 \neq 4^m(8n + 7)$; *true if b is odd.*
$(1, 1, 2, 3)$, $6(7a - b^2) \neq 49^m(7n + e)$, $e = 3, 5, 6$; *true if either a or b is prime to 7.*
$(1, 1, 2, 4)$, *b odd or double an odd.*
$(1, 1, 3, 3)$, *b odd or double an odd,* $8a - b^2 \neq 9^m(3n + 2)$.
$(1, 2, 2, 2)$, $7a - b^2 \neq 4^m(16n + 14)$.
$(1, 2, 2, 3)$, *b odd or double an odd.*

We have gone to considerable extra trouble to secure the same inequalities (4) in all cases in Theorems 1, 2. The advantage is that a single analysis of the inequalities applies to all types of applications.

Theorems 2-5 cover all cases with $t \leqq 7$.

THEOREM 2. *Let (2) and (4) hold. Then (1) with $s = 5$ have integral solutions $> -k$ for $(1, 1, 1, 1, j)$, $j = 1, 2, 3$, and $(1, 1, 1, 2, 2)$.*

While Theorems 1 and 2 are proved by use of fundamental identities and the theory of ternary quadratic forms, a simple combination of those theorems yields

THEOREM 3. *For the four sets of c_i in Theorem 2, equations (1) have integral solutions $\geqq 0$ if $b \geqq 0$, $a \equiv b \pmod 2$, and*

$$ta \geqq b^2, \quad (t - 2)(a - 2) < b^2.$$

Since the interval in which b^2 lies is here roughly double of that in (4), Theorem 3 is much more advantageous for problems like (3) than Theorem 1.

THEOREM 4. *Let S be a set of c_i derived from one of the four sets σ in Theorem 2 by annexing one or more 1's. As before, let t denote Σc_i for σ, but let T denote Σc_i for S. Restrict S so that $T \leqq 8$. Then there exist integral solutions $\geqq 0$ for S if $b \geqq 0$, $a \equiv b \pmod 2$, and*

$$(t - 2)(a - 2) < b^2,\ Ta \geqq b^2 \text{ if } T < 8,\ Ta > b^2 \text{ if } T = 8.$$

THEOREM 5. *For the four sets in Theorem 2, equations (1) have integral solutions $> -k$ if we assume (2) and*

$$ta \geqq b^2, \quad (t - 2)(a - 2) < b^2 + 4b(k - 1) + 2t(k - 1)^2.$$

Similarly for the sets in Theorem 4 if we replace $(t - 2)(a - 2) < b^2$ by our final inequality.

THEOREM[*] 6. *When* $T \geqq 8$, *Theorem* 4 *holds if we replace* $Ta > b^2$ *by* $Ta > b^2 + G$, *where* G *is the greatest integer* $\leqq T(T-8)/4$.

In § 23 there is a note on my paper in Vol. 55 of this journal.

2. *The fundamental identity.* Denote $c_1 + \cdots + c_s$ by t_1 (the former t). Define $t_2, t_3, \cdots$ by

$$t_k = t_{k-1} - c_{k-1}. \tag{5}$$

By (1), $t_1 a - b^2 = \phi_1$, where

$$\phi_k = \sum_{i=k}^{s} c_i(t_k - c_i)x_i^2 - 2 \sum_{i<j}^{k,\ldots,s} c_i c_j x_i x_j. \tag{6}$$

By completing the square on the terms in x_k, we get

$$(t_k - c_k)\phi_k \equiv c_k F_k^2 + t_k \phi_{k+1}, \tag{7}$$

$$F_k = (t_k - c_k)x_k - \sum_{j=k+1}^{s} c_j x_j. \tag{8}$$

We employ the abbreviation

$$\Pi(i,j) = (t_i - c_i)(t_{i+1} - c_{i+1}) \cdots (t_j - c_j), \qquad j \geqq i.$$

By repeated use of (7) we may eliminate $\phi_2, \phi_3, \cdots, \phi_n$ from (7) for $k = 1$ and obtain

$$\Pi(1,n)\phi_1 = \sum_{i=2}^{n} \Pi(i,n) t_1 t_2 \cdots t_{i-2} c_{i-1} F^2{}_{i-1} + t_1 t_2 \cdots t_{n-1}(c_n F_n^2 + t_n \phi_{n+1}).$$

This may also be proved by induction on n by multiplying it by $t_{n+1} - c_{n+1}$ and employing (7) with $k = n + 1$. We take $n = s - 2$ and apply

$$t_{s-1} = c_{s-1} + c_s, \qquad \phi_{s-1} = c_{s-1} c_s (x_{s-1} - x_s)^2.$$

We obtain our fundamental identity:

$$\begin{aligned} \Pi(1, s-2)(t_1 a - b^2) = \sum_{i=2}^{s-2} \Pi(i, s-2) t_1 t_2 \cdots t_{i-2} c_{i-1} F^2{}_{i-1} \\ + t_1 \cdots t_{s-3} c_{s-2} F^2{}_{s-2} + t_1 \cdots t_{s-2} c_{s-1} c_s (x_{s-1} - x_s)^2. \end{aligned} \tag{9}$$

By (8) and (5),

$$F_k = F_{k+1} + (t_k - c_k)(x_k - x_{k+1}). \tag{10}$$

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[*] Proof by Theorem 4 as in the writer's paper on Polygonal Numbers, etc., *Quarterly Journal of Mathematics*, 1934-35.

I. PROOF OF THEOREM 1.

3. *Case* $s=4$. Replace t_1 by $t=c_1+c_2+c_3+c_4$. Write

$$g=c_2+c_3+c_4 \quad \text{for} \quad t_2=t_1-c_1.$$

Write F for F_1, v for F_2. Then (9) and (8) give

$$(11) \qquad g(c_3+c_4)(ta-b^2)=(c_3+c_4)c_1F^2+tc_2v^2+tgc_3c_4W^2,$$

$$(12) \qquad F=gx_1-c_2x_2-c_3x_3-c_4x_4, \quad v=(c_3+c_4)x_2-c_3x_3-c_4x_4, \quad W=x_3-x_4.$$

By (10) with $k=1$, we have $F=v+gD$, $D=x_1-x_2$. After eliminating F from (11), we may cancel the factor g and get

$$(13) \qquad (c_3+c_4)(ta-b^2)=(c_1-c_2)v^2 + c_1(c_3+c_4)(2vD+gD^2)+c_3c_4tW^2.$$

We shall often use this identity transformed into one involving only squares.

4. *Case* $c_3+c_4=c_1+c_2$. Cancellation of the factor c_3+c_4 from (13) gives

$$(14) \qquad ta-b^2=V^2+2c_1c_2D^2+2c_3c_4W^2, \qquad V=v+c_1D.$$

Denote the second member of (14) by f. Solving $b=\Sigma c_ix_i$ with the linear equations having V, D, W as left members, we get

$$(15) \qquad tx_1=b+V+2c_2D, \quad tx_2=b+V-2c_1D, \quad tx_3=b-V+2c_4W, \quad tx_4=b-V-2c_3W.$$

The x_i are all integers if and only if

$$(16) \qquad b+V-2c_1D\equiv 0 \pmod{t}, \quad b-c_1D-c_3W\equiv 0 \pmod{t/2},$$

since the double of the second function is tx_2+tx_4.

The four x_i in (15) are all $>-k$ if

$$(17) \qquad -V-2c_2D, \quad -V+2c_1D, \quad V-2c_4W, \quad V+2c_3W$$

are all $<b+tk$. Now

$$(18) \qquad (y+2c_iz)^2+2c_i(y-z)^2\equiv(1+2c_i)(y^2+2c_iz^2).$$

Take $y=|V|$, $z=|W|$, $i=3$ or 4. By (18) and (14), the square of $|V|+2c_i|W|$ is $\leqq(1+2c_i)f$. Similarly, for $i=1$ or 2, the square of

$|V| + 2c_i|D|$ is $\leqq (1+2c_i)f$. Let c denote the largest c_i. Hence every integral x_i will be $> -k$ if $(1+2c)(ta-b^2) < (b+tk)^2$, viz.,

$$(19) \qquad (1+2c)ta < (2+2c)b^2 + 2btk + t^2k^2.$$

In all our examples (§§ 4-9) under our present case $c_3+c_4=c_1+c_2$, we shall take $c_3=1$. Then c_4 is the largest c_i, and $2(1+c)=t$. Thus (19) becomes (4_2).

By way of example we complete the discussion in Cauchy's case in which each $c_i=1$. By (14), $V\equiv b \pmod 2$.

First, let a and b be odd. Then $2\equiv 2D^2+2W^2 \pmod 8$, whence $D+W$ is odd. Permuting D and W if necessary, we may take D even. Since $b\pm V$ are even, one of them (say $b+V$ by choice of the sign of V) is a multiple of 4. Thus (16) hold.

Second, let a and b be even. Then $b=2B$, $V=2v$, $-4B^2\equiv 4v^2+2D^2+2W^2 \pmod 8$. Thus $W\equiv D \pmod 2$, whence $2W^2\equiv 2D^2 \pmod 8$, $B+v-D$ is even. Thus (16) hold.

Write $y=D+W$, $z=D-W$. Then $f=V^2+y^2+z^2$. Conversely, if $4a-b^2$, which is $\equiv 0$ or $3 \pmod 4$, is a sum of three squares, they are all even or all odd, whence $y\equiv z \pmod 2$, and f is equivalent to (14).

In case each $c_i\geqq 2$, we may improve on (19). Instead of (18) we now employ

$$(y+2c_iz)^2+(2c_i/mn)(my-nz)^2\equiv C_iQ_i,$$
$$C_i=1+2c_im/n, \qquad Q_i=y^2+(2c_in/m)z^2.$$

For $i=1$ or 2, take $y=|V|$, $z=|D|$. Then $Q_i\leqq f$ if $c_in\leqq c_1c_2m$. For $i=1$, the least C_1 is its value $1+2c_1/c_2$ for the maximum value c_2m of n. Similarly, or by symmetry, we see that the squares of the four functions (17) are all $\leqq Gf$, where G is the greatest of

$$(20) \qquad 1+2c_1/c_2, \qquad 1+2c_2/c_1, \qquad 1+2c_3/c_4, \qquad 1+2c_4/c_3.$$

If c_4 be the greatest c_i, evidently c_3 is the least. Hence if some c_i is 1, then $c_3=1$ and $G=1+2c_4$, and we have the results leading to (19). But if each $c_i>1$, we have an improvement on (19). For example, let $c_1=c_2=3$, $c_3=2$, $c_4=4$. Then $G=1+8/2=5<1+2c=9$.

5. (1, 2, 2, 3). Let $c_1=c_2=2$, $c_3=1$, $c_4=3$. Then $t=8$, $f=V^2+8D^2+6W^2$ shall represent $8a-b^2$.

First, let a and b be odd. Since f shall represent $8a-b^2\equiv -1$ (mod 8), V and W are odd. By choice of the sign of W, $b-W-2D\equiv 0 \pmod 4$, which is (16_2). Since $b \pm V$ are even, we may choose the sign of V so that $b + V = 4m$. But

$$6W^2 \equiv 6, \qquad 8 - b^2 \equiv V^2 + 8D^2 + 6 \pmod{16}.$$

Elimination of V gives $1 \equiv b^2 - 4mb + 4D^2 \pmod 8$. Hence $D \equiv m$ (mod 2), $b + V \equiv 4D$ (mod 8), which is (16_1).

Second, let a and b be even. Then $b = 2B$, $V = 2v$, $W = 2w$. Then $-B^2 \equiv v^2 + 2D^2 + 6w^2 \pmod 4$, $v \equiv B$, $B + D + w \equiv 0 \pmod 2$. Hence (16_2) holds. Here (16_1) is $B + v - 2D \equiv 0 \pmod 4$. In case v is odd this may be satisfied by choice of the sign of v. Hence if B is odd, the x_i are all integers. But if B is even the method fails in general.

The form * f represents every positive integer $2m + 1$, $3m$ or $4m$ which is of neither of the forms $8n + 3$, $4^m(8n + 5)$. If a and b are odd, $8a - b^2$ is $\equiv -1 \pmod 8$ and is represented by f. Next, let $a = 2A$, $b = 2B$, B odd. Then $8a - b^2 = 4N$, $N = 4A - B^2 \equiv 4A - 1 \not\equiv 5 \pmod 8$. Thus $4N$ is represented by f.

6. (1, 1, 3, 3). Let $c_1 = c_3 = 1$, $c_2 = c_4 = 3$. Then $t = 8$, $f = V^2 + 6D^2 + 6W^2$ shall represent $8a - b^2$.

First, let a and b be odd. Then V and $D + W$ are odd. By permuting D and W and changing their signs, we may take W odd, $D = 2d$, $D + W \equiv b$ (mod 4). By choice of the sign of V, $b + V = 4U$. Then

$$8 - b^2 \equiv (4U - b)^2 + 24d^2 + 6 \pmod{16},$$
$$0 \equiv 1 - b^2 \equiv 4U + 4d^2 \pmod 8,$$

whence $U \equiv d \pmod 2$. Hence (16) hold.

Second, let a be even, $b = 2B$. Then $V = 2v$. Taking (14) modulo 16, we get

$$-2B^2 \equiv 2v^2 + 3D^2 + 3W^2 \pmod 8. \tag{21}$$

Hence $D + W$ is even. First, let D and W be odd. Then (21) holds if and only if $B + v$ is odd. One of the even integers $B + v \pm D$ is a multiple of 4. Hence by choice of the sign of D, (16_1) holds. Similarly for $2B - D \pm W$ and (16_2).

Next, when $b = 2B$, let $D = 2d$, $W = 2w$. Then (21) becomes $-B^2 \equiv v^2 + 6d^2 + 6w^2 \pmod 4$, which holds if and only if $v \equiv B$, $d + w \equiv B \pmod 2$. The double of the latter gives (16_2). But (16_1) is equivalent to $B + v + 2d \equiv 0 \pmod 4$. If v is odd, this holds by choice of

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* B. W. Jones, *unpublished Chicago thesis*, 1928, p. 122.

the sign of v. This proves Theorem 1 unless B is even. In fact (Jones, p. 82), f represents exclusively every positive integer except $8n+3$, $9^m(3n+2)$.

7. $(1,1,2,2)$. Let $c_1=c_4=2$, $c_2=c_3=1$. Then $t=6$, $f=V^2+4D^2+4W^2$ shall represent $6a-b^2$ Hence $-b^2\equiv V^2+D^2+W^2$ (mod 3). If $b\equiv 0$ (mod 3), V, D, W are all prime to 3 or all divisible by 3. By choice of signs, $V\equiv D$, $W\equiv D$ (mod 3), whence (16) hold modulo 3. Next, let $b^2\equiv 1$ (mod 3). Then two of V, D, W are prime to 3 and the third is divisible by 3. If both D and W are prime to 3, V is divisible by 3 and, by choice of signs, $D\equiv b$, $W\equiv -b$ (mod 3). If one of D and W is divisible by 3, permute them and change signs; then $W\equiv 0$, $D\equiv -b$, $V\equiv b$ (mod 3). In all cases (16) hold modulo 3. Evidently $V\equiv b$ (mod 2). Hence the x_i are all integers.

But f represents * exclusively all positive integers except $4n+2$, $4n+3$, $4^m(8n+7)$. If a and b are odd, $6a-b^2$ is $\equiv 1$ (mod 4) and is represented by f. This proves Theorem 1.

Miss Griffiths † proved the theorem for $k=1$ with my condition $5a<b^2+2b+6$ replaced by the more restrictive one $15a\leqq 3b^2+b$ (for $b=3$, she has no a, while I get $a=3$).

8. $(1,1,2,4)$. Take $c_1=4$, $c_2=2$, $c_3=c_4=1$. Then $g=4$, $t=8$. Then (11) is the product of 8 by $f=F^2+2v^2+4W^2$ shall represent $8a-b^2$. Here $b=4x_1+2x_2+x_3+x_4$, $F=4x_1-2x_2-x_3-x_4$, $v=2x_2-x_3-x_4$, $W=x_3-x_4$. Solving, we get

$$8x_1=b+F,\qquad 8x_2=b-F+2v,\qquad 8x_3,\qquad 8x_4=b-F-2v\pm 4W.$$

The form f represents ‡ exclusively all positive integers except $4^m(16n+14)$.

First, let a and b be odd. Since $8a-b^2$ is $\equiv 7$ or -1 (mod 16), it is represented by f if positive. Here F is odd, $-1\equiv v^2+2W^2$ (mod 4), whence v and W are odd. Then $8-b^2\equiv F^2+2+4$ (mod 16). Write $F\equiv xb$ (mod 8). Then $x=2\xi+1$, $2\equiv b^2(4\xi^2+4\xi+2)$ (mod 16), $\xi^2+\xi\equiv 0$ (mod 4), $\xi\equiv 0$ or -1 (mod 4), $x\equiv\pm 1$ (mod 8). By choice of the sign of F, we have $F\equiv -b$ (mod 8). By choice of the sign of v, $v\equiv -b$ (mod 4). Hence each x_i is an integer.

Second, let a be even and $b=2B$. Since the method fails if B is even, let B be odd. Then $8a-b^2$ is $\equiv -4$ (mod 16) and is represented by f if positive. Also, $F=2h$, $v=2n$, $-1\equiv h^2+2n^2+W^2$ (mod 4). Hence n

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[*] Jones, p. 75. Proof due to Dickson.

[†] I. Her identity is half of (13) for $c_1=c_2=2$, $c_3=c_4=1$, with $\xi=v+D$.

[‡] Dickson, *Bulletin of the American Mathematical Society* (1927), p. 67.

and $h + W$ are odd. Now f is symmetric in h and W. Permuting them if necessary, we may take h odd, W even. By choice of the sign of F we have $h \equiv -B \pmod 4$. The x_i are all integers.

The x_i are all $> -k$ if $|F| + 2|v| + 4|W| < b + 8k$. This linear function is $\leqq \sqrt{7f}$ by the following lemma with with $d_1 = 1$, $d_2 = 2$, $d_3 = 4$, $d_4 = 0$.

LEMMA. *If* $L = d_1y_1 + \cdots + d_4y_4 \geqq 0$ *and each* $d_i \geqq 0$, *then*

$$L \leqq \{(d_1 + d_2 + d_3 + d_4)(d_1y_1^2 + \cdots + d_4y_4^2)\}^{1/2}.$$

This follows from the identity

$$L^2 + \sum_{i<j}^{1,\ldots,4} d_id_j(y_i - y_j)^2 \equiv (d_1 + \cdots + d_4)(d_1y_1^2 + \cdots + d_4y_4^2).$$

9. (1, 1, 1, 2). If $c_2 = 2$, (11) is the double of

$$4(5a - b^2) = F^2 + 5v^2 + 10W^2.$$

This and the formulas for F, v, W are the relations used by Pall.* If we use $x_i > -1$ instead of his $x_i \geqq 0$, we get $4a < b^2 + 2b + 5$, which yields more pairs a, b than his $4a \leqq b^2$.

In all our further problems with $s = 4$ we employ identity (13), each with its own device.

10. (1, 1, 1, 3). Let $c_2 = 3$. Here (13) is the double of

$$6a - b^2 = 2v^2 + 2vD + 5D^2 + 3W^2.$$

Write $x = v - D$, $y = v + 2D$. Hence $f = x^2 + y^2 + 3W^2$ shall represent $6a - b^2$,

$$x = -x_1 + 3x_2 - x_3 - x_4, \quad y = 2x_1 - x_3 - x_4, \quad W = x_3 - x_4.$$

Solving these with $b = x_1 + 3x_2 + x_3 + x_4$, we get

$$6x_1 = b - x + 2y, \quad 6x_2 = b + x, \quad 6x_3, \quad 6x_4 = b - x - y \pm 3W.$$

Each $x_i > -k$ if $x - 2y$, $-x$, $x - y \mp 3W$ are $< b + 6k$. By the lemma, $|x| + |y| + 3|W| \leqq \sqrt{5f}$. But for the first function we use

$$(m + 2n)^2 + (2m - n)^2 \equiv 5(m^2 + n^2), \quad \{|x| + 2|y|\}^2 \leqq 5f.$$

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* Griffiths, II, for $k = 1$.

Evidently $5f < (b + 6k)^2$ and $t = 6$ give (4_2).

The form f represents* exclusively all positive integers except $9^m(9n + 6)$.

Since $6a - b^2 \equiv 1$ or $0 \pmod 4$, we may permute x, y if necessary and take $x \equiv b$, $y \equiv W \pmod 2$. Next, we may take $x \equiv -b$, $y \equiv -b \pmod 3$. Hence the x_i are all integers.

11. (1, 1, 2, 3). Let $c_1 = c_3 = 1$, $c_2 = 3$, $c_4 = 2$. The double of (13) is

$$N = 6(7a - b^2) = \xi^2 + 7v^2 + 28W^2 = f, \quad \xi = 6D + v,$$
$$v = 3x_2 - x_3 - 2x_4, \quad W = x_3 - x_4, \quad D = x_1 - x_2.$$

$F = \xi^2 + 7v^2 + 7z^2$ represents (Jones, p. 123) every positive integer $\equiv 0$ or $3 \pmod 4$ except $49^m(7n + e)$, $e = 3$, 5 or 6. Since $a \equiv b \pmod 2$, $F = N \equiv 0 \pmod 4$ implies $z = 2W$. Then F becomes f.

Now $N \equiv b^2 \not\equiv e \pmod 7$. If a and b are not both divisible by 7, N is not divisible by 49. Hence N is represented by F and hence by f.

Our linear equations and $b = x_1 + 3x_2 + x_3 + 2x_4$ have the solution $7x_1 = b + v + 6D$, $7x_2 = b + v - D$, $21x_3 = M + 14W$, $21x_4 = M - 7W$, where $M = 3b - 3D - 4v$. We may choose the sign of ξ so that $\xi \equiv -b \pmod 7$. By choice of the signs of v and w, we have $v \equiv -W$, $\xi \equiv v \pmod 3$. Evidently $\xi \equiv v \pmod 2$. Hence $\xi = 6D + v$ yields an integer D. Write μ for $b - D + v$. Then $\mu \equiv b + \xi \equiv 0$, $M \equiv 3\mu \pmod 7$. Hence the x_i are all integers.

From the values of the x_i we eliminate D in terms of ξ. The x_i are all $> -k$ if $-\xi < b + 7k$, and if $\xi - 7v$, $\xi + 7v - 28W$, $\xi + 7v + 14W$ are all $< 6(b + 7k)$. By the Lemma, the square of $|\xi| + 7|v| + 28|W|$ is $\leqq 36f$. The condition is (4_2).

12. (1, 1, 1, 5). For $c_4 = 5$, (13) is the double of

$$N = 3(8a - b^2) = x^2 + 12D^2 + 20W^2 \equiv G,$$

where $x = v + 3D$. Write $y = 2D$. We get $f = x^2 + 3y^2 + 20W^2$, which represents (Jones, p. 136) every positive multiple of 3 which is of neither of the types $4n + 2$, $25^m(25n \pm 10)$. Here

$$8x_1, \; 8x_2 = b + x \pm 2y, \quad 24x_3 = 3b - x + 20W, \quad 24x_4 = 3b - x - 4W.$$

Take† a and b odd. Then $f = N \equiv -3 \pmod 8$ gives x odd, $y = 2D$.

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* Dickson, *Bulletin of the American Mathematical Society* (1927), p. 64.

† If a and b are even, we cannot conclude that y is even. But we do not know what G represents.

By $N \equiv G \pmod 8$, $D + W$ is odd. By choice of the sign of x, $b + x = 4m$. By choice of the sign of W, $x + W \equiv 0 \pmod 3$. We find that $N \equiv G \pmod{16}$ implies that $m \equiv D \pmod 2$. Hence the x_i are all integers. Each is $> -k$ if $-3x \pm 3D$, $x - 20W$, $x + 4W$ are $< 3(b + 8k)$. By the lemma, these hold if $21G = 21N < 9(b + 8k)^2$, viz., (4_2).

13. $(1, 1, 1, 4)$. For $c_2 = 4$, (13) is

$$2\phi = 5v^2 + 4vD + 12D^2 + 7W^2, \qquad \phi = 7a - b^2.$$

Then

$$6\phi = f = x^2 + 14v^2 + 21W^2, \qquad x = 6D + v.$$

By choice of the sign of x, $x \equiv -b \pmod 7$. Also, $v \equiv x \equiv W \pmod 2$. Hence in

$$7x_1 = b + v + 6D, \quad 7x_2 = b + v - D, \quad 14x_3, \quad 14x_4 = 2b - 2D - 5v \pm 7W,$$

each x_i is an integer. Also, each $x_i > -k$, if $-x < b + 7k$, $x - 7v$ and $x + 14v \pm 21W$ are $< 6(b + 7k)$. These conditions hold by the lemma if $6\phi < (b + 7k)^2$, viz., (4_2).

In 2ϕ, write [*] $v = X + V$, $W = X - V$, $D = -Z$. Then

$$\phi = 6X^2 + 6V^2 + 6Z^2 - 2XY - 2XZ - 2VZ$$

is reduced and of Hessian 196. All the remaining improperly primitive reduced forms of Hessian 196 represent 2, 4 or 8. Since ϕ is the only form of its genus, it is regular and hence represents exclusively the positive integers not of the forms $7n + 1$, $7n + 2$, $7n + 4$, $4^m(8n + 7)$, $2n + 1$. It represents none of the latter since $f \neq 7n + 3$, $7n + 5$, $7n + 6$, $4^m(16n + 10)$. Hence ϕ represents every $7a - b^2$ except $4^m(8n + 7)$, $2n + 1$.

14. $(1, 2, 2, 2)$. Let $c_4 = 1$. Then (13) is

$$\begin{aligned} 3\phi &= 4v^2 + 12vD + 30D^2 + 14W^2 \\ &= x^2 + 21D^2 + 14W^2, \qquad x = 2v + 3D. \end{aligned}$$

Here

$$7x_1 = b + v + 5D, \quad 7x_2 = b + v - 2D, \quad 21x_3 = R + 7W, \quad 21x_4 = R - 14W,$$

where $R = 3b - 6D - 4v$. Since $\phi = 7a - b^2$, $x \equiv \pm 2b$, $x \equiv -2b \pmod 7$ by choice of the sign of x. Similarly, $v = W \pmod 3$. Hence the x_i are all integers. Next, each $x_i > -k$ if

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[*] Proof obtained by B. W. Jones, by means of unpublished tables and his paper, *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 111-124.

$-x \pm 7D < 2(b+7k)$, $x+7W$ and $x-7W/2$ are $< 3(b+7k)/2$. By the lemma, $(r+7s)^2 \leqq 8(r^2+7s^2)$. Thus $|x| + 7|D| \leqq \sqrt{8f}$, which is $< 2(b+7k)$ if (4_2) holds. By the former inequality

$$|x| + 7|W| < \sqrt{9f} < 3(b+7k)^2/2.$$

The following is due to Jones. We may write $v = 3z - x$, $D = y - z$, $W = -x$. Then $\phi = 2f$,

$$f = 3x^2 + 5y^2 + 5z^2 - 4yz - 2xz - 2xy.$$

This f is reduced and every further reduced form of the same Hessian 49 represents 1 or 2. As in § 13, f is regular and represents exclusively the positive integers $\neq 7n+1$, $7n+2$, $7n+4$, $4^m(8n+7)$. Hence ϕ represents all $7a - b^2 \neq 4^m(16n+14)$.

II. PROOF OF THEOREM 2.

15. *Case* $s = 5$. In the fundamental identity (9), we write P, Q, R for F_1, F_2, F_3 and obtain

$$\begin{aligned}(t_1-c_1)(t_2-c_2)(t_3-c_3)(t_1a-b^2)&\\ \equiv (t_2-c_2)(t_3-c_3)c_1P^2 &+ (t_3-c_3)t_1c_2Q^2\\ &+ t_1t_2c_3R^2 + t_1t_2t_3c_4c_5W^2,\end{aligned} \tag{22}$$

where $W = x_4 - x_5$. Solving the linear equation with the left members b, P, Q, R, W, we get

$$\begin{aligned}&t_1x_1 = b + P, \quad t_1t_2x_2 = t_2b - c_1P + t_1Q,\\ &t_1t_2t_3x_3 = t_2t_3b - c_1t_3P - c_2t_1Q + t_1t_2R,\\ &(c_4+c_5)t_1t_2t_3x_4 = G + c_5t_1t_2t_3W,\\ &(c_4+c_5)t_1t_2t_3x_5 = G - c_4t_1t_2t_3W,\end{aligned} \tag{23}$$

where

$$G = (t_3 - c_3)\{t_2t_3b - c_1t_3P - c_2t_1Q\} - c_3t_1t_2R.$$

16. $(1, 1, 1, 1, 2)$. Let $c_1 = 2$. Cancel 12 from (22). Hence

$$2(6a - b^2) = P^2 + Q^2 + 2R^2 + 6W^2,$$

$$\begin{aligned}&6x_1 = b + P, &&12x_2 = 2b - P + 3Q, &&12x_3 = 2b - P - Q + 4R,\\ &12x_4 = H + 6W, &&12x_5 = H - 6W, &&H = 2b - P - Q - 2R.\end{aligned}$$

Since we cannot prove that one of P, Q is $\equiv b \pmod 2$, we eliminate $P = Q + 4D$, where $D = x_1 - x_2$. Hence *

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* Also by starting with $c_2 = 2$, choosing the sign of Q so that $P = Q + 5y$, dividing by 5 and writing $x = Q + y$.

$$f = x^2 + 4D^2 + R^2 + 3W^2 \quad \text{shall represent} \quad 6a - b^2,$$

where $x = Q + 2D$. Thus $Q = x - 2D$, $P = x + 2D$, and

$$6x_1 = b + x + 2D, \quad 6x_2 = b + x - 4D, \quad 6x_3 = b - x + 2R,$$
$$6x_4, \quad 6x_5 = b - x - R \pm 3W.$$

We assume $a \equiv b \pmod 2$. If x and R are both $\equiv b + 1 \pmod 2$, then $6a - b^2 \equiv 6b - b^2 \equiv f \pmod 4$ becomes $1 + (1 + b)^2 \equiv 3W^2$, which is impossible. Permuting x and R if necessary, we may take $x \equiv b \pmod 2$. Then $R \equiv W \pmod 2$.

First, let $b \equiv 0 \pmod 3$. Then x, D, R are all prime to 3 or all divisible by 3. By choice of the sign of D we have $x - D \equiv -b \pmod 3$.

Second, let $b^2 \equiv 1 \pmod 3$. Then exactly two of x, D, R are prime to 3. By choice of the signs of x and D, we have $x - D \equiv -b \pmod 3$.

In both cases, $R^2 \equiv (D + b)^2$. By choice of the sign of R, we take $R \equiv -b - D \pmod 3$. Hence the x_i are all integers.

Each $x_i > -k$ if $-x - 2D$, $-x + 4D$, $x - 2R$, $x + R \mp 3W$ are all $< b + 6k$. By the lemma $|x| + |R| + 3|W|$ and $|x| + 4|D|$ are $\leqq \sqrt{5f}$. The same is true of $|x| - 2|R|$ since

$$(X + 2F)^2 + (2X - F)^2 \equiv 5(X^2 + Y^2).$$

But $5f < (b + 6k)^2$ with $t = 6$ give (4_2).

Since f represents all positive integers, we have Theorem 2.

17. $(1, 1, 1, 1, 3)$. Let $c_2 = 3$. Cancel 6 from (22). Hence

$$f = P^2 + 7Q^2 + 7R^2 + 21W^2 \quad \text{shall represent} \quad M = 6(7a - b^2),$$
$$7x_1 = b + P, \quad 42x_2 = 6b - P + 7Q, \quad 42x_3 = 6b - P - 7Q + 14R,$$
$$42x_4, \quad 42x_5 = H \pm 21W, \quad H = 6b - P - 7Q - 7R.$$

We have $0 \equiv f \pmod 4$. Since $Q \equiv R \equiv P + 1 \pmod 2$ is impossible, we permute Q and R if necessary and have $Q \equiv P$, whence $R \equiv W \pmod 2$.

Since $0 \equiv P^2 + Q^2 + R^2 \pmod 3$, P, Q, R are all prime to 3 or all divisible by 3. By choice of the signs of Q and R, we get $Q \equiv P, R \equiv P \pmod 3$. By choice of the sign of P, $P \equiv -b \pmod 7$.

Hence the x_i are all integers. They are $> -k$ if $-P < b + 7k$, while $P - 7Q$, $P + 7Q - 14R$ and $P + 7Q + 7R \mp 21W$ are $< 6(b + 7k)$. The first and third are $\leqq \sqrt{36f}$ by the lemma. The same is true of the second since

$$(x + 7y + 14z)^2 + 7(x - y)^2 + 7(2x - z)^2 + (14y - 7z)^2$$
$$\equiv 36(x^2 + 7y^2 + 7z^2).$$

But $M < (b + 7k)^2$ yields (4_2) since $t = 7$.

When $a \equiv b \pmod 2$ and M is positive it will be proved to be represented by f. Take $P \equiv b \pmod 7$, P prime to 6. Then $N = M - P^2 \equiv 0 \pmod 7$, $N \equiv -1$ (mod 4 and 3). Hence $N = 12j - 1$, $j = 3 + 7l$ and $N/7 = 5 + 12l$, if positive, is represented by $Q^2 + R^2 + 3W^2$, since the latter represents every positive integer except $9^m(9n+6)$, and $5 + 12l$ is prime to 3. It remains to treat $N < 0$. Since P was determined modulo 42, we may take $|P| \leqq 21$.

It remains to treat $0 < M < 21^2$. We take $W = 0$. Since M is a multiple of 4, it is represented by the ternary $(1, 7, 7)$ unless $M = 49^m(7n + e)$, $e = 3$, 5 or 6 (Jones, p. 136). But if M is prime to 7, $M \equiv b^2 \not\equiv e \pmod 7$ and M is represented. Likewise if M (and hence b) is divisible by 7 and a is not, since M is then not divisible by 49. But if a and b are multiples of 7, M is a multiple of $6 \cdot 7^2 \cdot 2 > 21^2$.

18. $(1, 1, 1, 2, 2)$. Let $c_2 = c_5 = 2$. Cancel 6 from (22). Thus

$$f = 2P^2 + 7Q^2 + 7R^2 + 56W^2 \quad \text{shall represent} \quad 12(7a - b^2),$$

$$\begin{aligned} 7x_1 &= b + P, & 42x_2 &= 6b - P + 7Q, & 84x_3 &= 12b - 2P - 7Q + 21R, \\ 84x_4 &= H + 56W, & 84x_5 &= H - 28W, & H &= 12b - 2P - 7Q - 7R. \end{aligned}$$

Since $P^2 \equiv b^2$, we may change the sign of P and get $P \equiv -b \pmod 7$.

Since $0 \equiv 2P^2 - Q^2 - R^2 \pmod 8$, P, Q, R are all even or all odd. If they are odd, one of the even $R \pm Q$ is $\equiv 2 \pmod 4$. By choice of the sign, $R + Q \equiv 2P \pmod 4$. The same follows if P, Q, R are even.

Since $P^2 + W^2 \equiv Q^2 + R^2 \pmod 3$ and since f is symmetric in Q, R we may take $Q^2 \equiv P^2$ unless $Q^2 \not\equiv P^2$, $R^2 \not\equiv P^2$. But $u^2 \not\equiv v^2$ implies that one square is $\equiv 0$ and the other $\equiv 1$, whence $uv \equiv 0$. If $P \not\equiv 0$, then $Q \equiv R \equiv 0$, $1 + W^2 \equiv 0$, which is impossible. Hence $P \equiv 0$, $Q^2 \equiv R^2 \equiv 1$, whence $W^2 \equiv 2$, which is impossible. Thus $Q^2 \equiv P^2$, $R^2 \equiv W^2$. Choice of signs gives $Q \equiv P$, $R \equiv -W$. Whence $H \equiv -R \equiv W$.

Hence the x_i are all integers. They are $> -k$ if $-P < b + 7k$ and $2P - 14Q$, $2P + 7Q - 21R$, $2P + 7Q + 7R + hW$ are $< 12(b + 7k)$ for $h = 28$ and -56. The square of the last function is $\leqq 72f$ by the lemma. For the second function,

$$(2x + 7y + 21z)^2 + (21y - 7z)^2 + (\alpha x - \beta z)^2 + (jx - ny)^2 \\ = 72(2x^2 + 7y^2 + 7z^2)$$

if $j^2 = n^2 = \beta^2 = 14$, $\alpha^2 = 126$, whence $\alpha\beta = 2 \cdot 21$, $jn = 2 \cdot 7$. Finally, for the first function, $(x + 7y)^2 + 7(x - y)^2 = 8(x^2 + 7y^2)$, whence the square of the double of $|P| + 7|Q|$ is $\leqq 32(2P^2 + 7Q^2) \leqq 32f$.

We saw that y and z are integers in $Q + R = 2y$, $Q - R = 2z$. Write $2W = w$. Hence

$$g = P^2 + 7y^2 + 7z^2 + 7w^2 \quad \text{shall represent} \quad M = 6(7a - b^2).$$

Conversely this implies that y, z, w are not all odd, whence, after permuting them, w is even. For, if all odd, $0 \equiv P^2 + 3 \cdot 7$, $P^2 \equiv 3 \pmod 4$, which is impossible.

To prove that every positive M is represented by g, take $P \equiv b \pmod 7$, P odd. Then $N = M - P^2 \equiv 0 \pmod 7$ and $N \equiv -1 \pmod 4$. Thus $N = 7(1 + 4m)$. But if $m \geqq 0$, $1 + 4m$ is a sum of three squares, whence M is represented by g. Since P was determined modulo 14, we may take $|P| \leqq 7$. It remains to treat $0 \leqq M < 49$. Take $w = 0$. The ternary $(1, 7, 7)$ represents M unless $M = 49^m(7n + e)$, $e = 3, 5, 6$. Here $m = 0$, and $M \equiv b^2 \not\equiv e \pmod 7$.

19. $(1, 1, 1, 1, 1)$. By (10), $Q = R + 3u$, $u = x_2 - x_3$. Eliminate Q from (22) and write $x = R + u$, $u + W = y$, $u - W = z$. We get $4(5a - b^2) = P^2 + 5x^2 + 5y^2 + 5z^2$. Proceed as in the preceding cases.*

III. PROOF OF THEOREM 3.

20. Our theorem follows from Theorem 2 unless

$$(24) \qquad (t-1)a \geqq b^2 + 2b + t, \qquad (t-2)(a-2) < b^2.$$

In part of the proof we shall assume that $c_1 = 1$ and take $x_1 = 1$, and examine the conditions that $\alpha = a - 1$, $\beta = b - 1$ shall satisfy the conditions of Theorem 1 with a, b, t replaced by α, β, $t - 1$, respectively, which we cite as Theorem 1′. Evidently $\alpha \equiv \beta \pmod 2$. Also $\beta \geqq 0$ unless $b = 0$, whence $a < 2$. But $a = 1 \not\equiv b \pmod 2$. For $a = b = 0$, Theorem 3 is true. Next, $(t-1)\alpha \geqq \beta^2$ if $(t-1)a \geqq b^2 - 2b + t$, which follows from (24_1). Finally, $(t-2)\alpha < \beta^2 + 2\beta + t - 1$ reduces to (24_2). Note that the latter implies

$$(25) \qquad (t-2)a < b^2 + 2b + t - 1$$

if $b^2 + 2(t-2)$ is $\leqq$ the second member and hence if $t - 3 \leqq 2b$. The latter is true for our cases $t = 5, 6, 7$ unless $b = 0$ or 1. Then (24) both hold only when $b = 1$, $a = 2$, while $a \equiv b \pmod 2$.

$(1, 1, 1, 2, 2)$. We need only the case β odd of Theorem 1′. If a and b are odd, take $x_1 = 0$. In view of (24_1) and (25), our theorem follows from Theorem 1′. Next, if a and b are even, we take $x_1 = 1$. Then $\beta = b - 1$ is odd. The proof above shows that our theorem follows from Theorem 1′.

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* Somewhat different from the earlier proof sketched by G. Pall, *Quarterly Journal of Mathematics*, vol. 2 (1931), pp. 136-141. When a and b are even, he proved that for $(1, 1, 1, 1)$ and $k = 1$, we may replace (4_2) by the weaker condition

$$8a < 3b^2 + 8b + 16.$$

$(1, 1, 1, 1, 2)$. We need Theorem 1′ only when either α or β is prime to 5. If a or b is prime to 5, take $x_1 = 0$. If a and b are both divisible by 5, take $x_1 = 1$, whence β is prime to 5.

$(1, 1, 1, 1, 3)$. We need Theorem 1′ only when β is prime to 3, whence a positive $6\alpha - \beta^2$ is represented by $(1, 1, 1, 3)$. According as b is prime to 3 or divisible by 3, take $x_1 = 0$ or $x_1 = 1$.

IV. Proof of Theorem 4.

21. First, let $S = (1, \sigma)$, whence $T = t + 1$, $5 \leqq t \leqq 7$. Hence by Theorem 3 there exist solutions $\geqq 0$ with $x_1 = 0$ if $ta \geqq b^2$. There remains the case

$$ta < b^2 \leqq (t+1)a \qquad (= \text{deleted if } t = 7). \tag{26}$$

Since $a \geqq 1$, $b^2 > 5$, $b^2 \geqq 3^2$. Hence $a \neq 1$, $b^2 \geqq 2t$, $b \geqq 4$. We may write

$$b = (t+1)x + r \begin{cases} r = 0, \pm 1, \pm 2, -3 & \text{if } t = 5; \\ r = 0, \pm 1, \pm 2, \pm 3 & \text{if } t = 6; \\ r = 0, \pm 1, \pm 2, \pm 3, -4 & \text{if } t = 7; \end{cases} \tag{27}$$

since the values of r form a complete set of residues modulo $t + 1$.

Since $b \geqq 4$, $x > 0$, $x \geqq 1$. Take $x_1 = x$. Write

$$\alpha = a - x^2, \qquad \beta = b - x = tx + r.$$

Hence $\beta > 0$. In $b^2 \leqq (t+1)a$ replace b by its value (27). Thus

$$a \geqq (t+1)x^2 + 2xr + I,$$

where I is the least integer $> r^2/(t+1)$ if $t = 7$, but the least one $\geqq$ that fraction of $t = 5$ or 6. Then $I > r^2/t$ or $I \geqq r^2/t$ in the respective cases. Hence

$$ta \geqq t(t+1)x^2 + 2txr + r^2,$$

with $=$ deleted if $t = 7$. In the latter case, $t\alpha > \beta^2$; while $t\alpha \geqq \beta^2$ if $t = 5$ or 6. From $ta < b^2$, we have

$$t(t-2)(a-2) < (t-2)\{(t+1)^2x^2 + 2(t+1)xr + r^2 - 2t\},$$

which is term by term $< t\{(t^2 + t - 2)x^2 + 2txr + r^2\}$, whence

$$(t-2)(\alpha - 2) < \beta^2.$$

Since we have verified the conditions for Theorem 3 with a, b replaced by α, β, we have proved Theorem 4 when $S = (1, \sigma)$.

Taking σ to be the resulting S, we see that our proof establishes Theorem 4 for $(1, S)$; etc.

V. PROOF OF THEOREM 5.

22. The relations $X_i = x_i + k - 1$ establish a $(1,1)$ correspondence between A, B and solutions $X_i > -1$ of

$$A = \Sigma c_i X_i^2, \qquad B = \Sigma c_i X_i$$

and a, b, and solutions $x_i > -k$ of (1), where

$$a = A - 2(k-1)B + t(k-1)^2, \qquad b = B - (k-1)t.$$

We see that $tA \geqq B^2$ implies $ta \geqq b^2$ and conversely. Also, $A \equiv B \pmod 2$ implies $a \equiv b$. Moreover, the case $k = 1$ of (4_2) in capital letters implies (4_2). Also, $B \geqq 0$ implies the inequality in (2). Hence Theorem 1 follows from the case $k = 1$ of it.

What we really desire is the new fact that $(t-2)(A-2) < B^2$ implies the final inequality in Theorem 5. This completes the proof of Theorem 5.

VI. NOTE ON A PREVIOUS PAPER.

23. Let $3^n = 2^n q(n) + r(n)$, $0 < r < 2^n$. The statement* that $2^n > q + r + 3$ was proved only in case $3r(n-1) > 2^n$. But it was used in the paper only when $n < 36$ and is then seen to be true by the values listed. For $n = 36$, $q/2^{46} < .000032$. But $q(n)/2^n$ decreases when n increases. Hence the statement holds for $n \geqq 36$ if $d = r/2^n < .99996$. For $36 \leqq n \leqq 180$, we have $d > .98$ only when

$$n = 105, \quad d = .98559, \qquad n = 140, \quad d = .980417,$$
$$n = 157, \quad d = .992746, \qquad n = 163, \quad d = .995501.$$

Hence the statement is true for $n \leqq 180$. For $n = 181$, $q/2^n$ is a decimal whose first 22 digits are zero. Hence the statement fails after $n = 180$ only in case there should occur a d all of whose first 22 digits are 9.

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* *American Journal of Mathematics*, vol. 55 (1933), p. 593.

PORISTIC DOUBLE BINARY FORMS.

By JOSEPHINE H. CHANLER.

Introduction. It is our purpose to exhibit poristic double binary forms of order (2, 4) up to and including those admitting configurations $\Delta^{2,4}_{8,16}$. A geometric representation of possible configurations is shown (§ **1**). Where possible, we study the involution curve (presently defined); in particular we do so for the Lüroth quartic (§ **2**) and for quartic curves admitting checkerboard configurations (§§ **4-7**). In the latter case an extension is made to the Jonquières curve. Factorable forms and those admitting imprimitive grouping are not discussed (1, § 4). We employ the methods and theorems developed by Coble in his studies of multiple binary forms (1, 2).

If a value t_1 is substituted in a double binary form $F_{k,\kappa} \equiv (\alpha t)^k (a\tau)^\kappa = 0$, there are determined κ values of τ, each of which determines $k-1$ values of t in addition to t_1, etc. If we can obtain a closed set of n t's and ν τ's such that each t determines κ of the τ's and each τ k of the t's, the form admits a configuration $\Delta^{k,\kappa}_{n,\nu}$. A poristic form admits ∞^1 such configurations.

In the form $F_{k,\kappa} = 0$ $(k \leqq \kappa)$, let t be the parameter of an $S_{k-1}\xi$ of the rational norm curve N^k in S_k, and let τ be the parameter of a point x on the rational curve R^κ of order κ in S_k. $F_{k,\kappa} = 0$ is then the incidence condition of the dual elements of the two curves. If the form is poristic, the values of t are arranged in sets of n belonging to an involution $I_1{}^n$; the $\binom{n}{k}$ sets of k values of t in each set of n determine $\binom{n}{k}$ points of an *involution curve* of order $\binom{n-1}{k-1}$. Of these points ν lie on the rational constituent R^κ of the involution curve (1, § 5).

Important forms connected with $F_{k,\kappa} = 0$ are the complementary form $F_{n-k,\nu-\kappa} = 0$ by which $t(\tau)$ determines the $\nu - \kappa$ τ's ($n - k$ t's) not related to it under $F_{k,\kappa} = 0$, and those forms giving rise to the covariant porisms (1, § 3).

1. Determination of configurations. We must arrange the n t's into ν couples so that each t occurs in κ couples. The whole must admit a group of permutations transitive on the t's (2, p. 360). It is convenient to think of the n t's as the vertices of a regular polygon; the diagonals joining ν couples of the vertices denote the ν τ's of the configuration.

There are no configurations of the types $\Delta^{2,4}_{3,6}$ $\Delta^{2,4}_{4,8}$, since in neither case can ν distinct couples be formed from the n t's. The one possible configuration

$\Delta^{2,4}_{5,10}$ is represented by the vertices and diagonals of a pentagon **(2)**. There is also only one configuration of type $\Delta^{2,4}_{6,12}$ **(3)**. For $\Delta^{2,4}_{7,14}$ there are two arrangements. For the one case,

$$\begin{aligned}\tau\colon\ & (12)(13)(14)(15)(23)(26)(27)\\ & (34)(35)(46)(47)(56)(57)(67)\end{aligned}$$

the vertices divide into two primitive sets 1, 3, 6, 7 and 2, 4, 5. For the other,

$$\begin{aligned}\tau\colon\ & (12)(23)(34)(45)(56)(67)(71)\\ & (14)(47)(73)(36)(62)(25)(51)\end{aligned}$$

we obtain by rational processes from a study of the covariant porisms the forms G_1, G_2, G_3, where a vertex t is related by G_1 to the two adjacent vertices, by G_2 to the two most distant, and by G_3 to those once removed. The forms are derived by the method employed by Coble in (2). Each of these forms correlates t's which produce a common τ in a Poncelet $F_{2,2}$ form. Hence if the points $1, 2, 3, \cdots 7$ are on a conic, the lines $(12), (23), \cdots (71)$ circumscribe one conic, and the lines $(14), (47), (73), \cdots (51)$ another. Since these 14 lines circumscribe our involution (line) quartic, the latter degenerates, and the original form is factorable.

For $\Delta^{2,4}_{8,16}$ we confine ourselves to the checkerboard type discussed in **(4-7)**.

2. The Lüroth quartic and its $F_{2,4}$ (1, pp. 9-10). We first obtain the equation.

By setting up an involution $I_1{}^5 \equiv (\alpha t)^5 + \lambda(\beta t)^5 = 0$ among the tangents t of a norm conic, we generate the Lüroth quartic curve

$$(1) \qquad Q \equiv \{(\alpha t)^5(\beta t')^5 - (\alpha t')^5(\beta t)^5\}/(t - t') = 0,$$

with coördinates

$$(2) \qquad tt' = x_0, \quad t + t' = 2x_1, \quad 1 = x_2.$$

If Q is rational, the incidence condition of line of C and point of Q is an $F_{2,4}$ form of the type desired. To provide for the nodes of Q, $I_1{}^5$ must contain three quintics each consisting of a squared quadratic and a linear factor. Thus we have an identity:

$$(3) \qquad \begin{aligned}(l_1t + l_2)(a_0t^2 + 2a_1t + a_2)^2 + (m_1t + m_2)(b_0t^2 + 2b_1t + b_2)^2&\\ + (n_1t + n_2)(c_0t^2 + 2c_1t + c_2)^2 \equiv 0&\end{aligned}$$

among three such members of $I_1{}^5$. The condition that linear forms l, m, n may exist is:

$$(4)\quad \begin{vmatrix} a_0^2 & 0 & b_0^2 & 0 & c_0^2 & 0 \\ 4a_0a_1 & a_0^2 & 4b_0b_1 & b_0^2 & 4c_0c_1 & c_0^2 \\ 4a_1^2+2a_0a_2 & 4a_0a_1 & 4b_1^2+2b_0b_2 & 4b_0b_1 & 4c_1^2+2c_0c_2 & 4c_0c_1 \\ 4a_1a_2 & 4a_1^2+2a_0a_2 & 4b_1b_2 & 4b_1^2+2b_0b_2 & 4c_1c_2 & 4c_1^2+2c_0c_2 \\ a_2^2 & 4a_1a_2 & b_2^2 & 4b_1b_2 & c_2^2 & 4c_1c_2 \\ 0 & a_2^2 & 0 & b_2^2 & 0 & c_2^2 \end{vmatrix} = 0.$$

This is the condition that the squared quadratics be apolar to the polar quartics of a binary quintic. Taking these polar quartics as the fundamental involution of a ternay triple-point quartic, we have the theorem:

(5) *Any 3 of the 4 sections of a triple-point quartic curve $\bar{Q}$ by its 4 double tangents will serve as squared quadratics for the Lüroth form desired.*

Now let the parameters t of points on $\bar{Q}$ be attached to the tangents of the norm conic C (^{5}, § 27). To the line sections of $\bar{Q}$ correspond polar foursides (circumscribed about C) of a pencil of point conics apolar to C. This pencil determines a quadratic transformation (^{5}, § 27) the Darboux coördinates of whose fixed points are given by the quadratics of the double tangents of $\bar{Q}$. The diagonal triangle of this basis fourpoint circumscribes the norm conic, for to its sides correspond the parameters of the triple point of $\bar{Q}$. Hence

(6) *Given four points in the plane, ∞^2 conics may be inscribed in the diagonal triangle. For each, as a norm conic, four Lüroth quartics may be obtained, each with nodes at 3 of the 4 original points.*

Take the basis fourpoint as the vertices and center of an equilateral triangle. Let the Darboux coördinates of the center be $(0, \infty)$, while the sides of the diagonal triangle have parameters t_1, t_2, t_3. The quadratics and linear forms giving the nodes and corresponding bitangents follow:

	Quartic curve	Quadratics (nodes)	Linear forms (bitangents)
(7)	Q_{123}	q_1, q_2, q_3	L_{23}, L_{31}, L_{12}
	Q_{0ij}	q_0, q_i, q_j	L_{ij}, L_{0j}, L_{0i}

where, if $s_1 = \Sigma t_1$, $s_2 = \Sigma t_1 t_2$, $s_3 = t_1 t_2 t_3$,

$$(8)\quad \begin{aligned} &q_0 = t,\ q_i = 2t_i t^2 - (t_i^2 + s_2)t + 2s_3,\ L_{0i} = t_i t - t_j t_k, \\ &L_{ij} = (s_2 - 3t_i t_j)t + t_i t_j (s_1 - 3t_k),\ L_{ij} = L_{ji},\ i \neq j \neq k \neq 0. \end{aligned}$$

Developing the ternary equation for Q_{012} and setting $\sigma_1 = t_1 + t_2$, $\sigma_2 = t_1 t_2$ we have:

$$
\begin{aligned}
Q_{012} \equiv\ & 8\sigma_2\{t_3\sigma_1 - 2\sigma_2\}x_0^3x_1 + 4\sigma_1\{t_3^2(\sigma_2 - \sigma_1^2) + t_3\sigma_1\sigma_2 + \sigma_2^2\}x_0^3x_2 \\
& + 4t_3^2\sigma_1\sigma_2^2\{t_3^2 + t_3\sigma_1 + \sigma_2 - \sigma_1^2\}x_0x_2^3 \\
& + 8t_3^3\sigma_2^3\{-2t_3 + \sigma_1\}x_1x_2^3 + 16\sigma_2^2\{-2t_3 + \sigma_1\}x_0^2x_1^2 \\
& + 16\sigma_2^2t_3^3\{t_3\sigma_1 - 2\sigma_2\}x_1^2x_2^2 + \{t_3^4\sigma_1^3 + t_3^3(2\sigma_1^4 - 4\sigma_1^2\sigma_2) \\
& + t_3^2(\sigma_1^5 - 10\sigma_1^3\sigma_2) + t_3(2\sigma_1^4\sigma_2 - 4\sigma_1^2\sigma_2^2) + \sigma_2^2\sigma_1^3\}x_0^2x_2^2 \\
& - 8\sigma_2\{t_3^2(2\sigma_2 - 2\sigma_1^2) + t_3(\sigma_1^3 - 3\sigma_1\sigma_2) + \sigma_2\sigma_1^2\}x_0^2x_1x_2 \\
& - 8t_3^2\sigma_2\{t_3^2\sigma_1^2 + t_3(\sigma_1^3 - 3\sigma_1\sigma_2) \\
& + 2(\sigma_2^2 - \sigma_1^2\sigma_2)\}x_0x_1x_2^2 = 0.
\end{aligned}
\tag{9}
$$

Q_{023} and Q_{031} may be obtained by the cyclic permutation of 1, 2, 3.

To indicate a parameter τ on Q_{012}, we display its equation as the result of eliminating the parameter between two projective pencils of conics (on the nodes and touching respectively the nodal tangents at q_0):

$$
Q_{012} \equiv (l_1C' + C_1)(l_2C' + C_2) - kC'^2 = 0,
\tag{10}
$$

where

(11)

$$
\begin{aligned}
& k = l_1l_2,\ \alpha^2 = \sigma_1 - 2t_3,\ \beta^2 = -t_3^3(t_3\sigma_1 - 2\sigma_2), \\
& C' = x_0^2 - t_3\sigma_1x_0x_2 + t_3^2\sigma_2x_2^2, \\
& C_1 = 4\sigma_2(\alpha x_0 - \beta x_2)x_1 + \sigma_1\sigma_2(\alpha t_3^2 + \beta)x_2^2 - \sigma_1(\alpha s_2 - \beta)x_0x_2, \\
& C_2 = 4\sigma_2(\alpha x_0 + \beta x_2)x_1 + \sigma_1\sigma_2(\alpha t_3^2 - \beta)x_2^2 - \sigma_1(\alpha s_2 + \beta)x_0x_2, \\
& \alpha\beta l_1 = \alpha t_3\{t_3(t_1^2 + t_2^2) - \sigma_1\sigma_2\} + \beta\{t_3\sigma_1 - 2\sigma_2\}, \\
& \alpha\beta l_2 = -\alpha t_3\{t_3(t_1^2 + t_2^2) - \sigma_1\sigma_2\} + \beta\{t_3\sigma_1 - 2\sigma_2\}.
\end{aligned}
$$

By solving the pencils of conics for x_i in terms of τ, we write

$$
F_{2,4} \equiv \Sigma\xi_ix_i = 0,
\tag{12}
$$

where $\xi_0 = 1,\quad \xi_1 = -2t,\quad \xi_2 = t^2.$

We now study the configuration of nodes and double tangents. Let q_i, q_j, q_k be the nodes of one quartic. (q_i represents as usual either a quadratic or the point determined by it.) The line joining q_i, q_j passes through one vertex of the diagonal triangle and meets the opposite side in a point whose tangents to the norm conic are that side and the L_{ij} paired with q_k. The parameters of L_{0i} and L_{jk} are harmonic with respect to t_j, t_k. From the equation of Q_{012} and the symmetry of the nodes we see that the nodal tangents are harmonic with respect to the tangents to the norm conic from that node. This exemplifies the first case of Morley's theorem ([6], p. 282):

(13) *A Lüroth quartic can acquire a double point in two ways. In the one, the lines at the double point are apolar to the points on a double line. In the other the lines at the double point meet the curve again on a line of the curve.*

We may note here:

(14) *In the system of four Lüroth quartics every pair have in common two nodes, one bitangent, and the four points at which the tangents from the common node touch the conic. The nodal tangents at a common node are pairs of an involution.*

(15) *If a norm conic and two nodes for a Lüroth quartic be chosen, the locus of the third is a quartic with double points at the two fixed nodes. When the third node is selected, the set of Lüroth quartics is determined: for in the net of point conics on the three nodes only one pencil is apolar to the norm conic* ([5], pp. 240-242).

We next study the binary quintic. Since $q_0 = t$, the quintic is in Hammond's form ([3], p. 305)

$$(16) \qquad \begin{aligned} (\alpha t)^5 &\equiv at^5 + 5bt^4 + 5et + f \\ &\equiv s_0^2 s_2 t^5 - 5s_0^2 s_3 t^4 + 5s_0 s_3^2 t - s_1 s_3^2, \end{aligned}$$

where $\qquad s_0 = 1, \quad s_1 = \Sigma t_1, \quad s_2 = \Sigma t_1 t_2, \quad s_3 = t_1 t_2 t_3.$

If we use the fundamental system of the quintic ([3], pp. 309-310) the nodal parameters of the Lüroth quartic are given by a covariant of degree 4 and order 8:

$$(17) \qquad C'_{4,8} \equiv C_{2,2} C_{2,6} - C_{1,5} C_{3,3}.$$

This octavic covariant breaks up into four quadratics for the general quintic; if the points of any one such quadratic are taken as the reference points $(0, \infty)$, the quintic assumes Hammond's form. The parameters of the six bitangents are given by

$$(18) \qquad 9C'_{12,6} \equiv -3I_4 C_{3,3} C_{5,3} + 4C_{5,3} C_{2,2} C_{5,1} + C^2_{2,2} C_{8,2}.$$

We are led to an interesting interpretation of of I_4's vanishing. If

$$(19) \quad I_4 \equiv a^2 f^2 - 10abef + 9b^2 e^2 \equiv s_0^4 s_3^4 (s_1 s_2 - s_0 s_3)(s_1 s_2 - 9s_0 s_3) = 0,$$

and the roots of the canonizant are distinct, either

$$(20.\ 1, 2) \qquad s_1 s_2 - s_0 s_3 = 0, \quad \text{or} \quad s_1 s_2 - 9s_0 s_3 = 0.$$

In either case,

(21) *The vanishing of the invariant I_4 of a quintic whose canonizant has no repeated factor is the necessary and sufficient condition:*

(a) *that 2 and therefore 3 of the quadratic factors of its $C'_{4,8}$ be mutually apolar:*

(b) *that 2 and therefore 3 of the linear factors of $C'_{12,6}$ coincide.*

For a geometric interpretation let q_1, q_2, q_3 be the mutually apolar quadratics. Then Q_{123} degenerates into the lines of a triangle self-polar with respect to the norm conic C, and the line $L_{12} = L_{23} = L_{31}$, which is the polar line of the triangle with respect to the point q_0.

(22) *If the parameters of the contact points of 2 bitangents of a triple point quartic are apolar, a third bitangent exists such that the three sets of parameters of contact points are mutually apolar.*

By further examination of the covariants, we arrive at the theorem:

(23) *The identical vanishing of the covariant* $C'_{12,6} + 27C_{3,3}C'_{9,3}$ *where* $C'_{9,3}$ *is the cubicovariant of the canonizant* $C_{3,3}$ *is the necessary and sufficient condition that* $C'_{12,6}$ *have 2 factors in common with* $C_{3,3}$. $C'_{12,6}$ *then factors into the canonizant and its cubicovariant. Two roots of the binary quintic are given by one of the quadratics* q_i.

The geometric interpretation is clear if we consider that case in which the bitangents not on Q_{123} form the diagonal triangle. The norm conic is the circle inscribed in the triangle. The configuration admits a rotation of period 3 about the circle's center under which the circle and Q_{123} are transformed into themselves, while Q_{012}, Q_{023}, Q_{031} are permuted cyclically.

3. The poristic $F_{2,4}$ with configurations $\Delta^{2,4}_{6,12}$. We first obtain the configuration. Representing the t's by the faces of a cube and the τ's on a t by the edges on the corresponding face, we may prove by a study of the covariant porisms that the opposite faces are projective, as are also the opposite edges. If we take the projectivities to be $t' = -t$, $\tau' = -\tau$, we have the configuration:

$$(1) \qquad \begin{array}{ll} \text{For} & t = \pm l, \ \tau = \pm \alpha, \mp \beta, \mp \gamma, \mp \delta; \\ \text{for} & t = \pm m, \ \tau = \pm \mu, \mp \alpha, \mp \gamma, \mp \lambda; \\ \text{for} & t = \pm n, \ \tau = \pm \delta, \mp \beta, \mp \lambda, \mp \mu. \end{array}$$

We now obtain the equations of condition for $F_{2,4}$ and its complementary form. To have the solutions indicated,

$$(2) \qquad \begin{aligned} F_{2,4} \equiv & \, r(\tau - \alpha)(\tau + \beta)(\tau + \gamma)(\tau + \delta)/(t - l) \\ & + s(\tau - \mu)(\tau + \alpha)(\tau + \gamma)(\tau + \lambda)/(t - m) \\ & + p(\tau - \delta)(\tau + \beta)(\tau + \lambda)(\tau + \mu)/(t - n) = 0, \end{aligned}$$

while its expanded form must be one of the types:

$$(3) \qquad t^2(a_{24}\tau^4 + a_{22}\tau^2 + a_{20}) + t(a_{13}\tau^3 + a_{11}\tau) + a_{04}\tau^4 + a_{02}\tau^2 + a_{00} = 0,$$

$$(4) \qquad t^2(a_{23}\tau^3 + a_{21}\tau) + t(a_{14}\tau^4 + a_{12}\tau^2 + a_{10}) + a_{03}\tau^3 + a_{01}\tau = 0.$$

Let a_1, a_2, a_3, a_4 represent the elementary symmetric functions of

$-\alpha, \beta, \gamma, \delta$; b_1, b_2, b_3, b_4 those of $-\mu, \alpha, \gamma, \lambda$; c_1, c_2, c_3, c_4 those of $-\delta, \beta, \lambda, \mu$. If case (3) appears we have

$$\begin{aligned}
&ra_1 + sb_1 + pc_1 &&= 0,\\
&ra_3 + sb_3 + pc_3 &&= 0,\\
&mnra_1 + nlsb_1 + lmpc_1 &&= 0,\\
(5)\qquad &mnra_3 + nlsb_3 + lmpc_3 &&= 0,\\
&(m+n)ra_4 + (n+l)sb_4 + (l+m)pc_4 &&= 0,\\
&(m+n)ra_2 + (n+l)sb_2 + (l+m)pc_2 &&= 0,\\
&(m+n)r + (n+l)s + (l+m)p &&= 0.
\end{aligned}$$

Hence, if there are no coincidences in (1),

$$(6)\qquad a_1b_3 - a_3b_1 = 0, \quad a_1c_3 - a_3c_1 = 0, \quad b_1c_3 - b_3c_1 = 0,$$

$$(7)\qquad \begin{vmatrix} 1 & 1 & 1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

If case (4) appears,

$$\begin{aligned}
&r + s + p &&= 0,\\
&ra_2 + sb_2 + pc_2 &&= 0,\\
&ra_4 + sb_4 + pc_4 &&= 0,\\
(8)\qquad &mnr + nls + lmp &&= 0,\\
&mnra_2 + nlsb_2 + lmpc_2 &&= 0,\\
&mnra_4 + nlsb_4 + lmpc_4 &&= 0,\\
&(m+n)ra_1 + (n+l)sb_1 + (l+m)pc_1 &&= 0,\\
&(m+n)ra_3 + (n+l)sb_3 + (l+m)pc_3 &&= 0,
\end{aligned}$$

whence

$$(9)\qquad a_2 = b_2 = c_2; \quad a_4 = b_4 = c_4.$$

The complementary $F_{4,8}$ contains a form obtained from $F_{2,4}$ by changing the sign of τ. The residual factor correlates to a face t the edges perpendicular to t, and so may be written

$$(10)\quad f(\tau^2 - \lambda^2)(\tau^2 - \mu^2)/(t^2 - l^2) - g(\tau^2 - \alpha^2)(\tau^2 - \gamma^2)/(t^2 - n^2) = 0,$$

where the correlations impose the condition

$$(11)\qquad \begin{vmatrix} 1 & \beta^2 + \delta^2 & \beta^2\delta^2 \\ 1 & \alpha^2 + \gamma^2 & \alpha^2\gamma^2 \\ 1 & \lambda^2 + \mu^2 & \lambda^2\mu^2 \end{vmatrix} = 0.$$

To study form (3), we eliminate α in (6), obtaining a quadratic in γ whose discriminant is a perfect square. After factors implying a degenerate configuration are discarded, (11) becomes

$$(12)\qquad (\delta-\beta)(\lambda+\mu)(\lambda\mu+\beta\delta)-2\beta\delta(\lambda^2+\mu^2)+2\lambda\mu(\beta^2+\delta^2)=0.$$

(7) results in the same equation. From equations (5):

$$(13)\qquad l:m:n=(-\alpha_1+\alpha_2+\alpha_3):(\alpha_1-\alpha_2+\alpha_3):(\alpha_1+\alpha_2-\alpha_3),$$

where $\alpha_1=(b_2-c_2)/r,\ \alpha_2=(c_2-a_2)/s,\ \alpha_3=(a_2-b_2)/p;$

$$(14)\qquad ra_1:sb_1:pc_1=(\kappa_1 b_1+\kappa_2 c_1):-\kappa_1 b_1:-\kappa_2 c_1.$$

(13) and (14) satisfy all but the third and fourth of equations (5). Substituting in either, and putting in the values of the α's and of $r:s:p$, we obtain the resolvent quadratic:

$$(15)\qquad \begin{aligned}&(b_2-c_2)^2a_1{}^2\kappa_1\kappa_2+(a_2-b_2)(b_2-c_2)\kappa_1{}^2a_1b_1\\&+(c_2-a_2)(b_2-c_2)\kappa_2{}^2a_1c_1\\&-(\kappa_1b_1+\kappa_2c_1)\{(c_2-a_2)^2b_1\kappa_2+(a_2-b_2)^2c_1\kappa_1\}\\&+(c_2-a_2)(a_2-b_2)(\kappa_1b_1+\kappa_2c_1)^2=0.\end{aligned}$$

Set

$$(16)\qquad \begin{aligned}&A=b_2-c_2,\quad B=c_2-a_2,\quad C=a_2-b_2\\&s_0=-Aa_1-Bb_1-Cc_1,\quad s_1=-Aa_1+Bb_1+Cc_1,\\&s_2=Aa_1-Bb_1+Cc_1,\quad s_3=Aa_1+Bb_1-Cc_1.\end{aligned}$$

With proper simplifications a solution of (15) may be written

$$(17)\qquad \kappa_1:\kappa_2=c_1\{\sqrt{s_0s_1s_2s_3}-s_2s_3\}:b_1(s_1+s_2)s_3.$$

Setting these values in (14), we get

$$(18)\qquad \begin{aligned}r:s:p=&-(\sqrt{s_0s_1s_2s_3}+s_1s_3)/a_1\\&:(\sqrt{s_0s_1s_2s_3}-s_2s_3)/b_1:s_3(s_1+s_2)/c_1.\end{aligned}$$

From (13)

$$(19)\qquad \begin{aligned}l:m:n=&\{-2(s_3+s_1)\sqrt{s_0s_1s_2s_3}+s_3s_0(s_1-s_2)+s_1s_2(s_3-s_0)\}\\&:\{2(s_3+s_2)\sqrt{s_0s_1s_2s_3}-s_3s_0(s_1-s_2)+s_1s_2(s_3-s_0)\}\\&:\{s_3(s_0s_1+s_0s_2+2s_1s_2)-s_1s_2(s_3-s_0)\}.\end{aligned}$$

The factor of proportionality in $r:s:p$ may be assigned at will; that in $l:m:n$ is fixed by a choice of unit point for the t's. When the unit point for the τ's is also chosen, we thus have the form and one configuration given in terms of two absolute constants—say β,δ. One absolute constant is left for the form.

To study form (4), we get from (9):

$$(20)\qquad \beta\delta=\mu\lambda=\alpha\gamma=\kappa,$$

$$\begin{array}{lll} & \gamma+\alpha=k_1(\gamma-\alpha), & \lambda+\mu=k_3(\gamma-\alpha) \\ (21) & \lambda+\mu=k_2(\lambda-\mu), & \gamma+\alpha=k_2(\beta-\delta) \\ & \beta+\delta=k_3(\beta-\delta), & \beta+\delta=k_1(\lambda-\mu) \end{array}$$

If we solve for the quantities in terms of α, k_i we have from (20)

$$k_2^2=1/(1-k_1^2), \quad k_3^2=1/(1-k_2^2)=(k_1^2-1)/k_1^2, \ k_1^2=1/(1-k_3^2). \tag{22}$$

Set

$$\beta+\delta=u_1, \ \beta-\delta=u_2, \ \gamma+\alpha=v_1, \quad \gamma-\alpha=v_2, \ \lambda+\mu=w_1, \ \lambda-\mu=w_2. \tag{23}$$

Then

$$\begin{array}{ll} & a_1=u_1+v_2, \quad b_1=v_1+w_2, \quad c_1=w_1+u_2, \\ & a_3=-\kappa(u_1-v_2), \quad b_3=-\kappa(v_1-w_2), \quad c_3=-\kappa(w_1-u_2), \\ (24) & u_1^2-u_2^2=v_1^2-v_2^2=w_1^2-w_2^2=4\kappa, \\ & u_1^2-w_1^2=u_2^2-w_2^2=u^2-w^2, \\ & w_1^2-v_1^2=w_2^2-v_2^2=w^2-v^2, \\ & v_1^2-u_1^2=v_2^2-u_2^2=v^2-u^2. \end{array}$$

From (8):

$$(m+n)r : (n+l)s : (l+m)p=(v_1u_2-w_1w_2) : (w_1v_2-u_1u_2) : (u_1w_2-v_1v_2), \tag{25}$$

$$r : s : p=(\kappa_1+\kappa_2) : -\kappa_1 : -\kappa_2, \tag{26}$$

whence, using (21), (24), we have

$$\begin{array}{ll} \imath : n : l = & \kappa_1\kappa_2k_2(u^2-w^2)+\kappa_2(\kappa_1+\kappa_2)k_3(v^2-u^2)-\kappa_1(\kappa_1+\kappa_2)k_1(w^2-v^2) \\ 27) & : \kappa_1\kappa_2k_2(u^2-w^2)-\kappa_2(\kappa_1+\kappa_2)k_3(v^2-u^2)+\kappa_1(\kappa_1+\kappa_2)k_1(w^2-v^2) \\ & : -\kappa_1\kappa_2k_2(u^2-w^2)-\kappa_2(\kappa_1+\kappa_2)k_3(v^2-u^2)-\kappa_1(\kappa_1+\kappa_2)k_1(w^2-v^2). \end{array}$$

We have now only to satisfy the fourth, fifth and sixth of equations (8). Substitute in any of these, employing (22), and expressing the differences u^2-w^2, w^2-v^2, v^2-u^2 in terms of α, k_i. The solutions of the resolvent are upon simplification:

$$\kappa_1 : \kappa_2=-\sqrt{k_1^2-1}\{k_1^2+\sqrt{k_1^2-1}\} : k_1\{k_1+k_1\sqrt{k_1^2-1}-i\sqrt{k_1^2-1} \tag{28}$$

$$\kappa_1 : \kappa_2=-k_1\{k_1^2+\sqrt{k_1^2-1}\} : \{k_1+k_1\sqrt{k_1^2-1}-i\sqrt{k_1^2-1}\}. \tag{28°}$$

When α is determined by the choice of unit point for τ we have the form and one configuration given in terms of the absolute constant k_1.

The form (4), which is free of constants, may be more satisfactorily obtained by the study of the degenerate configurations. A possible arrangement is:

(I) two configurations in which

$$t:\ l=m=n,$$
$$\tau:\ \beta=\gamma=\lambda,\ \alpha=\delta=\mu.$$

(II) one configuration in which

$$t:\ l=-l,\ m=-m,$$
$$\tau:\ \beta=-\delta,\ \lambda=\mu,\ \alpha=-\alpha,\ \gamma=-\gamma.$$

In each of (I) we have the equivalent of 4 double roots for the t's and 8 for the τ's; $\alpha, -\alpha$ are the parameters of a node of the quartic; $l, -l$ represent common lines at common points of Q and C. In (II) there are 2 double roots for t and 6 for τ; α, γ are the parameters of a node; $n, -n$ are two bitangents of Q.

In (II) set

$$(29)\qquad l=-l=0,\ m=-m=\infty,\ \alpha=-\alpha=0,\ \gamma=-\gamma=\infty.$$

In homogeneous coördinates

$$(30)\qquad \begin{aligned} F_{2,4}\equiv{}& r\tau_1\tau_2(\tau_1^2-\beta^2\tau_2^2)(t_1t_2-nt_2^2)+s\tau_1\tau_2(\tau_1^2-\mu^2\tau_2^2)(t_1^2-nt_1t_2)\\ &+p\{\tau_1^4+2(\beta+\mu)\tau_1^3\tau_2+(\beta^2+\mu^2+4\beta\mu)\tau_1^2\tau_2^2\\ &+2\beta\mu(\beta+\mu)\tau_1\tau_2^3+\beta^2\mu^2\tau_2^4\}t_1t_2=0.\end{aligned}$$

To bring the equation into form (4), we must have

$$(31)\qquad r:s:p=2n\mu(\beta+\mu):2\beta(\beta+\mu):n(\beta-\mu).$$

The complementary form imposes the condition

$$(32)\qquad \beta^4=\mu^4,\ \text{i. e.,}\ \beta=\mu i \qquad (i^2=-1).$$

If we set $\mu=1$, $n=1$, we have, after simplifying,

$$(33)\qquad F_{2,4}\equiv t^2(2\tau^3-2\tau)+t(\tau^4+4i\tau^2-1)+2i\tau^3+2i\tau=0.$$

The parametric equations for Q are:

$$(34)\qquad x_0:x_1:x_2=4i\tau(\tau^2+1):-(\tau^4+4i\tau^2-1):4\tau(\tau^2-1).$$

The discriminant of $F_{2,4}=0$ as a quadratic in t is the perfect square $(\tau^4-4i\tau^2-1)^2$, corresponding to the fact that C is a contact conic of Q. The nodal parameters in the configurations (I) are given by the quadratics:

$$(35)\qquad \begin{aligned}&(1-2i-i\sqrt{3})\tau^2+1+2i-i\sqrt{3}=0,\\ &(1-2i+i\sqrt{3})\tau^2+1+2i+i\sqrt{3}=0.\end{aligned}$$

The lines on the nodes are

$$(36)\qquad 4\beta^2x_2-(\beta^2-1)^2x_0=0,\ 4\bar{\beta}^2x_2-(\bar{\beta}^2-1)^2x_0=0,\ x_1=0,$$

where $$\beta^2 = i(2+\sqrt{3}),\ \bar{\beta}^2 = i(2-\sqrt{3}).$$

From (34), (35), (36) we may easily prove:

(37). *The nodal tangents at the isolated node* $(0, 1, 0)$ *are harmonic with respect to the tangents from* $(0, 1, 0)$ *to the norm conic. At each of the other nodes the nodal tangents and the tangents to the norm conic are pairs of an involution whose fixed members are the lines from that node to the other two.*

4. Forms admitting checkerboard configurations. If we set up an involution $I_1{}^n \equiv (\gamma t)^n + \lambda(\delta t)^n = 0$ among the tangents t of the norm conic C, the locus points (t, t') such that

$$(\gamma t)^n + \lambda(\delta t)^n = 0, \quad (\gamma t')^n - \lambda(\delta t')^n = 0,$$

is an -ic $Q_n \equiv (\gamma t)^n(\delta t')^n + (\gamma t')^n(\delta t)^n = 0$ admitting ∞^1 configurations $\Delta^{2,n}_{2n,n^2}$, whose lines are tangent to the norm conic. If Q_n is rational, the incidence condition of points of Q_n with lines of C is the $F_{2,n}$ desired. In the following we assume known the properties and theorems developed by Professor Coble concerning the poristic $F_{2,n}$ and the curves involved (^{1}, §§ 9, 10).

5. The triple point quartic. To construct the $F_{2,4}$, take

(1) $$(\theta t)^4 \equiv t^3(t-1) = 0, \quad (\phi t)^4 \equiv (t-\lambda) = 0.$$

Then

(2) $$(\gamma t)^4 \equiv (\theta t)^4 + \sqrt{\mu}(\phi t)^4 = 0, \quad (\delta t)^4 \equiv (\theta t)^4 - \sqrt{\mu}(\phi t)^4 = 0.$$

The curve Q has the ternary equation (use **2** (2)):

(3) $$Q \equiv x_0{}^3(x_0 - 2x_1 + x_2) - \mu x_2{}^3(x_0 - 2\lambda x_1 + \lambda^2 x_2) = 0.$$

Cutting out the points of Q by the pencil of lines $x_0 - \tau x_2 = 0$, we get

(4) $$x_0 : x_1 : x_2 = (2\tau^4 - 2\lambda\mu\tau) : (\tau^4 + \tau^3 - \mu\tau - \lambda^2\mu) : (2\tau^3 - 2\lambda\mu).$$

Recalling the line coördinates for the conic, we have

(5) $$F_{2,4} \equiv \Sigma\xi_i x_i \equiv \tau^4 - \lambda\mu\tau - t(\tau^4 + \tau^3 - \mu\tau - \lambda^2\mu) + t^2(\tau^3 - \lambda\mu) = 0.$$

To get the complementary form, we obtain the completely poristic form $F_{8,16}$ and divide it by $F_{2,4}$.

The two absolute constants exhibited in $F_{2,4} = 0$ are the conic constant λ and the quartic constant μ. Given the four lines $t_1 = t_2 = t_3$, t_4, $t_5 = t_6 = t_7$, t_8, the conic may be any one of the pencil determined by them: the quartic is any one of a pencil with a triple point at $\tau_{15}, \cdots, \tau_{37}$, and flex points at τ_{18}, τ_{45}, whose tangents t_8, t_4 meet again on the curve at τ_{48}. Taking t_1, t_5, t_4 as the sides of the reference triangle and t_8 as the unit line, we have the ternary equation

$$x_0^3 x_2 - \mu x_1^3 (x_0 + x_1 + x_2) = 0. \tag{6}$$

We may then state the theorem:

(7) *The necessary and sufficient condition that a nondegenerate triple-point quartic admit* ∞^1 *checkerboard configurations* $\Delta^{2,4}_{8,16}$ *whose lines are tangent to a conic is that two flex tangents meet on the curve, forming a proper fourline with the lines from the triple point to the flexes. The conic may be any one of the pencil inscribed in the fourline.*

We can now prove:

(8) *A sufficient condition that a given form* $F_{2,4}$ *be poristic with checkerboard configurations is that it admit a degenerate configuration with the set of t's* $\left\{\begin{matrix} t_1 = t_2 = t_3, t_4 \\ t_5 = t_6 = t_7, t_8 \end{matrix}\right\}$, *there being no other equalities in the set.*

For we have the two distinct values $\tau_{45} = \tau_{46} = \tau_{47}$, $\tau_{18} = \tau_{28} = \tau_{38}$. Furthermore, the remaining τ's are distinct from the foregoing, and they have no more than three distinct values, which may all be obtained by substituting t_1, or t_4 in the form. Then $F_{2,4} = 0$ may be interpreted as the incidence condition for the dual elements of a conic and quartic satisfying the conditions of theorem (7).

We now study the quartic's fundamental pencil. ([7], pp. 35-41; [8], pp. 301-304.) For a simple parametric form cut out the points of (6) by the pencil $x_0 + \tau x_1 = 0$. We get

$$x_0 : x_1 : x_2 = -\tau(\tau^3 - \mu) : (\tau^3 - \mu) : -\mu(\tau - 1). \tag{9}$$

Noting that the flex points $(0, \infty)$ comprise the Hessian of the cubic giving the triple point parameters, we may write for Q:

$$x_0 : x_1 : x_2 = (\tau^3 - 1) : \tau(\tau^3 - 1) : \tau(a\tau^2 + b\tau + c). \tag{10}$$

$b = 0$ is the necessary and sufficient condition that either $ax_0 - x_2 = 0$, or $cx_1 + x_2 = 0$ be a flex tangent. Hence:

(11) *If there is a flex at one of the Hessian points of the cubic giving the parameters of the triple point, there is a flex at the other, and the two flex tangents meet on the curve.*

(12) *If two flex tangents meet on the curve at a point distinct from the flexes, the flex points comprise the Hessian of the cubic giving the triple point parameters, and the line joining the two flex points meets the curve again in a pair of points harmonic to the flexes.*

This pair is such that tangents drawn from them to the quartic form self-apolar sets ([8], p. 299).

The equations for Q may be taken:

$$x_0 : x_1 : x_2 = (\tau^3 - 1) : \tau(\tau^3 - 1) : \tau(\tau^2 - c). \tag{13}$$

The fundamental pencil becomes:

$$c\tau^4 - 4\tau^3 - 4c\tau + 1 + 6\lambda\tau^2 = 0, \tag{14}$$

with the concomitants:

$$\begin{aligned} &I_2 = 0, \text{ denoting a triple point,} \\ &I'_2 = -36c,\ I_4 = (-36c)^2,\ I_6 = 27(c^3 - 1)^2 \neq 0, \end{aligned} \tag{15}$$

else the line sections would have a common root. Q has therefore no cusp at the triple point. Since

$$I_4 - (I'_2 - 36I_2)^2 = 0, \tag{16}$$

four flexes, given by

$$c\tau^4 - 2\tau^3 + 2c\tau - 1 = 0, \tag{17}$$

lie on a line, namely

$$x_0 + cx_1 - 3x_2 = 0. \tag{18}$$

The condition for three collinear points is

$$s_2{}^2 - cs_1{}^2 - s_1s_3 - cs_2s_3 + cs_2 + s_1 = 0, \tag{19}$$

in terms of the symmetric functions of the parameters of these points.

The pencil of poristic line conics, which I have not found in the literature, is

$$c\xi_2{}^2 + \xi_1\xi_2 + c\xi_2\xi_0 + \lambda\xi_0\xi_1 + 0, \tag{20}$$

or in point coördinates,

$$x_0{}^2 + c^2x_1{}^2 + \lambda^2x_2{}^2 + 2c(2\lambda - 1)x_0x_1 - 2\lambda cx_1x_2 - 2\lambda x_2x_0 = 0. \tag{21}$$

The intersections of this pencil with the quartic curve are given by the system of octavics,

$$\begin{aligned} c^2\tau^8 + 2c(\lambda - 1)\tau^7 + (\lambda - 1)^2\tau^6 + 2c^2(\lambda - 1)\tau^5 - 2c(\lambda^2 + 2\lambda - 2)\tau^4 \\ + 2(\lambda - 1)\tau^3 + c^2(\lambda - 1)^2\tau^2 + 2c(\lambda - 1)\tau + 1 = 0. \end{aligned} \tag{22}$$

The three degenerate line conics are: for $\lambda = \infty$, the product of the two flex points $(0, \infty)$; for $\lambda = 0$, the product of the triple point and the meeting point $\tau = 1/c$ of the two flex tangents; for $\lambda = 1$, the product of two points on the conic κ of osculant theory (7, p. 45):

$$(23) \qquad 16I_2 \equiv x_0^2 + c^2x_1^2 - 15x_2^2 - 14cx_0x_1 + 14x_0x_2 + 14cx_1x_2 = 0.$$

A well-known pencil is determined by the conic on the flex points

$$(24) \qquad x_2(x_0 + cx_1 - 3x_2) = 0,$$

and the locus of self-apolar sections

$$(25) \qquad x_0^2 + c^2x_1^2 + 9x_2^2 - 14cx_0x_1 + 6x_0x_2 + 6cx_1x_2 = 0,$$

a poristic conic. The conic on the bitangent contacts,

$$(26) \qquad \begin{aligned} &\{(4 + 2\sqrt{3})x_0 + (-4 + 2\sqrt{3})cx_1 - 6\sqrt{3}x_2\} \\ &\{(4 - 2\sqrt{3})x_0 + (-4 - 2\sqrt{3})cx_1 + 6\sqrt{3}x_2\} = 0, \end{aligned}$$

counts twice as a degenerate member in the pencil. The pencil therefore passes through three points and touches the line (18) at $(3c, 3, 2c)$, a point on the line

$$(27) \qquad x_0 - cx_1 = 0$$

joining the triple point to the point $\tau = 1/c$. The two factors of (26) are harmonic with respect to (18) and (27). The conic (7, p. 45):

$$(28) \qquad 4I'_2 \equiv x_0^2 + c^2x_1^2 + 81x_2^2 - 14cx_0x_1 - 18x_0x_2 - 18cx_1x_2 = 0$$

is the locus of flex lines of the cubic osculants.

We next study the quartic from the standpoint of Landry's quintic and its system of concomitants (4, pp. 102-104). For equations (4) the apolar quintic is

$$(29) \qquad \tau^5 - 5\lambda\tau^4 + 10\lambda^2\tau^3 - 10\lambda\mu\tau^2 + 5\lambda^2\mu\tau - \lambda^3\mu = 0.$$

We have thus in general a pencil of binary quintics in (11) correspondence with the quartic curves; however, for $\mu = 0$, $\mu = \infty$, $\mu = \lambda^2$, the curve equations fail.

Given the quintic $(ft)^5 = 0$, we can calculate

$$(30) \qquad \begin{aligned} &i = (f, f)^4, \\ &j = -(f, i)^2, \quad \text{the canonizant giving the triple point,} \\ &s = s_1s_2 = (i, j)^2, \text{ Hessian of canonizant.} \end{aligned}$$

We may take for line sections of the quartic curve (4, p. 102):

$$x_0 : x_1 : x_2 = s_1 \cdot j : s_2 \cdot j : s \cdot i. \tag{31}$$

Taking

$$j = \tau^3 - 1, \quad s = \tau, \quad i = a'\tau^2 + b'\tau + c', \tag{32}$$

we see from previous theorems that the necessary and sufficient condition for the quartic to be poristic is that

$$B = (s, i)^2 = 0. \tag{33}$$

Equations (31) then reduce to (13). The quintic and its system of concomitants follow:

$$\begin{aligned}
f &= c^2\tau^5 - 5c\tau^4 + 10\tau^3 - 10c^2\tau^2 + 5c\tau - 1, \\
H &= (f, f)^2 = \tau(c\tau^4 - 2\tau^3 + 2c\tau - 1), \\
i &= (f, f)^4 = 6(1 - c^3)(\tau^2 - c), \\
A &= (i, i)^2 = -72(1 - c^3)^2 c, \\
j &= -(f, i)^2 = -6(1 - c^3)^2(\tau^3 - 1), \\
s &= (j, j)^2 = -72(1 - c^3)^4(t), \\
B &= (s, i)^2 = 0, \\
c &= (s, s)^2 = -2592(1 - c^3)^8, \\
\alpha &= -(j, i)^2 = -36(1 - c^3)^3(c\tau + 1), \\
-\theta &= (s, i) = 216(1 - c^3)^5(\tau^2 + c), \\
-\beta &= (\alpha, i) = 216(1 - c^3)^4(\tau + c^2), \\
\gamma &= (s, \alpha) = -1296(1 - c^3)^7(c\tau - 1), \\
\delta &= (\theta, \alpha) = -7776(1 - c^3)^8(\tau - c^2), \\
-R &= (\alpha, \delta) = 279936(1 - c^3)^{11}(1 + c^3), \\
M &= -3c, \\
N &= 93312(1 - c^3)^{10}c.
\end{aligned} \tag{34}$$

We may now prove:

(35) *The necessary and sufficient condition that a nondegenerate triple point quartic be poristic with checkerboard configurations is that four flexes lie on a line.*

We have already proved the condition necessary. To prove it sufficient, consider first that the branches at the triple point are distinct. Putting the apolar quintic in Hammond's form ([8], p. 306), we use Morley's relationships ([7], p. 45) to calculate the conic on the flexes $I'_2 - 4I_2 = 0$. For it to degenerate, its discriminant and therefore the invariant B of the quintic must vanish. Whence the curve is poristic. If there is a cusp at the triple point, the quartic cannot be poristic. In the case of a simple cusp we show by an algebraic calculation that the collinearity of four flexes implies a degenerate curve. Should all the parameters at the triple point coincide, only two actual flexes remain; hence four flexes cannot lie on a line.

6. Extension to Jonquières curves of higher order. The method exactly parallels that of **5**. Take for the n-ics of the multiple point configuration $t^{n-1}(t-1)$, $(t-\lambda)$. Then

$$(1)\qquad C_n \equiv x_0^{n-1}(x_0-2x_1+x_2)-\mu x_2^{n-1}(x_0-2\lambda x_1+\lambda^2 x_2)=0,$$

with parametric equations

$$(2)\qquad x_0 : x_1 : x_2 = (2\tau^n-2\lambda\mu\tau) : \tau^n+\tau^{n-1}-\mu\tau-\lambda^2\mu) : (2\tau^{n-1}-2\lambda\mu).$$

$$(3)\qquad F_{2,n} \equiv \tau^n-\lambda\mu\tau-t(\tau^n+\tau^{n-1}-\mu\tau-\lambda^2\mu)+t^2(\tau^{n-1}-\lambda\mu)=0.$$

Landry's $(2n-3)$ -ic is

$$\begin{aligned}&(c\tau)^{2n-3} \equiv \tau^{2n-3}-\tbinom{2n-3}{1}\lambda\tau^{2n-4}+\tbinom{2n-3}{2}\lambda^2\tau^{2n-5}+\cdots+(-1)^{n-3}\tbinom{2n-3}{n-3}\lambda^{n-3}\tau^n\\ (4)\quad &+(-1)^{n-2}\tbinom{2n-3}{n-2}\lambda^{n-2}\tau^{n-1}+(-1)^{n-1}\tbinom{2n-3}{n-1}\lambda\mu\tau^{n-2}+(-1)^n\tbinom{2n-3}{n}\lambda^2\mu\tau^{n-3}\\ &+\cdots+\tbinom{2n-3}{2n-4}\lambda^{n-2}\mu\tau-\lambda^{n-1}\mu.\end{aligned}$$

Its canonizant is

$$(5)\qquad \tau^{n-1}-\lambda\mu.$$

While all the theorems of **5** have their analogues here, the results are often too unwieldy to be useful. The following extensions may be noted:

(6) *The necessary and sufficient condition that a Jonquières curve of order n admit ∞^1 checkerboard configurations $\Delta_{2n,n^2}^{2,n}$ whose lines are tangent to a conic is that it possesses two $(n-1)$-point-contact tangents which meet again on the curve, forming a proper fourline with the lines from the $(n-1)$-fold point to the contacts of the tangents. The conic may be any one of the pencil inscribed in the fourline.*

(7) *The n further flexes of the entire pencil of poristic n-ics lie on the line*

$$(n+n\lambda-2)x_0+4\lambda(1-n)x_1+\lambda(n+n\lambda-2)x_2=0.$$

(8) *The $(n-1)$-ic giving the parameters of the multiple point of a poristic Jonquières curve is cyclic, and the fixed points of the cycle are the $(n-1)$ point contacts.*

(9) *If the $(n-1)$-ic of the multiple point of a Jonquières curve is cyclic, and if there is an $(n-1)$-point contact at one of the fixed points of the cycle, there is an $(n-1)$-point contact at the other also, and the tangents meet on the curve, which is therefore poristic.*

7. A trinodal quartic with checkerboard configurations. In the pencil of binary quartics

$$(1) \qquad (t-a)^2(t-b)^2+\lambda(t-1)^2(t-d)(t-e)=0,$$

pair the members by the involution $\lambda\lambda'=k$. If we impose the conditions that $\lambda=-1$, $\lambda=-k$ produce respectively the double roots $t=\infty$, $t=0$, we have

$$(2) \qquad \begin{aligned} &ab=m,\ a+b=(k-m^2)/(k-m),\\ &de=m^2/k,\ d+e=2(m-m^2)/(k-m). \end{aligned}$$

The one condition on the pencil that it contain a perfect square thus leaves two absolute constants for the quartic Q and the $F_{2,4}$ form. If $(\gamma t)^4$, $(\delta t)^4$ are members of the pencil given by $\lambda=k^{1/2}$, $\lambda=-k^{1/2}$, the equation of Q may be written

$$(3) \qquad (\gamma t)^4(\delta t')^4+(\gamma t')^4(\delta t)^4=0,$$

or setting $x_0=1$, $x_1=-(t+t')$, $x_2=tt'$, we have

$$(4) \qquad \begin{aligned} Q\equiv{} & x_1^2(k-m)^2\{m^3(m^3-3mk+2k^2)x_0^2-k(2m^3-3m^2k+k^2)x_2^2\}\\ & +x_1\{2m^3(k-m^2)(k-m)^3x_0^3\\ & +2m^2(k-m^2)(k-m)(2k^2-3mk+m^3)x_0^2x_2\\ & -2k(k-m^2)(k-m)(2m^3-3m^2k+k^2)x_0x_2^2\\ & -2k(k-m^2)(k-m)^3x_2^3\}+m^4(k-m)^4x_0^4\\ & -2m^2(k-m)^2\{2(m-1)k^2+m^2k-m^4\}x_0^3x_2\\ & +\{m^8-10m^6k+16m^5k^2+9m^4k(1-k)\\ & -16m^3k^3+10m^2k^4-k^5\}x_0^2x_2^2\\ & -2k(k-m)^2\{2m^3(m-1)-m^2k+k^2\}x_0x_2^3\\ & -k(k-m)^4x_2^4=0. \end{aligned}$$

To indicate the parameter on the curve, we follow the method of **2**, expressing Q's equation as the result of eliminating the parameter between two projective pencils of conics. We have

$$(5) \qquad Q\equiv\{l_1c'+(\Sigma x_i)c_1\}\{l_2c'+(\Sigma x_i)c_2\}-\lambda c'^2=0,$$

where, if

$$(6) \qquad \begin{aligned} &\kappa_1^2=m^3(m^3-3mk+2k^2),\ \kappa_2^2=k(2m^3-3m^2k+k^2),\\ &c'=(k-m)x_2^2-(k-m^2)x_0x_2+(k-m)mx_0^2,\\ &c_1=(k-m)(\kappa_1x_0-\kappa_2x_2),\ c_2=(k-m)(\kappa_1x_0+\kappa_2x_2),\\ &\kappa_1\kappa_2ml_1=\kappa_1m\{\kappa_2^2-k(k-m^2)(k-m)\}+\kappa_2\{m^3(k-m^2)(k-m)-\kappa_1^2\},\\ &\kappa_1\kappa_2ml_2=-\kappa_1m\{\kappa_2^2-k(k-m^2)(k-m)\}+\kappa_2\{m^3(k-m^2)(k-m)-\kappa_1\\ &\lambda=k(k-m)^2-\kappa_2^2+(l_1-l_2)\kappa_2+l_1l_2. \end{aligned}$$

We can now prove the theorem:

(7) *Given a poristic trinodal quartic with one isolated node, the nodal tangents at each node are harmonic with respect to the tangents from that node to the poristic conic.*

That the theorem holds at the isolated node is evident from (4). To prove it for the others, send $(1, -a-1, a)$, $(0, 1, 0)$, $(1, -b-1, b)$ into $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$ by the transformation

$$(8) \qquad \begin{aligned} x_0 : x_1 : x_2 = {} & \{x_0' - x_2'\} \\ & : \{-(b+1)x_0' + (b-a)x_1' + (a+1)x_2'\} : \{bx_0' - ax_2'\}. \end{aligned}$$

If we then set up the products of the nodal tangents at $(0, 0, 1)$, $(1, 0, 0)$, and the products of the tangents from those nodes to the poristic conic, we find the necessary apolarity conditions satisfied.

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CONCERNING IRREDUCIBLY CONNECTED SETS AND IRREDUCIBLE REGULAR CONNEXES.[†]

By R. L. Wilder.

The notions of a *point set M irreducibly connected about a point set K* and *basic set about which M is irreducibly connected* were introduced and studied to some extent by H. M. Gehman.[‡] The present paper is an investigation of these same notions with the emphasis placed on non-closed sets.[§]

The only spacial assumptions necessary are explicitly stated in the hypothesis of each theorem, in each case the set under consideration being considered as a general topological space without reference to any imbedding space. In general, only the notion of limit point is necessary, *connectedness* being defined in terms of limit point according to the usual Lennes-Hausdorff definition. In the case of Theorems 11-14 it is necessary to assume that the set under consideration is a "neighborhood" space in which it is true that if *P* is a limit point of a point set *M*, and *R* is a neighborhood of *P*, then *R* contains infinitely many points of *M*. Otherwise, we need no such properties as are implied in denumerability axioms, covering theorems, etc. (Whenever references are made to theorems of other papers needed in proofs, it may be readily ascertained that such theorems hold true under our assumptions.) As pointed out below, "regular" is used in the sense of *local connectedness* (connected neighborhoods), and not in the sense in which it is usually employed, as, for instance, in metrization theory. As might be expected, however, where restrictions are placed upon the set about which the space is irreducibly connected, it turns out that the character of the space is considerably simplified (see, for instance, Theorems 8, 9 and its corollary, and 13, below).

We have by no means exhausted the subject, and a number of problems may suggest themselves to the reader.

Theorem 1. *In order that a connected set M should be irreducibly connected about a subset K, it is necessary and sufficient that if P is any point of*

[†] Presented to the American Mathematical Society April 7, 1928. Our attention was recently called to certain typographical errors in the original abstract of this paper (*Bulletin of the American Mathematical Society*, vol. 34, p. 426, No. 21): In line 8, "n" should read "n + 1", and in line 11, the first "M" should be "K."

[‡] "Concerning irreducibly connected sets and irreducible continua," *Proceedings of the National Academy of Sciences*, vol. 12 (1926), pp. 544-547. Also see other papers by the same author cited below.

[§] Gehman's work was chiefly on closed sets of points.

$M - K$, then $M - P$ is the sum of two mutually separated sets each of which contains at least one point of K.

Proof of necessity: Clearly $M - P$ is not connected, else M is not irreducibly connected about K. Hence $M - P$ is the sum of two mutually separated sets M_1 and M_2. Each of the latter sets contains at least one point of K, for if M_1, say, contains all of K, the connected[†] set $M_1 + P$ is a proper connected subset of M containing K.

Proof of sufficiency: Suppose M is not irreducibly connected about K. Then M contains a non-vacuous subset, N, such that $M - N$ is connected and contains K. Let P be a point of N. By hypothesis, $M - P$ is the sum of two mutually separated sets M_1 and M_2, each of which contains at least one point of K. If we let those points of M_i $(i = 1, 2)$ that do not belong to N be denoted by T_i, the set $M - N$ is the sum of the two mutually separated sets T_1 and T_2, and a contradiction results.

It will be noted that Gehman's Theorem 3 (*loc. cit.*) to the effect that K contains all the non-cut-points of M is a corollary of the above theorem; also that instead of a *point* P we might use a *connected subset* P of $M - N$ in the above theorem.

THEOREM 2. *In order that a connected set M should be irreducibly connected about a subset K consisting of n points, where n is a positive integer, it is necessary and sufficient that if P is any point of $M - K$, then $M - P$ is the sum of at least two, but not more than n, distinct components each of which contains at least one point of K.*

Proof of necessity: That $M - P$ has at least two components follows, of course, from Theorem 1. If $M - P$ has more than n components, at least one of these, say C, fails to contain a point of K. But $M - C$ is connected[‡] and a proper subset of M containing K, violating the condition that M is irreducibly connected about K. In the same way we may show that every component of $M - P$ contains at least one point of K.

The sufficiency of the condition stated in the theorem follows from Theorem 1.

It will be noted that in the proof just given it is also shown that if a point set M is irreducibly connected about a set K and P is any point of $M - K$, then every component of $M - P$ contains at least one point of K; indeed,

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[†] B. Knaster and C. Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, vol. 2 (1921), pp. 206-255, Theorem VI. This paper also contains a valuable study of sets irreducibly connected about two points.

[‡] Knaster and Kuratowski, *loc. cit.*, Theorem X.

Theorem 1 could have been stated in terms of the *components* of $M - P$ instead of in terms of the separated parts of $M - P$—an essential distinction, since in general the components and the separated parts of a point set are not identifiable.

For purposes of proof below, we list the following Lemmas: 1. *If N is a connected subset of a connected set M, and $M - N$ is the sum of n mutually separated sets $M_1, M_2, \cdots, M_n$, then $M_i + N$ is connected $(i = 1, 2, \cdots, n)$.*[†] 2. *If a set M is the sum of exactly n components, then these components are mutually separated in M.* 3. *If a set M has at least $n (> 1)$ distinct components, then it is the sum of at least n mutually separated sets.* 4. *If a set M has only a finite number, n (> 1), of components, then for any positive integer $k \leqq n$, M is the sum of k mutually separated sets $M_1, M_2, \cdots, M_k$ such that $M_1, M_2, \cdots, M_{k-1}$ are arbitrary components of M.*

(The proof of Lemma 2 is obvious. Lemma 3 is clearly true for $n = 2$, and is easily proved for the general case by mathematical induction. In regard to Lemma 4: By Lemma 3, M is the sum of n mutually separated sets, $N_1, N_2, \cdots, N_n$. Each set N_i must be a component of M. Then for $i < k$, let $M_i = N_i$ and let $M_k = N_k + \cdots + N_n$.)

THEOREM 3. *Let M be irreducibly connected about a set of points K, and let P be a point of M such that $M - P = M_1 + M_2 + \cdots + M_k$, where the sets M_i are mutually separated and $k > 1$. Then if $M_i \cdot K = K_i$ $(i = 1, 2, \cdots, k)$, the set $M_i + P$ is irreducibly connected about $K_i + P$.*

Proof. The set $M_i + P$ is connected by Lemma 1. Suppose it has a proper connected subset, N, containing $K_i + P$. The set $M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n + P$ is connected, and consequently by adding N to this set we get a proper connected subset of M containing K, contradicting the hypothesis.

THEOREM 4. *If M is irreducibly connected about a set K consisting of n points and P is a point of M such that $M - P$ has n components, then if M_1 is one of those components, the set $M_1 + P$ is irreducibly connected about two points, viz. P and $M_1 \cdot K$.*

Proof. It is easily shown that $P \subset M - K$. By Theorem 2 the set $M_1 \cdot K$ is exactly one point. Theorem 4 then follows from Lemma 2 and Theorem 3.

THEOREM 5. *Under the same hypothesis as in Theorem 4, if Q is a point of M not in $K + P$, then $M - Q$ has only two components, and one of these*

[†] This is an obvious extension of Theorem VI of Knaster and Kuratowski, *loc. cit.*

components contains only one point of K, namely that point of K which lies with Q in the same component of $M - P$.

Proof. If the components of $M - P$ are $M_1, M_2, \cdots, M_n$, then by Theorem 2 and Lemma 2 the sets M_i are mutually separated. Let the point (Theorem 2) $M_i \cdot K$ be denoted by P_i. We may assume that $M_1 \supset Q$. Then

$$M - Q = (M_1 - Q) + (M_2 + \cdots + M_n + P).$$

Suppose $M - Q$ has at least three components. Since, by Theorem 2, $M - Q$ has at most n components, we can conclude (Lemma 4) that $M - Q$ is the sum of three mutually separated sets, K_1, K_2 and K_3, where K_1 and K_2 are arbitrary components of $M - Q$. As the set $M_2 + \cdots + M_n + P$ is connected, it lies wholly in one component of $M - Q$; we may therefore so assign the symbol K_1 as to denote the component of $M - Q$ containing $M_2 + \cdots + M_n + P$. Then $K_1 \supset P_2 + \cdots + P_n = K - P_1$. Accordingly $(K_2 + K_3) \cdot K = P_1$, and we may assume that $K_2 \supset P_1$. But then K_3 contains no point of K, a contradiction of Theorem 2.

Thus, with the result of Theorem 1, $M - Q$ has exactly two components, K_1 and K_2, where $K_1 \supset M_2 + \cdots + M_n + P \supset K - P_1$ and $K_2 \supset P_1$.

Definition. If M is a connected set and P a point of M such that $M - P$ can be arranged as the sum of $n + 1$ mutually separated sets (where n is a positive integer), then P will be called a *cut-point* of M *of class n*. (Thus, every cut-point is of class 1, and, in general, a cut-point of class n, where $n > 1$, is also a cut-point of classes $1, 2, \cdots, n - 1$, respectively.)

In particular we have shown in Theorems 2, 4 and 5 that a set M irreducibly connected about a set K consisting of n points has no cut-point of class n, and if it has a cut-point of class $n - 1$ all other points of $M - K$ are cut-points *only* of class 1. We shall investigate further the possible classes of the cut-points of such a set later on.

We shall use the term *basic set* for general connected sets in the sense introduced by Gehman; that is, K is a basic set about which M is irreducibly connected if M is not irreducibly connected about any proper subset of K. Although, as shown by Gehman, in the case of a compact continuum M the set of non-cut-points is a basic set about which M is irreducibly connected, the analogous theorem is not true for the general connected set, as simple examples show. However, we can prove the following:

THEOREM 6. *If a point set M has a basic set, K, about which it is irreducibly connected, then K is the set of non-cut-points of M, and every set about which M is irreducibly connected contains K.*

Proof. That K contains all the non-cut-points of M follows from

Theorem 1. Suppose P is a point of K that is a cut-point of M. Then $M - P$ is the sum of two mutually separated sets M_1 and M_2. Clearly, both M_1 and M_2 contain points of K. Let $M_i \cdot K = K_i$ $(i = 1, 2)$.

Now M is irreducibly connected about $K_1 + K_2 = K - P$. For suppose not. Then there exists a proper connected subset, N, of M that contains $K - P$. As N is connected and contains points of both M_1 and M_2, it must also contain P. Therefore N is a proper connected subset of M that contains K, contradicting the hypothesis that M is irreducibly connected about K. Thus M is irreducibly connected about $K - P$. But then K is not a basic set about which M is irreducibly connected. Consequently every point of K must be a non-cut-point of M and accordingly K is identically the set of non-cut-points of M.

In conclusion, if A is a set about which M is irreducibly connected, it follows from Theorem 1 that A contains the set of non-cut-points of M; that is, $A \supset K$.

COROLLARY. *A point set can have only one basic set about which it is irreducibly connected.*

We need the following lemma for the proofs of Theorems 7 and 8:

LEMMA 5. *Let M be a connected point set and P a point of M. Then in order that P should be a cut-point of M of class n it is necessary and sufficient that if x is any point of $M - P$ and M_1 is a separate* † *of $M - x$ which contains P, then P is a cut-point of $M_1 + x$ of class n.*

Proof of necessity: By hypothesis, $M - P$ is the sum of $n + 1$ separates K_i $(i = 1, 2, \cdots, n+1)$. Denote the set $M - (M_1 + x)$ by M_2. Since P is a point of M_1 and therefore not a limit point of M_2, the set $N_i = K_i \cdot M_1 (i = 1, 2, \cdots, n+1)$ is non-vacuous. We have that

$$(1) \qquad (M - P) \cdot (M_1 + x) = K_1 \cdot (M_1 + x) + \cdots + K_{n+1} \cdot (M_1 + x).$$

Now the point x is in only one of the sets K_i, say in K_1. Consequently (1) may be written:

$$(M_1 + x) - P = (N_1 + x) + N_2 + \cdots + N_{n+1},$$

where the sets in the right-hand member are separates of $(M_1 + x) - P$. Thus P is a cut-point of $M_1 + x$ of class n.

Proof of sufficiency: By hypothesis, $(M_1 + x) - P$ is the sum of separates $N_i (i = 1, 2, \cdots, n+1)$. Let $N_1 \supset x$, and denote $M - (M_1 + x)$ by M_2. For $i = 2, 3, \cdots, n+1$, the sets N_i and $N_1 + M_2 + x$ are mutually separated. Consequently $M - P$ is the sum of the separates $N_1 + M_2 + x$, $N_2, \cdots, N_{n+1}$.

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† If M is any point set and K a subset of M such that K and $M - K$ are mutually separated, we shall call K a *separate* of M.

THEOREM 7. *If K is a basic set about which M is irreducibly connected, and P is a point of M such that $M - P$ is the sum of n components M_i $(i = 1, 2, \cdots, n)$, then, letting $K \cdot M_i = K_i$, each set $K_i + P$ is a basic set about which $M_i + P$ is irreducibly connected.*

Proof. The set $M_i + P$ is irreducibly connected about $K_i + P$ by Lemma 2 and Theorem 3. By Theorem 6, every point of K is a non-cut-point of M, and by Lemma 5 every point of K_i is therefore a non-cut-point of $M_i + P$. Also, since M_i is connected, P is a non-cut-point of $M_i + P$. That no point of $M_i - K_i$ is a non-cut-point of $M_i + P$ follows from Lemma 5. Thus $K_i + P$ is identically the set of non-cut-points of $M_i + P$. Consequently, since $M_i + P$ is irreducibly connected about $K_i + P$ and every set about which $M_i + P$ is irreducibly connected must contain all points of $K_i + P$, the latter set must be a basic set about which $M_i + P$ is irreducibly connected.

THEOREM 8. *If a point set M has a basic set K about which it is irreducibly connected, such that K consists of n points, then the number of cut-points of M having class greater than 1 is at most $n - 2$.*

Proof. If $n = 2$, the theorem follows from Theorem 2. For the general case we use mathematical induction, assuming the theorem true for the cases where K consists of $2, 3, \cdots, n - 1$ points, respectively.

By Theorem 6 no point of K is a cut-point of M. Suppose P is a cut-point of M of class greater than 1. Then by Theorem 2, $M - P$ is the sum of components $M_1, M_2, \cdots, M_k$, where $2 < k \leqq n$. The sets $M_i \cdot K = K_i$ $(i = 1, 2, \cdots, k)$ are non-vacuous and by Theorem 7, $K_i + P$ is a basic set about which $M_i + P$ is irreducibly connected. For each i, let the number of points of K_i be denoted by n_i.

Clearly $n_i + 1 < n$ for each i, since $k > 2$. Hence the set $M_i + P$ contains at most $n_i + 1 - 2 = n_i - 1$ cut-points of class greater than 1. Since P is a non-cut-point of $M_i + P$, it follows from Lemma 5 that a cut-point of $M_i + P$ of class greater than 1 is a similar cut-point of M, and conversely a cut-point of M of class greater than 1 and distinct from P is a similar cut-point of some $M_i + P$. Thus the number of cut-points of M of class greater than 1 is at most $\sum_{i=1}^{k} (n_i - 1) + 1 = \sum_{i=1}^{k} n_i - k + 1 = n - k + 1$. As $k > 2$, this proves the theorem.

We have now shown the structure of sets that are irreducibly connected about a finite number of points to be fairly simple. One more fact that we wish to observe, however, in order to make the intuitive structure of these sets more complete, is that they always have *basic sets* about which they are irreducibly connected. This is, however, a special case of a much more general theorem concerning the existence of basic sets:

THEOREM 9. *If a point set M is irreducibly connected about a compact and closed point set K, such that $K - T$ is either vacuous or separable* † *(where T is the set of non-cut-points of M), then M has a basic set (the set T) about which it is irreducibly connected.*

Proof. By Theorem 1, $K \supset T$. If either $K \equiv T$ or M is irreducibly connected about T, the theorem is proved. Suppose that $K - T$ is non-vacuous and that M is not irreducibly connected about T. Then M has a proper connected subset $N \supset T$, where N may be vacuous if T is vacuous—we do not assume T non-vacuous. By hypothesis there exists a denumerable set of points $x_1, x_2, x_3, \cdots$ dense in $K - T$.

Not all points of $K - T$ lie in N, else N is a proper connected subset of M containing K. Without loss of generality, we may assume that x_1 is not in N. As x_1 is not a point of T, $M - x_1$ is the sum of two separates, M_1 and M_2, where $M_1 \supset N \supset T$. The set M_2 contains points of $K - T$, else $M_1 + x_1$ would be a proper connected subset of M containing K. Hence there is a first point, x_{n_1}, of the sequence of points $\{x_i\}$, in M_2.

Again, $M - x_{n_1}$ is the sum of two separates M_1^1 and M_2^1, where $M_1^1 \supset M_1 + x_1 \supset N \supset T$. Arguing as before, let x_{n_2} be the first point of the sequence $\{ x_i \}$ in M_2^1; $M - x_{n_2}$ is the sum of two separates M_1^2 and M_2^2 where $M_1^2 \supset M_1^1 + x_{n_1} \supset M_1 + x_1 \supset N \supset T$. And so on.

Let $M_2^i \cdot K = K_i (i = 1, 2, 3, \cdots)$. For each i, $K_i + x_{n_i}$ is a closed subset of $K - T$ and therefore a compact and closed set. Since, for each i, $K_i + x_{n_i} \supset K_{i+1} + x_{n_{i+1}}$, the product $\prod_{i=1}^{\infty} (K_i + x_{n_i})$ contains at least one point P. As P is a point of $K - T$, $M - P$ is the sum of two separates M_1^ω and M_2^ω, where $M_1^\omega \supset x_1$. As $P \subset K_i \subset M_2^i$ for all positive integers i, it is clear that $M_1^\omega \supset M_1^i + x_{n_i}$. Then M_1^ω contains all points x_i. On the other hand M_2^ω must contain points of $K - T$. This is impossible since the points $\{ x_i \}$ are dense in $K - T$. From this contradiction we conclude that either $K \equiv T$ or M is irreducibly connected about T, and the theorem is proved.

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† We use *separable* only in the sense that there exists a denumerable subset of M that is dense in M. It is easy to give an example, based on the ordinal numbers of the first and second class, to show the necessity for this assumption. As for the necessity of the assumption that K is compact, consider the set M of real numbers x such that $0 < x \leqq 1$ and the set K of numbers $x = 1/n$ where n is a natural number. Concerning "compact," see footnote in connection with Theorem 13.

Since, in the above proof, we made no assumptions concerning T, we have the following corollary of Theorem 9:

COROLLARY. *If a point set M is irreducibly connected about a separable, compact and closed point set K, then M has at least two non-cut-points.*

In connection with Theorem 9 and its corollary we may recall the theorem to the effect that every compact separable continuum is irreducibly connected about a closed set, namely the closure of its set of non-cut-points,† and the theorem that every such continuum has at least two non-cut-points.‡ The former is provable directly by a method similar to that used in proving Theorem 9 above, and the latter may therefore be considered its corollary.§

We can now complete our study of sets irreducibly connected about finite sets of points with the following theorem:

THEOREM 11. *If a point set M is irreducibly connected about a finite set of points K, then M is irreducibly connected about the points of K that are non-cut-points of M.*

Theorem 11 is, of course, a direct consequence of Theorem 9.

We conclude this paper with a few remarks concerning regular ¶ point sets.

Definition. A point set M will be called an *irreducible regular connexe* if there exists a point set K relative to which it satisfies the following condition: M is regular and connected and contains K, but M has no proper subset satisfying the same conditions. We shall also call M an *irreducible regular connexe about K* when we wish to specify the set K.

THEOREM 12. *For a point set M to be an irreducible regular connexe about a subset K it is necessary and sufficient that M be regular and irreducibly connected about K.*

Proof of necessity. Suppose M not irreducibly connected about K. Then by Theorem 1 there exists a point P of $M - K$ such that either (a) $M - P$ is connected or (b) $M - P$ is the sum of two separates, M_1 and M_2, such that

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† See Gehman, *loc. cit.*, Theorem 1. Gehman's proof depends upon the theorem of Moore cited below.

‡ See R. L. Moore, "Report on continuous curves from the viewpoint of analysis situs," *Bulletin of the American Mathematical Society*, vol. 29 (1923), pp. 289-302, Theorem C, and the earlier paper by the same author cited therein.

§ See also H. M. Gehman, "Concerning irreducible continua," *Proceedings of the National Academy of Sciences*, vol. 14 (1928), pp. 433-435.

¶ Regular = locally connected = "connected im kleinen." Although the word "regular" in this sense has gone out of fashion, we retain it in the present instance because of its brevity and because we do not wish to change the title of the paper as originally announced.

$M_1 \supset K$. But (a) cannot hold, since $M - P$ is regular and if connected also, then M is not an irreducible regular connexe about K. On the other hand, (b) cannot hold since $M_1 + P$ is regular [†] and as this set is a proper connected subset of M we again have a contradiction.

The sufficiency is obvious.

In view of Theorem 12, what we have proved above concerning irreducibly connected sets applies to irreducible regular connexes. In particular, a set about which M forms an irreducible regular connexe must contain the non-cut-points of M.

Probably the most important theorem concerning irreducible connexes is the following:

THEOREM 13. *An irreducible regular connexe M about a compact* [‡] *set K is itself compact.*

We assume that M is a space in which it is true that if P is a limit point of a set of points L, then a neighborhood of P contains infinitely many points of L.

Proof. Suppose M is not a compact space. Then it contains an infinite set N that has no limit point. As K is compact, not more than a finite number of points of N lie in K, and we may therefore assume that $N \cdot K = 0$.

In what follows we use the term "region" to denote a connected neighborhood. As M is regular, we may cover it by a set G of regions as follows: If P is a point of M, there is a region in M containing P and containing no point of N except possibly P. This is true for all points P, and we let G denote a collection of such regions covering M.

If K is connected there is no further problem since then K and M must be identical (Theorem 12). Suppose K has at least two but at most a finite number of components, $K_1, K_2, \cdots, K_k$. As M is connected, there exists for each i ($i = 1, 2, \cdots, k-1$) a simple chain† C_i of regions of G from a point of K_i to a point of K_{i+1}. The set of all points in chains C_i and in K is a connected set which contains at most a finite number of points of N, and which is therefore a proper connected subset of M containing K. But this contradicts the fact (Theorem 12) that M is irreducibly connected about K. We must conclude, then, that K has infinitely many components.

[†] See Theorem 5 of my paper "The non-existence of a certain type of regular point set," *Bulletin of the American Mathematical Society*, vol. 33 (1927), pp. 439-446.

[‡] By *compact* we mean that every infinite subset of K has a limit point in M, but not necessarily in K; in other words, K is *not* necessarily *self*-compact. In a euclidean space, then, this theorem would imply that if M is an irreducible regular connexe about a bounded set K whose closure belongs to M, then M is closed and bounded. We note, however, that whereas in the latter case M would automatically be an irreducible regular connexe about a closed and bounded set K ($= K$ plus its limit points), it is not, in general, true that if K is a compact subset of a general space M then $\bar{K}$ ($= K$ plus its limit points) is self-compact.

For the case where K is compact and closed and M satisfies the Borel-Lebesgue covering theorem with respect to K (i. e., if G is a set of neighborhoods of M covering K, then a finite subset of G covers K), a very simple proof can be given for Theorem 13—a proof which is a slight modification of the proof for the theorem in euclidean spaces as stated above, and which was given in my paper "On connected and regular sets," *Bulletin of the American Mathematical Society*, vol. 34 (1928), pp. 649-655 (Theorem 7).

Hereafter, if A and B are point sets, we denote by $C(A,B)$ a simple chain of regions of G from a point of A to a point of B. Also, we let P_1, P_2, $P_3, \cdots$ be an infinite sequence of distinct points of N, and henceforth identify N with their sum.

By Theorem 1, there exists for each natural number i a pair of components K_{i1}, K_{i2} of K that are separated in M by P_i. Involved in the pairs (K_{i1}, K_{i2}) there must be an infinite number of distinct components of K. For, as shown above, any finite number of components of K lie in a connected subset of M that contains only a finite number of points of N, and consequently the points P_i cannot each separate components of K that are all selected from a fixed finite set of such components.

There exists, then, in the sequence (K_{i1}, K_{i2}), a sequence of pairs of components of K, (K'_{i1}, K'_{i2}), $(i = 1, 2, 3, \cdots)$, such that some sequence K'_{ij} ($i = 1, 2, 3, \cdots$; j varying between the numbers 1 and 2 as i varies) consists of distinct components. As infinitely many of the j's are all 1's, or all 2's, we may assume that j has the value 1 infinitely many times. There exists, then, in the sequence (K_{i1}, K_{i2}), a sequence of pairs of components (K_{i1}^*, K_{i2}^*), $(i = 1, 2, 3, \cdots)$ such that the sets K_{i1}^* are all distinct.

We have two cases to consider: 1) Suppose only a finite number of the sets K_{i2}^* are distinct. Clearly, then, we may assume, without loss of generality, that the sets K_{i2}^* are all one and the same component of K. For each i, let x_i be a point of K_{i1}^*, and let P_i^* denote the point of the sequence $\{P_i\}$ originally assigned as separating the pair (K_{i1}^*, K_{i2}^*). As K is compact, there is a point X of M which is a limit point of the set of points x_i. Let Y be a point of K_{i2}^*. As there exists a $C(X, Y)$, M contains a connected set, H, consisting of $C(X, Y)$ and K_{i2}^*, as well as all sets K_{i1}^* that have points in R, the region of $C(X, Y)$ that contains X. But infinitely many sets K_{i1}^* belong to H, whereas only a finite number of points P_i^* lie in H. For i great enough, then, P_i^* cannot separate K_{i1}^* and K_{i2}^* in M, and we have a contradiction.

2) If infinitely many of the sets K_{i2}^* are distinct, there is a subsequence

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† See my paper "The non-existence . . . ," cited above.

of the sequence (K_{i1}^*, K_{i2}^*) whose sets K_{i2}^* are all distinct. We may therefore assume the sequence (K_{i1}^*, K_{i2}^*) to have this property. Let x_i be selected as before, with a limit point X. Let R_1 be a region of G containing X; and denote those sets K_{i1}^* that have points in R_1 by K_{i1}^{**}. The sets K_{i2}^* paired with sets K_{i1}^{**} we denote by K_{i2}^{**}. For each K_{i2}^{**} we assign a point $y_i \subset K_{i2}^{**}$. As K is compact, the set of points y_i has a limit point Y. Let R_2 be a region of G containing Y. There exists a $C(R_1, R_2)$, and the set H consisting of R_1, R_2, $C(R_1, R_2)$ and all sets K_{i1}^{**}, K_{i2}^{**} that have points in R_1 and R_2 forms a connected subset of M that contains only a finite number of points of N. But as we have selected the point Y, there are infinitely many pairs $(K_{i1}^{**}, K_{i2}^{**})$ such that *both* sets K_{i1}^{**}, K_{i2}^{**} lie in H, and we again have a contradiction.

Accordingly no such set as N exists and M is compact.

The importance of Theorem 13, aside from whatever intrinsic interest it may have, consists in the fact that it reduces the study of sets that are irreducible regular connexes about compact sets to a study of compact locally connected continua. A number of theorems have been found in this direction,† although in many cases it is an open question whether the theorems hold in the case of a space as general as that described in Theorem 13. It is interesting to note, however, that a perfectly separable set M which is an irreducible regular connexe about a finite point set K is a linear graph ‡ whose basic set is the set of non-cut-(end-) points of M and whose vertices are the points of the basic set and cut-points of class greater than one.

It might be of interest to analyze further the properties of a set M as defined in Theorem 13. For instance, the following theorem is readily proved:

THEOREM 14. *If M is an irreducible regular connexe about a compact set K, then every point of $M - \bar{K}$ is a cut-point of at most finite class; indeed, if P is such a point, $M - P$ has only a finite number of components.*

This theorem does not imply, of course, that the classes, as cut-points, of all points of $M - \bar{K}$ are a bounded set — simple examples show this not to be the case. But for a fixed point P of $M - \bar{K}$ there exists a natural number $n(P)$ such that $M - P$ is the sum of exactly $n(P)$ components.

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† Besides the papers of Gehman already cited, see, by the same author, "Irreducible continuous curves," *American Journal of Mathematics*, vol. 49, pp. 189-196, and "Concerning acyclic continuous curves," *Transactions of the American Mathematical Society*, vol. 29 (1927), pp. 553-568, Theorem 3. Also see L. Zippin, "On continuous curves irreducible about subsets," *Fundamenta Mathematicae*, vol. 20 (1933), pp. 197-205.

‡ See O. Veblen, "Analysis situs," *American Mathematical Society Colloquium Publications*, vol. 5 (1921), part II.

CHARACTERIZATIONS OF CERTAIN FINITE CURVE-SUMS.

By Norman E. Steenrod.

1. *Introduction.* Let K be a class of continuous curves[*] defined in some manner. Let the term K-curve denote an element of the class K. In this paper we are concerned with the following problem:

What conditions are necessary and sufficient in order that a given continuous curve C should be expressible as the sum of a finite number of K-curves?

Solutions of this problem have already been given for various classes K. If K is the class of n-dimensional curves, then C must be n-dimensional.[†] If K is the class of rational curves, so also must C be rational.[‡] If K is the class of continua which contain no continua of condensation, then C can contain no continua of condensation.[§] For no other of the principal classes K are the results so degenerate. Urysohn[¶] found that the connected sum of two perfect curves may fail to be a perfect curve.[**] More recently Whyburn[††] proved that the most general curve which, when added to a regular (or perfect) curve, always gives a regular (or perfect) curve as a sum, can contain no continum of condensation. In the same paper he gives an example of two acyclic curves whose connected sum is not a perfect curve.

We propose to give a solution of the above problem for each of the following five classes K of continuous curves:

(1) Perfect curves,
(2) Regular curves,

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[*] By a continuous curve is meant a compact, connected and locally connected, metric space.

[†] K. Menger, *Dimensionstheorie* (1928), p. 92.

[‡] P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, II, Verhand. Akad. v. Wet. te Amsterdam, Eerstie Sectie, vol. 13 (1928), No. 4, p. 23.

[§] S. Janiszewski, "Sur les continus irréductibles entre deux points," *Journal de l'École Polytechnique* (1912), pp. 79-170, Theorem VII.

[¶] P. Urysohn, *loc. cit.*, p. 46.

[**] A perfect curve is a continuous curve whose every sub-continuum is a continuous curve.

[††] G. T. Whyburn, "Concerning the addition of regular curves," *Monatshefte für Mathematik und Physik*, vol. 38 (1931), pp. 1-4.

(3) Baum im kleinen curves,[*]
(4) Boundary curves,[†]
(5) Acyclic curves.

In all that follows, unless stated to the contrary, the class K is to represent an arbitrary one of these five classes.

The set of all continuous curve which can be expressed as the sum of a finite number of K-curves may be considered as forming a class. Corresponding to each class that K may represent we obtain a new type of continuous curve. In the next section of this paper we characterize these new types, and in the concluding section we classify them with respect to the known types.

2. *The characterization.* Our principal results hinge on the notion of generalized derived aggregates. These sets were first defined and studied by Whyburn.[‡] For purposes of reference we state in detail the definition of these sets, and give two related theorems proved by Whyburn. For further results and applications we refer the reader to Whyburn's paper.

Generalized derived aggregates. Let C be any compact metric space, let A be a closed subset of C, and let K be any class of closed point sets. Then by the K-derivative $K(A)$ of A is meant the set of all points x of A such that for no neighborhood U of x is $A \cdot U$ contained in any K-set in C. Set

$$A_K{}^1 = K(A),\ A_K{}^2 = K(A_K{}^1), \cdots,\ A_K{}^n = K(A_K{}^{n-1}), \cdots,$$

in general, $A_K{}^\alpha = K(A_K{}^{\alpha-1})$ or $= \prod_{\beta<\alpha} K(A_K{}^\beta)$, according as the ordinal number α does or does not have an immediate predecessor. It follows that the derived aggregates are closed sets, and $A_K{}^\alpha \supset A_K{}^{\alpha+1}$.

THEOREM A. *In order that the compact metric space C should be the sum of a countable number of K-sets, where K is any given class of closed*

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[*] A baum im kleinen curve is a continuous curve such that each of its points possesses arbitrarily small neighborhoods whose closures are acyclic curves. These curves are characterized by the fact that they contain only a finite number of distinct simple closed curves. Cf. K. Menger, *Kurventheorie* (1932), p. 323.

[†] This name was suggested by G. T. Whyburn. See this Journal, vol. 56 (1934), p. 301. By a curve of this type we mean a continuous curve whose every true cyclic element is a simple closed curve. Such a curve is characterized by the fact that it is homeomorphic with the boundary of a plane domain. Cf. R. L. Wilder, "Concerning continuous curves," *Fundamenta Mathematicae*, vol. 7 (1925), pp. 340-377, Theorem 4. See also W. L. Ayres, "Continuous curves homeomorphic with the boundary of a plane domain," *Fundamenta Mathematicae*, vol. 14 (1929), pp. 92-95.

[‡] G. T. Whyburn, "On the decomposability of closed sets into a countable number of simple sets of various types," ***American Journal of Mathematics*, vol. 54 (1932),** pp. 169-175.

point sets, it is necessary and sufficient that $C_K{}^\beta = 0$ *for some ordinal* β *of the first or second class.*

THEOREM B. *In order that the compact metric space* C *should be the sum of* n *K-sets, where* n *is an integer, it is necessary that* $C_K{}^n = 0$.

Whyburn points out that the condition $C_K{}^n = 0$ is not in general sufficient for C to be the sum of n or any finite number of K-sets. We propose to show that, if C is a continuous curve, and K is the class 1, 2, 3, 4 or 5 of continuous curves, then the condition $C_K{}^n = 0$ is sufficient for C to be the sum of a finite number of K-curves.

We shall use the term *curve-set* to denote a closed set every component of which is a point or a continuous curve, and such that the curve components form a null family. A curve-set is said to be a *K-curve-set* if each of its components is a point or a K-curve; if K is the class of baum im kleinen curves, we require in addition that only a finite number of the components of the curve-set shall contain simple closed curves.

It has been shown by Zippin* that a 1-dimensional curve-set H, lying in a continuous curve C, may be imbedded in a continuous subcurve G of C which is irreducible about H. In every case a K-curve-set is 1-dimensional; so this theorem applies. It is a consequence of a theorem due to Gehman † that every true cyclic element ‡ of the irreducible continuous curve G is contained in H. Now the property of being a continuous curve of type 1, 2, 4 or 5 is cyclicly extensible.§ Thus, in these four cases, we infer that the irreducible continuous curve G is a K-curve. A baum im kleinen curve-set contains only a finite number of simple closed curves; this must also be true of a continuous curve irreducible about it. Thus, in every case, *a K-curve-set lying in a continuous curve* C *may be imbedded in a K-curve lying in* C.

A point set S is said to be an *open K-curve-set*, where K is the class 1, 2, 4 or 5, provided S is open relative to $\bar{S}$, every component of S is a point or a self-compact K-curve, and the curve components of S form a null family.

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* L. Zippin, "On continuous curves irreducible about subsets," *Fundamenta Mathematicae*, vol. 20 (1933), pp. 197-205.

† H. M. Gehman, "Concerning irreducible continua," *Proceedings of the National Academy of Sciences*, vol. 14 (1928), pp. 433-435.

‡ For an exposition of the theory of the cyclic elements of a continuous curve cf. C. Kuratowski and G. T. Whyburn, "Sur les éléments cycliques et leurs applications," *Fundamenta Mathematicae*, vol. 16 (1930), pp. 305-331.

§ The property P is said to be cyclicly extensible if, knowing that each cyclic element of the continuous curve C has property P, we are able to infer that C itself has property P. Cf. C. Kuratowski and G. T. Whyburn, "Sur les éléments cycliques . . . ," *loc. cit.*, pp. 322-323.

By *open baum im kleinen curve-set* we shall mean an open acyclic curve-set. It need hardly be remarked that $\bar{S}$ may fail to be a curve-set; or, if it is, it may fail to be a K-curve-set. We proceed to prove

LEMMA 1. *If X is a subset of a continuous curve C such that X is open relative to $\bar{X}$, and the K-derivative of $\bar{X}$ lies in $F(X)$,*[*] *where K is the class 1, 2, 3, 4 or 5, then X can be expressed as the sum of two open K-curve-sets which are closed in X.*

From the definition of baum im kleinen curve it follows that the derived aggregate of any set with respect to this class is identically the derived aggregate of the set with respect to the class of acyclic curves. So the proof for this case depends on that of the acyclic case.

Let P be a point of X. By hypothesis, there exists an open subset W of C containing P such that $W \cdot X$ lies in a K-curve K' which lies in C. Now K' is a rational curve.[†] Hence there exists an open subset R of K' containing P such that $W \supset \bar{R}$ and $F(R)$ is a countable set of points. Let U be an open subset of C containing P such that $W \supset U \supset R$ and $\bar{U} \cdot (K' - \bar{R}) = 0$.[‡] It follows that $X \cdot F(U)$ is a countable set of points. Thus it follows that, for a given positive ϵ and a point P of X, there exists an open subset U of C containing P of diameter $< \epsilon$ and such that $\bar{U} \cdot F(X) = 0$, $\bar{U} \cdot X$ lies in a K-curve in C, and $X \cdot F(U)$ is a countable set of points. We will call such an open set a U-region. We note that $\bar{U} \cdot X$ is a K-curve-set. Let $d_1 > d_2 > d_3 > \cdots$ be a sequence of positive numbers converging to zero. Let D_1 be the set of all points of X whose least distance d from $F(X)$ satisfies the inequality $d \geqq d_1$. In general, let D_i be the set of all points of X whose least distance d from $F(X)$ satisfies the inequality $d_{i-1} \geqq d \geqq d_i$. Then D_j $(j = 1, 2, \cdots)$ is a closed set, and may therefore be covered by a finite number of U-regions $U_1{}^j, U_2{}^j, \cdots, U^j{}_{k_j}$, all of diameter $< 1/j$. Let $T = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} X \cdot F(U_i{}^j)$. Then T is countable and closed in X. Any component of $X - T$ lies wholly in some region $U_i{}^j$, and therefore lies in a K-curve in C. If $X - T$ possessed infinitely many components of diameter greater than some positive ϵ, it would follow that infinitely many of them lie interior to some region $U_i{}^j$. But $X \cdot U_i{}^j$ lies in a K-curve; so this is impossible. Hence the components of $X - T$ form a null family. Let $P_1, P_2, P_3, \cdots$ be the points of T. There exists a U-region U_1 containing P_1 of diameter < 1. Since T is 0-dimensional, there exists a neighborhood V_1 of P_1 such that $U_1 \supset \bar{V}_1$ and $T \cdot F(V_1) = 0$. Then $X \cdot \bar{V}_1$ is a K-curve-set having no component of diameter > 1. Let P_{n_2} be the point of lowest subscript of T not contained in V_1. There exists a U-region U_2 containing P_{n_2} of diameter $< 1/2$, and such that $U_2 \cdot V_1 = 0$. And there exists a neighborhood V_2 of P_{n_2} such that $U_2 \supset \bar{V}_2$ and $T \cdot F(V_2) = 0$. Then $X \cdot \bar{V}_2$ is a K-curve-set having no component of diameter $> 1/2$. We assume as defined the regions $V_1, V_2, \cdots, V_{k-1}$. Let P_{n_k} be the point of lowest subscript of T not contained in $\sum_1^{k-1} V_i$. There exists a U-region U_k containing P_{n_k} of diameter $< 1/k$, and such that $U_k \cdot \sum_1^{k-1} V_i = 0$. And there exists a neighborhood V_k of P_{n_k} such that $U_k \supset \bar{V}_k$ and $T \cdot F(V_k) = 0$. Then $X \cdot \bar{V}_k$ is a K-curve-set having no component of diameter $> 1/k$. Set

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[*] $F(X) = \bar{X} - X$.

[†] That a perfect curve is a rational curve cf. G. T. Whyburn, "Concerning hereditarily locally connected continua," *American Journal of Mathematics*, vol. 53 (1931), pp. 374-384. Note also, on page 377, that the components of any subset of a perfect curve form a null family. We use this fact in the sequel.

[‡] For the existence of such an open set see, for example, K. Menger, *Dimensionstheorie*, p. 31.

$$S_1 = \sum_1^{\infty} X \cdot \bar{V}_i, \text{ and } S_2 = X - \sum_1^{\infty} X \cdot V_i.$$

Since $\bar{V}_i \cdot \bar{V}_j = 0$ $(i \neq j)$, the components of S_1 are points or self-compact K-curves, and they form a null family. Furthermore S_1 is closed in X. For, if P were a limit point of S_1 in X but not contained in S_1, P would be a limit point of T since the diameters of the regions V_i converge to zero. But T is closed in X, and S_1 contains T; so this is impossible. Thus S_1 is an open K-curve-set which is closed in X. It is obvious that S_2 is closed in X. Since $T \cdot S_2 = 0$, every component of S_2 lies in a component of $X - T$. From this it readily follows that the components of S_2 form a null family, and each such component is a point or a self-compact K-curve. Evidently $X = S_1 + S_2$, and the lemma is proved.

It should be observed that the argument of Lemma 1 still holds even though the set X is closed. Should this be the case, the sets S_1 and S_2, being closed, will be K-curve-sets.

We have need of one more lemma before proceeding to the proof of our principal theorem.

LEMMA 2. *Let B be a closed subset of a continuous curve C; let $a_1, a_2, \cdots, a_h$ be a finite number of closed subsets of B, and let N denote their sum. Then B can be expressed as the sum of h closed sets $A_1, A_2, \cdots, A_h$ such that, for every i, $N \cdot A_i = a_i$.*

Let $b_1 = \sum_{j=2}^{h} a_j$. Let U be an open subset of C such that $U \supset (b_1 - a_1 \cdot b_1)$ and $\bar{U} \cdot (a_1 - a_1 \cdot b_1) = 0$. Set $A_1 = B - U \cdot B$, and $B_1 = b_1 + \bar{U} \cdot B$. Then $B = A_1 + B_1$, $N \cdot A_1 = a_1$, and $N \cdot B_1 = b_1$. Let $b_2 = \sum_{j=3}^{h} a_j$. We repeat the argument and express B_1 as the sum of two closed sets A_2 and B_2 such that $(a_2 + b_2) \cdot A_2 = a_2$ and $(a_2 + b_2) \cdot B_2 = b_2$. It readily follows that $N \cdot A_2 = a_2$ and $N \cdot B_2 = b_2$. Continue in this manner. At the $(h-1)$st step we shall have defined two closed sets A_{h-1} and B_{h-1} such that $B_{h-2} = A_{h-1} + B_{h-1}$, $N \cdot A_{h-1} = a_{h-1}$, and $N \cdot B_{h-1} = b_{h-1} = a_h$. Now set $A_h = B_{h-1}$ and the lemma follows as stated.

We are now prepared to state and prove our principal theorem.

THEOREM 1. *Let K be a class of continuous curves, and let C be a continuous curve such that $C_K^n = 0$ where n is an integer. If K is the class of perfect, regular, or baum im kleinen curves, then C may be expressed as the sum of 2^{n-1} K-curves. If K is the class of acyclic, or boundary curves, then C may be expressed as the sum of 2^n K-curves; and if, as a special case, C_K^{n-1} is a K-curve-set, then C may be expressed as the sum of 2^{n-1} K-curves.*

We may suppose that n is the least integer such that $C_K^n = 0$. If K is the class of perfect, regular, or baum im kleinen curves, then C_K^{n-1} is a K-curve-set. For, if we assumed the contrary, we would find that C_K^{n-1} contains a point P possessing no neighborhood U such that $U \cdot C_K^{n-1}$ can be imbedded in any K-curve whatsoever. But this contradicts the fact that the K-derivative of C_K^{n-1} is vacuous. Thus, in these three cases, C_K^{n-1} may be imbedded in a K-curve lying in C. If K is the class of acyclic or boundary curves, C_K^{n-1} may fail to be a K-curve-set. We then set $X = C_K^{n-1}$; from Lemma 1 we find that C_K^{n-1} may be expressed as the sum of two K-curve-sets. Then, for these two cases, C_K^{n-1} may be imbedded in the sum of two K-curves lying in C.

We now assume that C_K^{i+1} $(i + 1 \leqq n - 1)$ may be imbedded in the sum of a finite number h of K-curves lying in C. Let these curves be a_1, $a_2, \cdots, a_h$, and let N denote their sum. Then N contains the K-derivative C_K^{i+1} of C_K^i. Set $X = C_K^i - N \cdot C_K^i$. Then X is open relative to $\bar{X}$. Furthermore the K-derivative of $\bar{X}$ lies in C_K^{i+1}, therefore in N, and finally in $F(X)$. By Lemma 1, X can be expressed as the sum of two open K-curve-sets S_1 and S_2 which are closed in X. Evidently $B = N + S_1$ is a closed set, and $a_1, a_2, \cdots, a_h$ are closed subsets. By Lemma 2, B may be expressed as the sum of h closed sets $A_1, A_2, \cdots, A_h$ such that, for every i, $N \cdot A_i = a_i$.

Then every component of A_i is either a_i or a closed connected subset of a component of S_1. Hence the components of A_i are points or K-curves, and they form a null family. If K is the class of baum im kleinen curves, we note that a_i is the only component of A_i which can contain a simple closed curve. Thus, in every case, A_i is a K-curve-set, and may, therefore, be imbedded in a K-curve lying in C. It follows that $N + S_1$ may be imbedded in the sum of h K-curves lying in C. The same being true of $N + S_2$, we obtain finally that $C_K{}^i$, which is contained in $N + S_1 + S_2$, may be imbedded in the sum of $2h$ K-curves lying in C. Having established the initial and the general step in the induction, the proof of Theorem 1 is complete.

Theorem 1 together with Whyburn's Theorem B gives us the characterizations we have been seeking.

THEOREM 2. *In order that the continuous curve C should be the sum of a finite number of K-curves, where K is the class of perfect, regular, baum im kleinen, boundary, or acyclic curves, it is necessary and sufficient that there should exist an integer n such that* $C_K{}^n = 0$.

3. *The classification.* A continuous curve which is the sum of a finite number of K-curves we shall call a *finite K-curve-sum.* Since, in the five cases considered, a K-curve is also a rational curve, we have, by the theorem of Urysohn, mentioned in the introduction, that a finite K-curve-sum is always a rational curve. Furthermore, this is the most that we can say; for, as noted in the introduction, the sum of two acyclic curves, and, consequently, the sum of two K-curves may fail to be a perfect curve.

There exist just two inclusion relations which are not immediate consequences of the definitions of these new classes.

THEOREM 3. *Every continuum containing no continuum of condensation may be expressed as the sum of two acyclic curves.*

By a theorem due to Urysohn,* a continuum containing no continuum of condensation is the sum of a closed 0-dimensional set F and a countable set of free open arcs which form a null family. Evidently the acyclic curve derivative of such a continuum is a subset of the 0-dimensional set F. But F is an acyclic curve-set. Thus the second derived aggregate is vacuous, and the first derived aggregate is an acyclic curve-set. This is the exceptional case noted in Theorem 1.

It follows from the definition of baum im kleinen curve that the acyclic curve derivative of such a continuous curve is vacuous. Thus we have

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* P. Urysohn, *loc. cit.*, p. 57.

THEOREM 4. *Every baum im kleinen curve may be expressed as the sum of two acyclic curves.*

It follows from Theorem 4 that two of these five classes of finite K-curve-sums are identical. That is to say: *finite acyclic curve-sums and finite baum im kleinen curve-sums form identical classes.*

It might be thought, in view of Theorems 3 and 4, that it would always be possible to express an arbitrary rational, perfect, or regular curve as the finite or countable sum of a number of K-curves of a more restricted type than itself. This is not true, and we establish the fact by a series of examples. The constructions used in these examples are all similar, and are special cases of a general construction. This general construction follows.

We suppose given a 1-dimensional cyclicly connected continuous curve β. In every case the curve β will have all the properties necessary for making the construction. We start with a 2-dimensional sphere S_0 lying in a 3-dimensional Euclidean space. Let I_0 be the interior of S_0, and P_0 a point of S_0. Set $C_0 = S_0 + I_0$. As a first step, we construct a curve β_1 similar to β (i. e. homeomorphic with β, and distances altered proportionately) which has only P_0 in common with S_0, and otherwise lies wholly in I_0. We now construct a countable set of spheres $S_1{}^1$, $S_1{}^2$, $S_1{}^3, \cdots$ such that (1) $S_1{}^i$ and $S_1{}^j$ $(i \neq j)$ have no points or interior points in common, (2) $S_1{}^i$ has exactly one point $P_1{}^i$ in common with β_1, and has no point of β_1 in its interior, (3) $S_1{}^i$ lies wholly in I_0, and has a diameter $< 1/2$ the diameter of S_0, (4) the point set $\Sigma P_1{}^i$ is everywhere dense on β_1. We note that the diameters of the spheres $S_1{}^i$ necessarily converge to zero. Let $I_1{}^i$ denote the interior of $S_1{}^i$. Now set

$$C_1 = \beta_1 + \sum_i (S_1{}^i + I_1{}^i).$$

This completes the first step. As the second step, we perform, for each i, the same construction on $S_1{}^i$ with respect to the point $P_1{}^i$ as we performed on S_0 with respect to the point P_0. Let $\beta_2{}^i$ be the curve similar to β which lies interior to $S_1{}^i$ save for the point $P_1{}^i$. Let $S_2{}^{ij}$ $(j = 1, 2, \cdots)$ denote one of the spheres lying in the interior of $S_1{}^i$, and let $P_2{}^{ij}$ denote its point of contact with $\beta_2{}^i$. Let $I_2{}^{ij}$ denote the interior of $S_2{}^{ij}$. Then set

$$C_2 = \beta_1 + \sum_i \beta_2{}^i + \sum_{i,j} (S_2{}^{ij} + I_2{}^{ij}).$$

This completes the second step. The general step is now quite obvious. In this manner we define a sequence of continua C_0, C_1, $C_2, \cdots$ which converge to a continuum C. For a given $\epsilon > 0$, there exists an integer n such that no sphere $S_n{}^{i \cdots k}$ of C_n has a diameter $> \epsilon$. From this fact two propositions follow (1) C is a continuous curve, (2) every true cyclic element of C is a curve similar to β. Finally we remark that any neighborhood of any point of C contains a true cyclic element of C.

Now let α be a class of continuous curves no one of which contains a subset homeomorphic with β. From the last remark of the preceding paragraph it follows that the α-derivative of C must be C itself. In view of Theorem A, it follows that C is not the sum of any countable number of α-curves.

There exists a regular curve which is the boundary of a plane domain but which is not the sum of any countable number of acyclic curves.

Let α be the class of acyclic curves. Let the curve β be a circle. In this case C will be a boundary curve and therefore homeomorphic with the boundary of a plane domain.

There exists a regular curve which is not the sum of any countable number of boundary curves.

Let α be the class of boundary curves. Let the curve β be a circle plus one of its diameters. C will be a regular curve since each of its true cyclic elements is a regular curve.

There exists a perfect curve which is not the sum of any countable number of regular curves.

Let α be the class of regular curves. We shall let β be a perfect curve, as the one constructed by Whyburn,* which contains infinitely many mutually exclusive continua of diameter unity. β is therefore not a regular curve. An application of Theorem 1 to Whyburn's example shows that it is the sum of two acyclic curves.

There exists a rational curve which is not the sum of any countable number of perfect curves.

Let α be the class of perfect curves. Let β be the rational curve, mentioned in the introduction, which is the sum of two acyclic curves, but which is not a perfect curve. C will be a rational curve since each of its true cyclic elements is a rational curve.

It will be noticed that in every case we chose the curve β so that it would be the sum of two acyclic curves, and therefore the sum of two curves of type α. Hence these examples suffice to establish the following proposition:

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* G. T. Whyburn, "Concerning points of continuous curves defined by certain im kleinen properties," *Mathematische Annalen*, vol. 102 (1929), pp. 313-336, article 7.

The property of being the sum of a finite or countable number of K-curves, where K is the class of perfect, regular, baum im kleinen, boundary, or acyclic curves, is not cyclicly extensible.

4. *Conclusion.* It is of interest to note that, if C is a continuous curve, we have proved the sufficiency of the condition $C_K^n = 0$ for all the principal classes K of continuous curves for which the condition is sufficient. While the condition is sufficient for the classes K of rational and n-dimensional curves, the proof is trivial. For, if some countable derived aggregate of C with respect to the class of rational curves is vacuous, it follows, from Theorem A, that C is the sum of a countable number of rational curves. Then C is itself a rational curve. A similar argument holds if K is the class of n-dimensional curves.

For the classes K of simple arcs, simple closed curves, node curves,* and continua containing no continua of condensation, the condition $C_K^n = 0$ is not sufficient for the continuous curve C to be the sum of a finite number of K-curves. To see that this is true, we consider a few examples.

Construct a continuous curve C by adding on to the unit interval a perpendicular ordinate of length $1/q$ for each point p/q where p and q are relatively prime integers and $p < q$. If K is the class of simple arcs or continua containing no continua of condensation, then $C_K^2 = 0$. But C contains a continuum of condensation, and is therefore not the sum of any finite number of K-curves.

To the unit interval plus the line segment $y = x$ $(0 \leqq x \leqq 1)$ add on a sequence of line segments joining the point $(1/q, 0)$ to the point $(1/q, 1/q)$ for every positive integer q. Let C be the resulting continuous curve, and let K be the class of simple closed curves. Then $C_K^2 = 0$; but no simple closed curve of C can contain more than two of the perpendicular line segments, so C is not the sum of any finite number of simple closed curves.

For the case of the node curve, we shall start with a cyclicly connected plane regular curve of order 3 recently constructed by Whyburn.† Let M denote this curve. It is evident that no ramification point of M is an im

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* A continuous curve M is a node curve if, for each $\epsilon > 0$, M is the sum of a finite number of ϵ-continua each having at most two points in common with the rest of M. Cf. G. T. Whyburn, "Concerning points of continuous curves defined by certain im kleinen properties," *loc. cit.*, article 5. Note, in particular, Theorems 20, 22, and 29 which we use in the sequel.

† G. T. Whyburn, "On the existence of totally imperfect and punctiform connected subsets in a given continuum," *American Journal of Mathematics*, vol. 55 (1933), pp. 146-152.

kleinen cycle point* of M. Thus M is a node curve. Also the im kleinen cycle points of M are dense in M. Let us select a countable dense set $P_1, P_2, P_3, \cdots$ of these im kleinen cycle points. Construct, for every integer n, perpendicular to the plane of M, a line segment of length $1/n$ with one end at P_n. Let C be the resulting continuous curve. Then $C_K{}^2 = 0$ if K is the class of node curves. Suppose C is the sum of a finite number of node curves $N_1, \cdots, N_n$. Then there exists a point P of C lying in M, an open subset U of C containing P, and an integer i such that $N_i \supset U$. But this is impossible since C is not a node curve im kleinen at any point of M.

Finally it should be remarked that the numbers 2^{n-1} and 2^n, obtained in Theorem 1, are not the least numbers which would in general suffice for the purposes of the theorem. In view of Theorem B, it is apparent that this least integer is not less than n. If K is the class of acyclic or boundary curves, examples may be found to show that this least integer is not less than $n + 1$. It seems quite possible that Theorem 1 would still be true if the numbers 2^{n-1} and 2^n were replaced by n and $n + 1$, respectively.

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* A point P of the continuous curve M is said to be an im kleinen cycle point of M if, for each $\epsilon > 0$, P lies on a simple closed curve of M of diameter $< \epsilon$.

PROPERTIES OF PLANE SETS AND FUNCTIONS OF TWO VARIABLES.

By DEANE MONTGOMERY.*

Recently mathematicians have been interested in a certain operation on plane sets and in the types of sets which may be obtained by means of its use.† This operation, which is an extension of the concept of projection, may be defined as follows. If E is any plane set and if p is any property of linear sets, $\Gamma_p(E)$ is defined to be the collection of all points on the x-axis such that vertical lines through these points cut E in a set having property p. For various types of sets E and various properties p, it is an interesting problem to determine the nature of the set $\Gamma_p(E)$. This is done in the present paper for certain properties additional to the ones already considered, and a condition is found under which the projection of a Borel set is a Borel set. A similar operation on functions, which has so far not been used, is also considered. If $f(x, y)$ is any function of two variables and p is any property of functions of one variable, $\Gamma_p(f)$ is defined to be the set of all points $\bar{x}$ on the x-axis such that $f(\bar{x}, y)$ has property p. When $f(x, y)$ is in the Baire classification the nature of the set $\Gamma_p(f)$ is determined for several different properties p among which may be mentioned the property of being Riemann integrable, the property of having limited variation, and the property of being of class one. Some of the methods are applied to obtain a result concerning the measurability of functions used by Tonelli in his definition of bounded variation of functions of two variables.

Instead of defining these operations on plane sets and on functions defined in the plane one might define them on a set E which is a subset of the combinatorial product of two metric spaces suitably restricted and on functions defined on such a product space. Many of the theorems which retain meaning in this general case could be proved essentially as they are here. A generalization in a different direction could be made by considering functions whose values are points in a metric or vector space. In this case also some of the present proofs would be adequate.

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† The references given here, and the articles to which they in turn refer, include most of the known results on this type of operation. Sierpinski, *Mathematica*, vol. 5 (1931), p. 49; Kuratowski and Szpilrajn, *Fundamenta Mathematicae*, vol. 18 (1932), p. 160; Hahn, *Reelle Funktionen* I (1932), p. 368; Kuratowski, *Topologie* I, p. 262. The notation used here is due to Sierpinski.

2. The property p in this section is the property of being of the second category.[*] In some of the proofs it is convenient to use the symbol $A \times B$ as indicating all points in the plane whose abscissas are in A and whose ordinates are in B. The set A is occasionally a single point. The symbol $(x = a)$ means all points lying on the line $x = a$.

THEOREM 1. *If E is open, $\Gamma_p(E)$ is open and if E is closed, $\Gamma_p(E)$ is an F_σ.*

When E is open, $\Gamma_p(E)$ is merely the projection of E and this is open. If E is closed, $(x = q) \cdot E$ is of the second category when and only when it contains an interval. From Theorem 9 of a previous paper,[†] it can be deduced that the points of E which lie on vertical intervals containing only points of E form an F_σ. Since the projection of an F_σ is an F_σ, the theorem follows.

THEOREM 2. *If E is an O_α, $\Gamma_p(E)$ is an O_α; and if E is an F_α, $\Gamma_p(E)$ is an $O_{\alpha+1}$.*[‡]

That the theorem is true when α is zero follows from the preceding theorem. The proof for the general case is by transfinite induction. There are two cases to consider and in each case the theorem is assumed for ordinals less than α and on that basis is proved for α. The first case is that in which α is of the first kind.

If E is an O_α, $E = \sum\limits_n E_n$ where E_n is an $F_{\alpha-1}$. By assumption, $\Gamma_p(E_n)$ is an O_α, and from the fact that $\Gamma_p(E) = \sum\limits_n \Gamma_p(E_n)$, it follows that $\Gamma_p(E)$ is an O_α.

If E is an F_α, CE is an O_α. Let all open intervals on the y-axis whose end points are rational be arranged in a sequence $\{d_m\}$ and let R_m be the set of all the points in the plane whose ordinates lie in the interval d_m. Let H_m be the complement of $\Gamma_p[R_m \cdot (CE)]$, the set H_m being therefore an F_α. For each point of q of H_m, $(CE) \cdot (q \times d_m)$ is of the first category so that $E \cdot (q \times d_m)$ is of the second category. Hence $\sum\limits_m H_m < \Gamma_p(E)$, and it will now be shown that $\Gamma_p(E) < \sum\limits_m H_m$. For this purpose let q be any point in

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[*] The term "second category" is not as descriptive as the term "inexhaustible" suggested by Denjoy, but because of its very general usage the first term is preferable.

[†] *Transactions of the American Mathematical Society*, vol. 35 (1933), p. 915.

[‡] For a discussion of Sets O_α and F_α see de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 132-139. It should be noted that by a set of a given class de la Vallée Poussin means a set which is at most of this class. The same convention is used in this paper, and a similar convention is followed with regard to the class of a function.

$\Gamma_p(E)$. The set E must then be uniformly of the second category[*] in some interval on $x = q$. There is then some d_n such that $E \cdot (q \times d_n)$ is of the second category uniformly in $(q \times d_n)$. Since every Borel set possesses the property of Baire,[†] the set CE must be of the first category in $q \times d_n$. Therefore q is in H_n and $\Gamma_p(E) < \sum_m H_m$. It follows that $\Gamma_p(E) = \sum_m H_m$ and that $\Gamma_p(E)$ is an $O_{\alpha+1}$. This concludes the proof in case α is of the first kind.

Suppose now that α is of the second kind and that the theorem is true for all ordinals less than α. According to de la Vallée Poussin,[‡] sets F_α and sets O_α, where α is of the second kind, are constructed as follows. From sets A already defined,[§] form all sets $A' = \Sigma A$ and all sets $A'' = \Pi A$, the sum and product being taken over all enumerable collections of sets A. Then

$$O_\alpha = \Sigma A_\alpha'' \quad \text{and} \quad F_\alpha = \Pi A'_\alpha.$$

Let Q be any A'_α. Since an A_β is also an F_β, $Q = \sum_n Q_n$ where Q_n is an F_{β_n} $(\beta_n < \alpha)$. Since $\Gamma_p(Q) = \sum_n \Gamma_p(Q_n)$, and since $\Gamma_p(Q_n)$ is an O_{β_n+1} and an F_{β_n+2}, it follows that $\Gamma_p(Q)$ is a set A'_α. By taking complements in a manner analogous to the one already used it can be shown that in case Q is an A_α'' that $\Gamma_p(Q) = \Sigma A_\alpha''$, that is, it is an O_α. Now if E is any O_α, $E = \sum_n E_n$ where E_n is an A''_α. Since $\Gamma_p(E_n)$ is an O_α, and since $\Gamma_p(E) = \sum_n \Gamma_p(E_n)$ it follows that $\Gamma_p(E)$ is an O_α. By taking complements as before it can be shown that when E is an F_α, that $\Gamma_p(E)$ is an $O_{\alpha+1}$. This completes the proof of Theorem 2.

If T is any linear O_α, there is a plane set E which is an O_α and for which $\Gamma_p(E) = T$. In order to obtain such a set E, it is only necessary to take all the points on all vertical lines passing through points of T.

If T is any linear $O_{\alpha+1}$ there is a plane set E which is an F_α and for which $\Gamma_p(E) = T$. To demonstrate this fact let i_n be the closed interval on the y-axis whose end points are $n + 1/4$ and $n + 3/4$. Since T is an $O_{\alpha+1}$, $T = \sum_n T_n$ where T_n is an F_α. A set E having the desired properties is the set $\sum_n (T_n \times i_n)$.

The above examples show that the determination of $\Gamma_p(E)$ given in the theorem is the best [*] determination that can be given. It would be interesting to know the best determination of $\Gamma_p(E)$ for analytic sets and their complements. The following theorems give a determination of $\Gamma_p(E)$ in these cases but it is not known whether or not this determination is the best one.

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[*] That is, of the second category at every point of the interval. See Banach, *Théorie des operationes linéaires*, p. 13.

[†] A set E is said to possess the property of Baire if there is no perfect set in which both E and CE are uniformily of the second category. For a proof that Borel and analytic sets possess this property see Lusin, *Leçons sur les ensembles analytiques*, p. 88 and p. 153.

[‡] See the previous reference.

[§] An A_α is a set which is both an F_α and an O_α.

In some of the remaining proofs it is convenient to use the method of evaluating classes which has been invented by Kuratowski and Tarski.[†] This method makes use of the logical symbols $\prod_x$ meaning "for all x"; $\sum_x$ meaning "there exists an x"; $'$ meaning "not." Where a set is defined in terms of these symbols one may obtain the class of the set by replacing these logical symbols by certain other symbols which operate on sets. Thus (where P means projection and C means complement) $\prod_x$ is replaced by CPC; $\sum_x$ is replaced by P; and $'$ is replaced by C. In case n runs over an enumerable set, $\sum_n$ is replaced by σ and $\prod_n$ by δ, σ and δ meaning the sum and product of an enumerable number of sets. The symbols $+$ and $\cdot$ have their usual logical meaning and replace themselves. In this symbolism the inclusion of A in B is written $[(x \epsilon A)' + (x \epsilon B)]$, ϵ denoting the inclusion of an element in a class. For brevity, an analytic set is written as A, the complement of such a set as CA, and so on. The letter A in this sense should not be confused with the A used previously which referred to the sets A of de la Vallée Poussin.

THEOREM 3. *If E is a CA, $\Gamma_p(E)$ is a PCA.*

Let U be a universal G_δ,[‡] that is a plane G_δ such that for any linear G_δ there is a q such that $U \cdot (x = q)$ is the given linear G_δ. If $E \cdot (x = q)$ is of the second category, it must contain a G_δ of the second category.[§] The formula for the set $\Gamma_p(E)$ may now be written.

$$x \epsilon \Gamma_p(E) \equiv \sum_{x_1} \{x_1 \epsilon \Gamma_p(U)\} \{ \prod_{y} [\overline{(x_1, y) \epsilon U}' + \overline{(x, y) \epsilon E}]\}.$$

Therefore $\Gamma_p(E) = P[G_{\delta\sigma}]\,[CPC(CG_\delta + CA)]$. A set which is a $CG_\delta + CA$ is a CA. The complement of a CA is an A, and the formula is thus reduced to $P[G_{\delta\sigma}]\,[CPA]$. The projection of an A is an A and a $G_{\delta\sigma}$ multiplied by a CA is a CA. Therefore $\Gamma_p(E)$ is a PCA.

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[*] The word "best" is here used as by Kuratowski to mean that $\Gamma_p(E)$ has been shown to belong to as simple a class as possible.

[†] See Kuratowski and Tarski, *Fundamenta Mathematicae*, vol. 17 (1931), p. 240, and Kuratowski, *Fundamenta Mathematicae*, vol. 17 (1931), p. 249; also Kuratowski *Topologie* I.

[‡] For the existence of universal sets see Sierpinski, *Fundamenta Mathematicae* vol. 14 (1929), pp. 88, 89.

[§] See Sierpinski, *Fundamenta Mathematicae*, vol. 4, p. 319.

By taking complements in a method analogous to one already used, it can be shown that if E is analytic, $\Gamma_p(E)$ is a $CPCA$.

THEOREM 4. *If E has the property of Baire, $\Gamma_p(E)$ has the property of Baire.*

The proof is by contradiction. If the theorem is not true there is a perfect set H on the x-axis on which $\Gamma_p(E)$ and $C\Gamma_p(E)$ are both uniformly of the second category. Let q be any point of $H \cdot \Gamma_p(E)$. Since $E \cdot (x = q)$ is of the second category it must be uniformly of the second category in some vertical interval d_q. Let K be the set of points on all such vertical intervals. By Lemma 2 of my previously cited paper, there exists a vertical interval d and a set M_1, uniformly of the second category in H_1, a portion of H, such that $M_1 \times d < K$. It follows that E is uniformly of the second category in $H_1 \times d$.[*] But $H_1 \cdot C\Gamma_p(E)$ is also uniformly of the second category in H_1, and therefore $[H_1 \cdot C\Gamma_p(E)] \times d$ is uniformly of the second category in $H_1 \times d$. For each $q \in C\Gamma_p(E)$, $E \cdot (q \times d)$ is of the first category; and therefore $(CE) \cdot (q \times d)$ is of the second category (in $q \times d$). Hence the set CE is uniformly of the second category in $H_1 \times d$. The fact that both E and CE are uniformly of the second category in $H_1 \times d$, a perfect set, is a contradiction to the hypothesis that E possessed the property of Baire, and the theorem follows.

3. In this section p_r is the property of having measure greater than or equal to r, p'_r is the property of having measure greater than r and p is the property of having positive measure. In some of the following theorems it is assumed that E is bounded and for convenience it will be assumed that E is in the square S whose corners are $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.

THEOREM 5. *If E is closed and in S, $\Gamma_{p_r}(E)$ is closed.*[†]

Let x_m be a sequence of points in $\Gamma_{p_r}(E)$ approaching a point x_0 and suppose that x_0 is not in $\Gamma_{p_r}(E)$. Since x_0 is not in $\Gamma_{p_r}(E)$ the set $E \cdot (x = x_0)$ is in a set $x_0 \times O$ where O is a set of points lying on open intervals of total length less than r. Since $E \cdot (x = x_n)$ has measure greater than or equal to r, there must be a point of E, say q_n, which is in $E \cdot (x = x_n)$ but which is not in $x_n \times O$. The set $\{q_n\}$ must have a limit point q on $x = x_0$. The point q is in E because E is closed but this point cannot be in $x_0 \times O$. From this contradiction the theorem follows.

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[*] See Kuratowski and Ulam, *Fundamenta Mathematicae*, vol. 19 (1932), p. 249.

[†] For a similar theorem concerning content see Hobson, *Theory of Functions of a Real Variable* I, Third Edition, p. 198.

THEOREM 6. *If E is open and in S, $\Gamma_{p'_r}(E)$ is open.*

The set $\Gamma_{p_{1-r}}(S-E)$ is closed. If x_0 is any point in this set, then $m[(S-E)\cdot(x=x_0)] \geq 1-r$. Hence $m[E\cdot(x=x_0)] \leq r$. If x_0 is in $C\Gamma_{p_{1-r}}(S-E)$, then $m[(S-E)\cdot(x=x_0)] < 1-r$, and $m[E\cdot(x=x_0)] > r$. Therefore $\Gamma_{p'_r}(E) = C\Gamma_{p_{1-r}}(S-E)$ and the theorem follows.

THEOREM 7. *If E is an F_α and in S, then $\Gamma_{p_r}(E)$ is an F_α; if E is an O_α and in S, then $\Gamma_{p'_r}(E)$ is an O_α.*

The proof of the theorem when α is zero follows from Theorems 5 and 6. The proof in the general case is by transfinite induction. Assume first that α is of the first kind and that the theorem is true for all ordinals less than α. If E is an O_α, $E = \sum_n E_n$ where E_n is an $F_{\alpha-1}$ and where for convenience the sequence of E_n's has been chosen as monotonic and increasing. If q is any point of $\Gamma_{p'_r}(E)$, q must be in $\Gamma_{p_r}(E_n)$ for a sufficiently large n. Therefore $\Gamma_{p'_r}(E) = \sum_n \Gamma_{p_r}(E_n)$, and from the fact that $\Gamma_{p_r}(E_n)$ is an $F_{\alpha-1}$ it follows that $\Gamma_{p'_r}(E)$ is an O_α. If E is an F_α, $\Gamma_{p_r}(E) = C\Gamma_{p'_{1-r}}(S-E)$ which proves that $\Gamma_{p_r}(E)$ is an F_α. The proof in case α is of the second kind is omitted because it can be made by a combination of the methods already given.

COROLLARY 1. *If E is an F_α, $\Gamma_p(E)$ is an $O_{\alpha+1}$; if E is an O_α, $\Gamma_p(E)$ is an O_α.*

If E is an F_α, divide the plane into an enumerable number of squares S_n whose edges are parallel to the axes and have length 1. Let $\{r_i\}$ be a sequence of positive numbers approaching zero. Then $\Gamma_p(E) = \sum_i \sum_n \Gamma_{p_{r_i}}(E\cdot S_n)$. Since $\Gamma_{p_{r_i}}(E\cdot S_n)$ is an F_α, the corollary follows. The proof in case E is an O_α is similar.

For the statement of the next two corollaries, let p_1 be the property of being of the second category and p_2 be the property of having positive measure. In general the projection of a Borel set is not a Borel set, but Corollary 2 gives a condition under which this is true.

COROLLARY 2. *If E is an F_α (O_α) such that every point of E lies on a vertical line on which E has either property p_1 or property p_2, then the projection of E is an $O_{\alpha+1}$ (O_α).*

This follows from the fact that the projection of E equals $\Gamma_{p_1}(E) + \Gamma_{p_2}(E)$. The conclusion of the corollary obviously remains the same if the vertical lines which cut E, but on which E does not have property p_1 or p_2, cut the x-axis in an $O_{\alpha+1}$ (O_α). This corollary has an application to the theory of implicit functions.* If $F(x, y)$ is a Baire function, $F(x, y) = 0$ defines y implicitly as a function of x. The field of definition of the implicit function is the projection of $E[F(x, y) = 0]$, this projection being in general not a Borel set.

COROLLARY 3. *If $F(x, y)$ is a function of class α such that every point of $E[F(x, y) = 0]$ is on a vertical line on which $E[F(x, y) = 0]$ possesses either property p_1 or property p_2, then the field of definition of the implicit function is an $O_{\alpha+1}$.*

THEOREM 8. *If E is a CA, $\Gamma_p(E)$ is a PCA.*

Every linear set of positive measure contains a closed set of positive measure, and therefore to make the proof, choose a universal closed set and proceed in a manner analogous to the proof of Theorem 3. Similar theorems can be proved for the properties p_r and p'_r.

If a set is of measure zero it is included in a G_δ of measure zero. Therefore if U is a universal G_δ, we have for any analytic set E,

$$x \,\epsilon\, C\,\Gamma_p(E) \equiv \sum_{x_1} [x_1 \,\epsilon\, C\,\Gamma_p(U)] \{ \prod_{y} [\overline{(x, y) \,\epsilon\, E} + \overline{(x_1, y) \,\epsilon\, U}]\}.$$

Therefore $C\,\Gamma_p(E)$ is a PCA, and $\Gamma_p(E)$ is a $CPCA$.

In case E is a set having closed vertical sections, it is possible to draw a somewhat stronger conclusion. The reasoning is based on the lemma given below.

LEMMA 1. *There exists an enumerable family of open sets $\{O_n\}$ each O_n of measure less than r such that any closed set of measure r is in some one of them.*

In order to define the family $\{O_n\}$ let $\{d_k\}$ be the set of all open intervals having rational end points. An O_n is obtained by adding together a finite number of d_k's of total length less than r and all O_n's may be obtained in this manner. This follows from the Heine-Borel theorem.

THEOREM 9. *If E is an analytic set having closed vertical sections, then $\Gamma_{p_r}(E)$ is an analytic set.*

The proof is obtained from the formula below.

$$x \,\epsilon\, \Gamma_{p_r}(E) \equiv \prod_{n} \sum_{y} [(x, y) \,\epsilon\, E]\, [(y \,\epsilon\, O_n)'].$$

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* For the theory of implicit functions see Lusin, *loc. cit.*, p. 222.

4. In this section p is the property of being everywhere dense and p' is the property of being nowhere dense.

THEOREM 10. *If E is any set, $\Gamma_p(E) = [P(O \cdot E)]_\delta$ where O is the symbol for an open set.*

Let $\{d_k\}$ be the set of all open intervals with rational end points.*

$$x \epsilon \Gamma_p(E) = \prod_k \sum_y [y \epsilon d_k]\,[(x, y) \epsilon E].$$

On application to special cases, this theorem yields such results as: if E is analytic, $\Gamma_p(E)$ is analytic; and if E is a CA, $\Gamma_p(E)$ is a PCA. Examples are easily constructed to show that these results are the best obtainable. By a slight variation of the above formula, it can be shown that if E is a Borel set or an analytic set, that $\Gamma_{p'}(E)$ is a CA, and that if E is a CA, $\Gamma_{p'}(E)$ is a $CPCA$. Examples show these also to be the best results possible.

5. The property p is here the property of being at the same time an F_σ and a G_δ. The necessary and sufficient condition that a set E be at the same time an F_σ and a G_δ is that for every perfect set H, there is some portion H_1, of H, which is contained entirely in either E or CE.† The truth of this condition is unchanged if the word closed is substituted for the word perfect. It is in this last form that the condition is used here. Let F denote any element of the space of all closed linear sets,‡ and let $\{d_k\}$ denote all open intervals with rational end points. Then the formula for determining $\Gamma_p(E)$ is given below.

$$x \epsilon \Gamma_p(E) \equiv \prod_F \sum_k \{ \sum_{y_1} (y_1 \epsilon F \cdot d_k) \}$$
$$\times \{ \prod_y [\overline{y \epsilon F \cdot d_k}' + \overline{(x, y) \epsilon E}] + \prod_y [\overline{y \epsilon F \cdot d_k}' + (x, y) \epsilon CE] \}.$$

From this formula, Theorems 11 and 12 follow. In the case of Theorem 12 it should be remembered that an A is a PCA.§

THEOREM 11. *If E is a Borel set, $\Gamma_p(E)$ is a CA.*

THEOREM 12. *If E is an A or a CA, $\Gamma_p(E)$ is a $CPCA$.*

These theorems give the best evaluation of $\Gamma_p(E)$ as is shown by the theorems below.

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* By taking closed intervals the O of the theorem could be changed to an F.

† Blue, *Mathematische Annalen*, vol. 102, p. 628.

‡ For a definition of this space see Kuratowski, *loc. cit.*, p. 259.

§ Lusin, *loc. cit.*, p. 284.

THEOREM 13. *If M is any linear set CA, there is a Borel set E such that* $\Gamma_p(E) = M$.

Since M is a CA, it follows that CM is an A, and as such, must be the set of values taken by a function $x = f(y)$ defined and continuous on the irrational points on the y-axis.* Let I be the image of this function, that is the set of all points (x_1, y_1) such that $x_1 = f(y_1)$. The set I is a G_δ and the projection of I is CM. Every point of I has an irrational ordinate. Let $\{r_n\}$ be the set of all positive rational numbers. From the set I form the set I_n by adding r_n to each ordinate of I and form I_{-n} by subtracting r_n from each ordinate of I. More explicitly if (x, y) is in I, $(x, y + r_n)$ is in I_n and $(x, y - r_n)$ is in I_{-n}. Let $I_0 = \sum_{n=1}^{\infty} (I_n + I_{-n})$. The set I_0 is a $G_{\delta\sigma}$ and every point in I_0 has an irrational ordinate. If q is any point of CM, I_0 is everywhere dense in $(x = q)$, but CI_0 which includes all points on this line having rational ordinates, is likewise everywhere dense in $(x = q)$. Therefore $I_0 \cdot (x = q)$ cannot be an F_σ and a G_δ, and the same is true of $(CI_0) \cdot (x = q)$. In letting E be either I_0, or CI_0 one has $\Gamma_p(E) = M$. In the one case E is a $G_{\delta\sigma}$ and in the other it is an $F_{\sigma\delta}$.

THEOREM 14. *If M is a $CPCA$, there is a set E_1 which is an A and a set E_2 which is a CA such that* $\Gamma_p(E_1) = \Gamma_p(E_2) = M$.

Sierpinski † has shown that CM, being a PCA, must be the set of values taken by a function $x = f(y)$ on a set K, where K is a CA which is a subset of the irrational numbers. The function $f(y)$ is continuous on K. Letting I be the image of the function, form the sets I_n and I_{-n} as before and from these the set I_0. Since I is a CA, I_0 is a CA. If $E_1 = CI_0$ and $E_2 = I_0$, the sets E_1 and E_2 satisfy the theorem.

Suppose that p' is the property of being an F_α. Let U be a universal F_α in the plane. If E is any set,

$$x \in \Gamma_{p'}(E) \equiv \sum_{x_1} \Big[\prod_{y} \{ \overline{(x, y) \in E}' + \overline{(x_1, y) \in U} \} \Big] \times \Big[\prod_{y} \{ \overline{(x_1, y) \in U}' + \overline{(x, y) \in E} \} \Big].$$

Hence if E is any Borel set, $\Gamma_p(E)$ is a PCA. The formula also enables us to show that if E is an A or a CA, that $\Gamma_p(E)$ is a $PCPCA$. It is not known whether or not these evaluations of $\Gamma_p(E)$ are the best possible when α is any transfinite ordinal.

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* Lusin, *loc. cit.*, p. 135.

† *Fundamenta Mathematicae*, vol. 11, p. 117. See also Lusin, *loc. cit.*, p. 275.

Let K denote the property of being connected. Assuming that null sets and sets of a single element are connected, one has for any set E,

$$x \epsilon C\Gamma_k(E) \equiv \sum_{y_1} \sum_{y_2} \sum_{y_3} [(x, y_1) \epsilon E] [(x, y_3) \epsilon E] [(x, y_2) \epsilon CE] [y_1 < y_2 < y_3].$$

Hence if E is a Borel set $\Gamma_k(E)$ is the complement of an analytic set and if E is analytic or the complement of an analytic set, $\Gamma_k(E)$ is a $CPCA$. Simple examples show that these evaluations are the best possible.

6. Let us turn now to operations on functions. In this section and in the seventh and ninth sections $f(x, y)$ is assumed to be defined on the finite rectangle $a \leq x \leq b$, $c \leq y \leq d$. In the remainder of the paper $f(x, y)$ is assumed to be defined throughout the entire plane. Let $(y_0, y_1, \cdots, y_{n+1})$ be any finite system of points in the interval (c, d) such that $y_0 = d$, $y_{n+1} = d$, and $y_0 \leq y_1 \leq y_2 \cdots \leq y_{n+1}$. The total variation of a function $t(y)$ defined on the interval (c, d) is the least upper bound of the sum $\sum_{i=1}^{n+1} | t(y_i) - t(y_{i-1}) |$ for all such finite systems of points as the one above. When the total variation is finite the function is said to be of limited variation, abbreviated as LV.

THEOREM 15. *If $f(x, y)$ is in the Baire classification, $\Gamma_{LV}(f)$ is a CA.*

It is convenient to introduce a function g_n defined as follows (where $y_0 = c$ and $y_{n+1} = d$):

$$g_n(x, y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n+1} | f(x, y_i) - f(x, y_{i-1}) |.$$

This function of the variables $(x, y_1, y_2, \cdots, y_n)$ must be of the same Baire class as f and it is defined for all points of an $(n+1)$ dimensional cell, $a \leq x \leq b$, $c \leq y_i \leq d$ $(i = 1, 2, \cdots, n)$. Let A_{mn} be the points of this cell at which g_n is greater than the integer m, and let E_n be the points of this cell for which $y_0 \leq y_1 \leq y_2 \cdots \leq y_{n+1}$. The set E_n is closed and A_{mn} is a Borel set. Consequently the set B_{mn}, which is the projection of $A_{mn} \cdot E_n$ on the x-axis, is an analytic set. Let $B_m = \sum_n B_{mn}$. The set B_m consists of all points $\bar{x}$ for which the total variation of $f(\bar{x}, y)$ is greater than m. The set $B = \prod_m B_m$ is therefore the set of all $\bar{x}$'s such that $f(\bar{x}, y)$ is of unlimited variation. The set B is analytic and since $\Gamma_{LV}(f)$ is the complement of B (with respect to the interval from a to b), it follows that $\Gamma_{LV}(f)$ is the complement of an analytic set.*

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* In the proof of this theorem and some of the following theorems the method of Kuratowski and Tarski might conveniently be used. The methods which are used were chosen because of their simplicity and because they involve little if any sacrifice in space.

COROLLARY 4. *If $f(x, y)$ is a function of class one, $\Gamma_{LV}(f)$ is a $G_{\delta\sigma}$.*

The set A_{mn} is in this case an F_σ so that $A_{mn} \cdot E_n$ and its projection B_{mn} are also F_σ's. Therefore B_m is an F_σ and B is an $F_{\sigma\delta}$. By taking complements it follows that $\Gamma_{LV}(f)$ is a $G_{\delta\sigma}$.

COROLLARY 5. *If $f(x, y)$ is lower semi-continuous, $\Gamma_{LV}(f)$ is an F_σ.*

In this case A_{mn} is open or at least it is open if the end points of intervals are neglected and the exclusion of such points will not affect the conclusion of the corollary since a finite number of points added to a G_δ gives another G_δ. For the proof of this corollary let E_n be all points of the $(n + 1)$ dimensional cell for which $y_0 < y_1 < y_2 \cdots < y_{n+1}$.[*] The set E_n is open (except for end points) and therefore B_{mn} and B_m are open; it follows that B is a G_δ and that $\Gamma_{LV}(f)$ is an F_σ.

In § 5 it has been shown that there exists a plane Borel set M_E whose projection on the x-axis is any specified analytic set E and such that for any point t in E, both M_E and its complement are everywhere dense in the line $x = t$. Let E lie in the interval (a, b) and let H_E be the part of M_E which lies between the parallels $y = c$ and $y = d$. The characteristic function $h_E(x, y)$ of H_E is in the Baire classification, and from the nature of H_E, it follows that $\Gamma_{LV}(h_E)$ is equal to the complement of E. This fact together with Theorem 15 is formulated in Theorem 16.

THEOREM 16. *A necessary and sufficient condition that a set E on the interval (a, b) be the complement of an analytic set is that there exist a function $f(x, y)$ in the Baire classification such that $\Gamma_{LV}(f) = E$.*

Tonelli [†] defines the function $\phi(\bar{x})$ as being the total variation of the function $f(\bar{x}, y)$ and gives a corresponding definition of $\psi(\bar{y})$.

THEOREM 17. *If $f(x, y)$ is in the Baire classification, $\phi(\bar{x})$ and $\psi(\bar{y})$ are measurable in the Lebesgue sense.*

It is evidently sufficient to give a proof for the case of $\phi(\bar{x})$. Define the set B_{rn} in the same manner as B_{mn} was defined in Theorem 1 except that now the number r may have any real value instead of being restricted to integral values as m was. As before let $B_r = \sum_n B_{rn}$. The set B_r is the set of all $\bar{x}$'s such that $f(\bar{x}, y)$ has total variation greater than r, that is, it is the set on which $\phi(\bar{x})$ is greater than r. From the fact that B_r is analytic it follows that B_r is measurable in the Lebesgue sense and hence that $\phi(\bar{x})$ is a measurable function.

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[*] In the proof of this corollary it is assumed that no two successive points of the finite set used in defining total variation are identical. This can be done without altering the value of the total variation.

[†] Tonelli, *Academia dei Lincei, Rendiconti* (6), vol. 3 (1926), p. 357. For a discussion of this and other definitions of limited variation see the paper by J. A. Clarkson and C. R. Adams, *Transactions of the American Mathematical Society*, vol. 35 (1933), p. 824. It is there shown that $\phi(\bar{x})$ and $\psi(\bar{y})$ are lower semi-continuous when $f(x, y)$ is continuous.

The function $\phi(\bar{x})$ may be outside the Baire classification even though $f(x, y)$ is in the Baire classification as is shown by the example leading up to Theorem 16. However it is possible to prove that $\phi(\bar{x})$ is in the Baire classification in a certain special case.

Corollary 6. *If $f(x, y)$ is of class one, the functions $\phi(\bar{x})$ and $\psi(\bar{y})$ are of class two.*

The set A_{rn} is here an F_σ. Therefore $A_{rn} \cdot E_n$ and its projection B_{rn} are F_σ's, and it follows that B_r is an F_σ. The set at which $\phi(\bar{x})$ is greater than or equal to r can be shown on the basis of this fact to be an $F_{\sigma\delta}$. Therefore the set at which $\phi(\bar{x})$ is less than r is a $G_{\delta\sigma}$. Since the set at which $\phi(\bar{x})$ is greater than r is an O_1 (in the notation of de la Vallée Poussin), it is also an O_2; since a $G_{\delta\sigma}$ is an O_2, $\phi(\bar{x})$ must be of class two at most.

7. A function $t(y)$ defined on the interval (c, d) is said to be absolutely continuous if for any positive ϵ there is a positive δ such that for any finite set of non-overlapping intervals (y_1, y_2), (y_3, y_4), $\cdots$, (y_{2n-1}, y_{2n})[*] lying in (c, d) and of total length less than or equal to δ, the sum

$$S = \sum_{i=1}^{n} | t(y_{2i}) - t(y_{2i-1}) |$$

is less than ϵ. The abbreviation AC will be used for this property.

Theorem 18. *If $f(x, y)$ is in the Baire classification, $\Gamma_{AC}(f)$ is the complement of an analytic set.*

In making the proof of this theorem it is convenient to define a function g_n somewhat like the function used in proving Theorem 1.

$$g_n(x, y_1, y_2, \cdots, y_{2n}) = \sum_{i=1}^{n} | f(x, y_{2i}) - f(x, y_{2i-1}) |.$$

This function is of the same class as $f(x, y)$ and is defined at all points of a $(2n + 1)$ dimensional cell $a \leq x \leq b$, $c \leq y_i \leq d$ for all i from 1 to $2n$. Let ϵ_m and δ_s be sequences of positive numbers approaching zero. Let A_{mn} be the points of the $(2n + 1)$ dimensional cell at which g_n is greater than ϵ_m, and let E_{ns} be the points of this cell for which $y_1 \leqq y_2 \leqq \cdots \leqq y_{2n}$, and $\sum_{i=1}^{n} | y_{2i} - y_{2i-1} | \leqq \delta_s$. Let B_{mns} be the projection on the x-axis of the set $E_{ns} \cdot A_{mn}$. Since E_{ns} and A_{mn} are both Borel sets, B_{mns} must be an analytic set. The set $B_{ms} = \sum_n B_{mns}$ is the set of all $\bar{x}$'s for which the sum S [taken for $f(\bar{x}, y)$] is greater than ϵ_m on some set of intervals of total length at most δ_s. Let $B_m = \prod_s B_{ms}$; this set is the set of all $\bar{x}$'s for which the sum S is greater than ϵ_m for a set of intervals of total length at most equal to *any* preassigned δ_s. The set $B = \sum_m B_m$ is therefore the set of all $\bar{x}$'s such that $f(\bar{x}, y)$ is not absolutely continuous. This is an analytic set and consequently its complement, $\Gamma_{AC}(f)$, is the complement of an analytic set.

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[*] There is no loss of generality in assuming $y_1 \leq y_2 \cdots \leq y_{2n}$.

A special result which may be proved in a similar manner is embodied in Corollary 7.

COROLLARY 7. *If $f(x, y)$ is of class one, $\Gamma_{AC}(f)$ is a $G_{\delta\sigma\delta}$.*

The example used to prove Theorem 16 together with Theorem 18 makes it possible to state Theorem 19.

THEOREM 19. *A necessary and sufficient condition that a set E on the interval (a, b) be the complement of an analytic set is that there exist a function $f(x, y)$ in the Baire classification such that $\Gamma_{AC}(f) = E$.*

8. Two lemmas will be established in this section and because of a later application they will be formulated for metric spaces in general instead of for Euclidean spaces only. If X and Y are two separable complete metric spaces,* their combinatorial product is denoted by $X \times Y$, and is the collection of all pairs of elements (x, y) where x is in X and y is in Y. If (x_1, y_1) and (x_2, y_2) are any two elements of this product space the distance between them is defined to be $[(x_1 x_2)^2 + (y_1 y_2)^2]^{1/2}$ where (pq) is a symbol denoting the distance between the two elements p and q of a metric space. Functions may be defined on metric spaces, and their theory is in many respects similar to the usual theory.† The functions used here will be defined on subsets of $X \times Y$ and will have real numbers as functional values. The necessary and sufficient condition that such a function be of class α is that the set at which the function is greater than (and smaller than) any real number is an O_α. A point $(\bar{x}, \bar{y})$ is a y-discontinuity of a function if $\bar{y}$ is a discontinuity of $f(\bar{x}, y)$. It may happen that a function is defined only on some closed subset of $X \times Y$.

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[*] For definition of terms concerning metric spaces see Hausdorff, *Mengenlehre* (1927).

[†] See Kuratowski, *Fundamenta Mathematicae*, vol. 17, p. 275.

LEMMA 2. *If $f(x, y)$ is any function in the Baire classification defined on a closed subset E of $X \times Y$, the set D_r of points at which $f(x, y)$ has a y-discontinuity greater than or equal to r (any positive real number) is an analytic set with closed vertical sections.*[*]

Let $g(x, y, y_1) = |f(x, y) - f(x, y_1)|$ where g is defined on all points (x, y, y_1) of $(X \times Y) \times Y$ which are such that (x, y) and (x, y_1) are in E.[†] The range of definition of the function g is therefore a closed set. Let R_m be the points of this closed set at which $g(x, y, y_1)$ is greater than $r - 1/m$ and let E_n be the points of this closed set at which $(y\, y_1) \leq 1/n$. The projection of the set $R_m \cdot E_n$ on $X \times Y$ is an analytic set which will be denoted by B_{mn}. The set B_{mn} is the set of all points (x, y) such that there is a point (x, y_1) in E for which $(y\, y_1) \leq 1/n$ and $|f(x, y) - f(x, y_1)| > r - 1/m$. If $B_n = \prod_m B_{mn}$, B_n is the set of all points (x, y) such that for any m, there is a point (x, y_1) for which $(y\, y_1) \leq 1/n$ and $|f(x, y) - f(x, y_1)| > r - 1/m$. Therefore $D_r = \prod_n B_n$, and D_r is analytic. The fact that D_r has closed vertical sections is demonstrated in the usual manner.

LEMMA 3. *If $f(x, y)$ is any function in the Baire classification defined on a closed subset E of $X \times Y$, the set D at which $f(x, y)$ has a y-discontinuity is an analytic set.*

If ϵ_n is any sequence of positive numbers approaching zero, then $D = \sum_n D_{\epsilon_n}$ which, in view of Lemma 2, completes the proof.

The symbol $(x = a)$ is used to denote all points of $X \times Y$ of the form (a, y).

The following lemma contains a sharpened form of a statement made in § 4.

LEMMA 4. *If R is any analytic set included in a closed subset E of $X \times Y$, the set A of points a in X such that $(x = a) \cdot R$ is nowhere dense in $(x = a) \cdot E$ is a CA.*

Let O_i be an enumerable family of open sets in Y such that any open set in Y is the sum of some combination of them. Such a family exists since Y is a metric separable space. For any a in A it is necessary, if there is an O_i

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[*] A vertical section of a set consists of all those points of the set which have a fixed "abscissa."

[†] The function $g(x, y, y_1)$ can be shown to be in the Baire classification as follows: The functions $f(x, y, y_1) = f(x, y)$ and $h(x, y, y_1) = f(x, y_1)$ are in the Baire classification on the space $E \times Y$, and therefore the absolute value of their difference is in the Baire classification on this set. The set on which $g(x, y, y_1)$ is defined is a closed subset of $E \times Y$, and therefore $g(x, y, y_1)$ is in the Baire classification on this set also.

such that $a \times O_i$ includes a point of E, that there is an O_j in O_i such that $a \times O_j$ includes a point of E but includes no point of R. Let O_{ni} denote the collection of O_i's which lie in O_n, and let A_n and A_{ni} denote the projections on X of $(X \times O_n) \cdot E$ and $(X \times O_{ni}) \cdot E$. The sets A_n and A_{ni} are F_σ's. Let B_{ni} be the projection on X of $(X \times O_{ni}) \cdot R$; the set B_{mi} is analytic. If a point a is in $A_{ni} - B_{mi}$, then $a \times O_{ni}$ includes a point of E but does not include a point of R. Therefore if a point a is in $A_n^{\bullet} = \sum\limits_i (A_{ni} - B_{ni})$ there is some O_{ni} such that $a \times O_{ni}$ includes a point of E but not a point of R. Since $A_{ni} - B_{ni}$ is the complement of an analytic set, A_n is also the complement of an analytic set. If a point a is in $\prod\limits_n A_n$, there is, for every O_n such that $a \times O_n$ includes a point of E, an O_{ni} (in O_n) which includes a point of E and does not include a point of R. It is seen that $A = \prod\limits_n A_n$ and therefore A is the complement of an analytic set.

9. It is assumed in this section that $f(x, y)$ is defined on the rectangle $a \leq x \leq b$, $c \leq x \leq d$. The property of being integrable in the Riemann sense will be denoted by R.

THEOREM 20. *If $f(x, y)$ is bounded and in the Baire classification, $\Gamma_R(f)$ is a CA.*

Using the notation of the proof of Lemma 3, $f(\bar{x}, y)$ will be integrable in the Riemann sense if and only if $(x = \bar{x}) \cdot D$ has measure zero; this will be true if and only if $(x = \bar{x}) \cdot D_{\epsilon_n}$ has measure zero for all n. Since vertical sections of D_{ϵ_n} are closed it is possible to apply Theorem 9. It may be deduced from this theorem that the set of $\bar{x}$'s for which $(x = \bar{x}) \cdot D_{\epsilon_n}$ has measure zero is the complement of an analytic set. By multiplication it follows that the set of $\bar{x}$'s such that $(x = \bar{x}) \cdot D$ has measure zero is the complement of an analytic set. This set is exactly $\Gamma_R(f)$ so that the theorem is proved.

The example used in the proof of Theorems 16 and 19 might be used to prove a similar theorem for the property R.

10. From this point on the function $f(x, y)$ will be assumed to be defined in the entire plane. The property of being pointwise discontinuous will be indicated by PD. By definition $f(\bar{x}, y)$ is point-wise discontinuous if and only if points of y-continuity are everywhere dense on $(x = \bar{x})$. The necessary and sufficient condition for this is that $(x = \bar{x}) \cdot D_{\epsilon_n}$ be nowhere dense for all n. If $f(x, y)$ is in the Baire classification it is possible to apply Lemma 4 where $X \times Y$ is now the Euclidean plane and the closed subset E in which D_{ϵ_n} lies is also the entire plane. According to this lemma the set of $\bar{x}$'s for which $(x = \bar{x}) \cdot D_{\epsilon_n}$ is nowhere dense in $(x = \bar{x})$ is the complement of an analytic

set. By multiplication the set of $\bar{x}$'s for which $(x = \bar{x}) \cdot D_{\epsilon_n}$ is nowhere dense in $(x = \bar{x})$ for all n is the complement of an analytic set; this is exactly the set $\Gamma_{PD}(f)$, so that Theorem 21 may be stated.

THEOREM 21. *If $f(x, y)$ is in the Baire classification, $\Gamma_{PD}(f)$ is the complement of an analytic set.*

For this property also a theorem similar to Theorems 16 and 19 might be stated.

The set of $\bar{x}$'s such that $f(\bar{x}, y)$ is *continuous* is also the complement of an analytic set. This follows from the fact that this set is the complement of the projection of the set D.

Let M denote the property of being non-decreasing.

THEOREM 22. *If $f(x, y)$ is a function in the Baire classification, $\Gamma_M(f)$ is a CA.*

Let A denote those points of (x, y, y_1) space for which $f(x, y) > f(x, y_1)$ and let B denote the points of this space for which $y < y_1$. Since A and B are both Borel sets the projection, W, of $A \cdot B$ on the (x, y) plane, is an analytic set. Let V be the projection of W on the x-axis. If x is in V there is some y such that $(\bar{x}, y)$ is in W, and therefore there is some y_1 such that (x, y, y_1) is in $A \cdot B$. This means that $y < y_1$, and $f(x, y) > f(x, y_1)$. It is seen that $\Gamma_M(f)$ is the complement of V. Since V is analytic $\Gamma_M(f)$ is the complement of an analytic set.

If $f(x, y)$ is continuous, $\Gamma_M(f)$ is closed; and if $f(x, y)$ is of class one, $\Gamma_M(f)$ is a G_δ. These facts follow as corollaries from the above method of proof.

THEOREM 23. *If $f(x, y)$ is in the Baire classification the set L of points $(\bar{x}, \bar{y})$ at which $f(x, y)$ has a unique finite y-limit is a CA.*

Let $g(x, y, h, k) = | f(x, y + h) - f(x, y + k) |$ where the function is defined at all points (x, y, h, k) of four dimensional space. This function is of the same class as $f(x, y)$. If ϵ_m and δ_n are two sequences of positive numbers approaching zero, let A_m be the set of points (x, y, h, k) at which $g(x, y, h, k) > \epsilon_m$ and let E_n be all points (x, y, h, k) such that $O < | h | < \delta_n$ and $O < | k | < \delta_n$. The set B_{mn} which is the projection on the (x, y) plane of $A_m \cdot E_n$ is an analytic set. If $B_m = \prod_n B_{mn}$ then B_m is the set of all points (x, y) for which $g(x, y, h, k) > \epsilon_m$ for some h and some k where the absolute values of h and k may be made less than any preassigned δ_n. A unique finite y-limit can not exist at such points. The set of all points at which a unique finite y-limit fails to exist is the set $B = \sum_m B_m$. The set of points L at which a unique finite

y-limit does exist is the complement of B and is therefore the complement of an analytic set.

By projecting the set L and its complement on the x-axis it is possible to obtain theorems concerning the set of $\bar{x}$'s for which $f(\bar{x}, y)$ has a unique finite limit at some point, at no point, or at every point. A special result concerning L may be obtained when $f(x, y)$ is of class one.

LEMMA 5. *If $g(x, y, h)$ is in the Baire classification, the set of points $(\bar{x}, \bar{y})$ for which $g(x, y, h)$ has a unique finite h-limit as h approaches zero is a CA.*

By the method of the proof of Theorem 23, it can be shown that the set K of points (x, y, h) at which $g(x, y, h)$ has a unique finite h-limit is the complement of an analytic set. The section of K by the plane $h = 0$ gives the set desired in the lemma.

THEOREM 24. *If $f(x, y)$ is in the Baire classification the set M of points $(\bar{x}, \bar{y})$ at which this function possesses a unique finite y-derivative is a CA.*

Let $g(x, y, h) = \dfrac{f(x, y + h) - f(x, y)}{h}$. This function, if we assume it is equal to any constant value, say 1, when $h = 0$, is defined on all of three dimensional space, and is in the Baire classification. By Lemma 4, the set $(\bar{x}, \bar{y})$ at which $g(x, y, h)$ has a unique finite h-limit as h approaches zero is the complement of an analytic set, and this is the set at which $f(x, y)$ has a unique finite derivative.

By projecting the set M and its complement on the x-axis it is possible to obtain results concerning the set of $\bar{x}$'s for which $f(\bar{x}, y)$ has a unique finite y-derivative at every point, at no point, or at some point.

In the proof of the next theorem use will be made of the space $\mathfrak{F}$ of all closed point sets on the y-axis. Elements of this space will be denoted by F. The property of being of the first class will be denoted by I. The x- and y-axes will be denoted by X and Y.

THEOREM 25. *If $f(x, y)$ is in the Baire classification, $\Gamma_I(f)$ is a CA.*

Let Q be all points (x, F, y) of the space $(X \times \mathfrak{F}) \times Y$ for which y is in F. The set Q is closed. The function $g(x, F, y)$ is defined everywhere on Q and is equal to $f(x, y)$. In order to prove that this function is in the Baire classification consider the function $h(x, F, y)$ defined on the whole of $(X \times \mathfrak{F}) \times Y$ and equal to $f(x, y)$. The function h is obviously in the Baire classification and therefore that part of it which is defined on Q is in the Baire classification. But that part of h which is on Q is the function g and therefore the function g is in the Baire classification.

If ϵ_n is a sequence of positive numbers approaching zero, let D_{ϵ_n} be the points of $(X \times \mathfrak{F}) \times Y$ at which $g(x, F, y)$ has y-discontinuities greater than or equal to ϵ_n. The set D_{ϵ_n} is an analytic set lying in the set Q. From Lemma 4 the set A_n of points $(\bar{x}, \bar{F})$ in $(X \times \mathfrak{F})$ such that* $(x = \bar{x}, F = \bar{F}) \cdot D_{\epsilon_n}$ is nowhere dense in $(x = \bar{x}, F = \bar{F}) \cdot Q$, is the complement of an analytic set. The set $A = \prod_n A_n$ is the complement of an analytic set and it is the set of points $(\bar{x}, \bar{F})$ in $X \times \mathfrak{F}$ such that $g(\bar{x}, \bar{F}, y)$ is pointwise discontinuous. Let B be the projection on X of the complement of A in $X \times \mathfrak{F}$. The set B is analytic. The complement of B, denoted by H, is the complement of an analytic set. If $\bar{x}$ is in H, then for every F, $g(\bar{x}, F, y)$ is pointwise discontinuous in y; that is for every F, $f(\bar{x}, y)$ is pointwise discontinuous on F. This is exactly the condition that $f(\bar{x}, y)$ be of class one, and therefore $\Gamma_I(f) = H$ which completes the proof of the theorem.

This theorem contains Theorem 11 as a corollary, but it does not contain Theorem 12. It would be interesting to know more concerning $\Gamma_\alpha(f)$ where α is the property of being of class α and α is greater than one. If $f(x, y)$ is a Baire function, $\Gamma_\alpha(f)$ is a projective set but it appears difficult to show that a given evaluation is the best possible. The example used in Theorem 16 shows that the evaluation in Theorem 25 is the best possible.

The remaining theorem treats a case different from the preceding ones in that the condition on $f(x, y)$ is of a different nature.

THEOREM 26. *If $f(x, y)$ is pointwise discontinuous, $C\Gamma_{PD}(f)$ is of the first category.*

If $\bar{x}$ is in $C\Gamma_{PD}(f)$ there is some positive number, $\epsilon_{\bar{x}}$ and some interval $d_{\bar{x}}$ on the line $x = \bar{x}$, such that $f(x, y)$ has a discontinuity greater than $\epsilon_{\bar{x}}$ at every point of the interval $d_{\bar{x}}$. Let Q_n be the points of $C\Gamma_{PD}(f)$ for which the chosen $\epsilon_{\bar{x}}$ is greater than $1/n$. Suppose that $C\Gamma_{PD}(f)$ is of the second category. Then there is an n such that Q_n is of the second category. Let D_n be all the points of all the $d_{\bar{x}}$'s which are associated with the $\bar{x}$'s in this set Q_n. By Lemma 2 of my previously cited paper it follows that there is some open set in the plane in which D_n is everywhere dense, and the function $f(x, y)$ can have no point of continuity in this open set. This contradicts the hypothesis and therefore the assumption that $C\Gamma_{PD}(f)$ was of the second category was wrong.

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* $(x = \bar{x}, F = \bar{F})$ denotes all points of $(X \times \mathfrak{F}) \times Y$ such that $x = \bar{x}$ and $F = \bar{F}$.

MATRIX DIFFERENTIAL EQUATIONS.[†]

By William M. Whyburn.

Systems of first order, linear, ordinary differential equations have been studied extensively through the use of matrix algebra.[‡] This algebra provides a simple notation for the differential systems and affords useful tools for work with the systems. Some recent work on non-linear systems of differential equations lead me to a linear system of the matrix form $dY/(dx)+A(x)Y+YA(x)=0$, where Y, A, $dY/(dx)$ are square matrices of n-rows. This equation naturally leads one to study the more general matrix equation $dY/(dx)+\sum_{i=1}^{m}A_i(x)YB_i(x)=R(x)$. The present paper is designed to indicate the essential relations between matrix equations of the above type and equations of the usual matrix form, namely, $dU/(dx)+A(x)U=R(x)$. Analogous algebraic equations have been studied extensively by a number of authors [§] and applications of this work have been made in quantum mechanics. The present paper includes some applications to non-linear differential systems. The notation and terminology of the paper are in agreement with MacDuffee's recent treatment of matrices in *Ergebnisse der Mathematik*, vol. 2, heft. 5. The work of the paper is confined to the real domain.

Consider the differential system

$$dY/(dx)+\sum_{i=1}^{m}A_i(x)YB_i(x)=R(x), \tag{1}$$

where Y, R, $A_i(x)$, $B_i(x)$, are square matrices of n-rows. The elements of the coefficient matrices $R(x)$, $A_i(x)$, $B_i(x)$, $(i=1,2,\cdots,m)$, are Lebesgue integrable on $X: a\leqq x\leqq b$ and a solution of a differential equation is understood to mean a matrix with absolutely continuous elements that satisfies the differential equation almost everywhere on X. Let I_n be the unit matrix of n-rows and let 0 represent the zero matrix of proper order when it is used in an equation. The equation

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[†] Presented to the American Mathematical Society December 2, 1933.

[‡] Volterra, *Memoria della Societa Italiane delle Scienze*, t. VI (1887) and t. XII (1899), developed the infinitesimal calculus of matrices. H. F. Baker, *Proceedings of the London Mathematical Society*, vol. 34 (1902), pp. 347-360, and vol. 35 (1903), pp. 333-378, used matrices extensively in a study of linear differential systems. See also an article by Birkhoff and Langer, *Proceedings American Academy of Arts and Science*, vol. 57 (1922), pp. 51-128, where other references to work along these lines are given.

[§] See C. C. MacDuffee's article in *Ergebnisse der Mathematik*, Vol. 2, Heft 5, Springer, 1933, for references to this work.

$$dY/(dx)+\sum_{i=1}^{m}A_i(x)YB_i(x)=0 \tag{2}$$

is the homogeneous equation corresponding to (1). We define

$$dZ/(dx)-\sum_{i=1}^{m}B_i(x)ZA_i(x)=0 \tag{3}$$

to be the adjoint equation of equation (2). We form the equations

$$dU/(dx)+\left[\sum_{i=1}^{m}B_i^T(x)\cdot\times A_i(x)\right]U=S(x) \tag{4}$$

$$dV/(dx)-V\left[\sum_{i=1}^{m}B_i^T(x)\cdot\times A_i(x)\right]=0, \tag{5}$$

Where U, V, and $S(x)$ are square matrices of n^2 rows. The notation $B_i^T(x)\cdot\times A_i(x)$ means the direct product (or the *left direct product*) of B_i^T with A_i. The matrix $S(x)$ is formed[†] from $R(x)$ by letting $s_{ij}=r_{k+1,i-kn}$, $(i, j=1, 2,\cdots, n^2)$, where $k\geqq 0$ is a whole number such that $kn<i\leqq(k+1)n$. Equation (5) is the adjoint of the homogeneous equation corresponding to equation (4).

THEOREM I. *Let* $U(x)$ *be any solution*[‡] *of equation* (4). *There corresponds a solution* $Y(x)$ *of equation* (1) *such that* $u_{ij}(x)=y_{k+1,i-kn}$, $(i, j=1,\cdots, n^2)$, *where* $k\geqq 0$ *is a whole number such that* $kn<i\leqq(k+1)n$. *Conversely, if* $Y(x)$ *is any solution of* (1), *the matrix* $U(x)$ *defined in the above manner is a solution of equation* (4).

Proof. Both parts of the theorem follow immediately from an examination of the equations satisfied by $U(x)$ and $Y(x)$. We observe that a solution matrix $U(x)$ of equation (4) has its columns alike.

An examination of equations (3) and (5) along with the observation that a solution matrix $V(x)$ of equation (5) has its rows alike yields

THEOREM II. *For each solution* $V(x)$ *of* (5), *there corresponds a solution* $Z(x)$ *of* (3) *such that* $v_{ji}(x)=z_{i-kn,k+1}(x)$, $(i, j=1, 2,\cdots, n^2)$, *where*

[†] We denote the element of the i-th row and j-th column of any matrix by the corresponding small letter with subscript ij. This is further indicated by the fact that in the equations specifying the quantities s_{ij}, $u_{ij}(x)$, $v_{ji}(x)$, the suffix j does not occur on the right hand side.

[‡] U, V, and S represent degenerate matrices in that they either have their columns alike or else their rows are alike.

$k \geqq 0$ is a whole number such that $kn < i \leqq (k+1)n$. Conversely, if $Z(x)$ is any solution of (3), *the matrix $V(x)$ defined in the above manner is a solution of* (5).

Fundamental Existence theorems and many properties of the solution functions for systems (1) and (3) follow directly from well-known theorems concerning systems (4) and (5) when Theorems I and II are taken into account. We show by direct substitution that if Y is a solution matrix for equation (2) then cY is also a solution of that equation, where c is an arbitrary constant. However, it does not follow that the product of a constant matrix by a solution of (2) yields a solution of that equation. It is well-known that there exists a matrix U^* of solutions of the homogeneous equation corresponding to (4) such that $d(U^*) \neq 0$ on X and such that the general solution of this homogeneous equation may be expressed in the form $U = U^* C$, where C is a constant matrix and $d(U^*)$ means the n^2-rowed determinant of U^* [the matrix U^* is composed of n^2 column solutions of the equation]. It follows at once that there exist n^2 solutions $Y_1, \cdots, Y_{n^2}$ of (2) such that if Y^* is a particular solution of (1) and Y is the general solution of (1), then $Y(x) = Y^*(x) + \sum_{i=1}^{n^2} c_i Y_i(x)$, where $c_1, \cdots, c_{n^2}$ are arbitrary (scalar) constants.

We note that the special case $m = 1$, $A_1(x) = A(x)$, $B_1(x) = I_n$ allows system (1) to break up into n systems (identical in form) of the type $dY/(dx) + A(x)Y = R(x)$. In this case, system (3) breaks up into n systems each of which has the form $dZ/(dx) - ZA(x) = 0$ and the adjoint relationship is consistent with the usual form of this relation. However, it is important to observe that the linear system remains of order n^2 and that a single equation $dY/(dx) + A(x)Y = R(x)$ cannot properly be regarded as completely equivalent to the original equation. This indicates that the matrix forms $dY/(dx) + A(x)Y = R(x)$ might be regarded as degenerate cases of the symmetric matrix equation (1) [despite the fact, as shown in Theorem I, that equation (1) may be regarded as equivalent in a sense to an equation of the form $dY/(dx) + A\,Y = R$]. This degeneracy offers a good explanation of the *vector* character of the matrices Y and R that occur in $dY/(dx) + AY = R$.

THEOREM III. *Let $Y(x)$ and $Z(x)$, respectively, be any solutions of* (2) *and* (3). *There exists a constant c such that $\sum_{i,j=1}^{n} y_{ij}(x) z_{ji}(x) \equiv c$ for all x on X. This may also be stated by saying that the sum of the main diagonal elements of YZ is constant on X.*

Proof. Let U and V be the solutions of (4) and (5), respectively, that correspond under Theorems I and II to Y and Z. Direct computation from (4) and (5) shows that $d(VU)/(dx) \equiv 0$ on X. Integration of this equation yields $VU = C$, a constant matrix, and interpretation of this result in terms of Y and Z yields Theorem III. This theorem may be proved without using (4) and (5) if one makes use of the component equations for (2) and (3). This proof, however, is slightly more complicated in detail.

We shall not pursue the study of equation (1) further although many interesting questions present themselves for investigation. For example, the distribution of the roots of $d(Y)$ on the interval X is of interest. An examination of the equation $dY/(dx) - AYA = 0$, where all of the elements of A are equal to unity, shows that it has a family of solutions such that $d(Y)$ has exactly one root on the real axis. Other examples may be given to emphasize the fact that the non-vanishing of $d(Y)$ at one point fails to insure its non-vanishing at other points of an interval (contrasting the well-known result for systems of the type $dY/(dx) + AY = 0$). We conclude the paper with some theorems which make use of systems of type (1) in a study of non-linear differential systems.

THEOREM IV. *Let $Y_1, \cdots, Y_{n^2+3}$ be solutions of the Riccati type*[†] *non-linear differential system*

$$dY/(dx) + YY = R(x), \tag{6}$$

where $R(x)$, Y, and $dY/(dx)$ are square matrices of n-rows and where the elements of $R(x)$ are Lebesgue integrable on X : $a \leqq x \leqq b$. If $d(Y_i - Y_j) \neq 0$ on X, $(i, j = 1, \cdots, n^2 + 3)$, $(i \neq j)$, there exist constants $c(i, j, k)$ such that

$$\sum_{\substack{i=1 \\ (i\neq j\neq k\neq i)}}^{n^2+3} c(i, j, k)[Y_i - Y_j]^{I}[Y_i - Y_k] \equiv 0, \quad (j, k = 1, \cdots, n^2 + 3), \tag{7}$$

hold as identities on X and at least one of the constants $c(i, j, k)$ in each equation of (7) is different from zero.

Proof. Let $V_i = Y_i - Y_j$, $(i \neq j)$, hence $Y_i = Y_j + V_i$, and substitute Y_i into equation (6). When use is made of the fact that Y_i and Y_j are solutions of (6), it is found that V_i satisfies

$$(dV_i)/(dx) + V_iY_j + Y_jV_i + V_iV_i = 0 \tag{8}$$

[†] See a paper of the author's in part 2 of the Commemoration Volume of the *Tôhoku Mathematical Journal*, vol. 38 (1933), pp. 447-450, concerning this equation.

while $U_i = V_i^I$, the inverse of V_i, satisfies the equation

$$(dU_i)/(dx) - Y_jU_i - U_iY_j - I_n = 0. \tag{9}$$

The matrix $W_{ik} = U_i - U_k$ is a solution of the equation

$$dW/(dx) - Y_jW - WY_j = 0. \tag{10}$$

Equations (9) and (10) are of type (1). For fixed j and k, $W_{ik}(x)$, $(i = 1, 2, \cdots, n^2+3)$, $(i \neq j \neq k)$, gives n^2+1 solutions of (11). It follows from Theorem I and the remarks that follow Theorem II that there exist constants $c(i, j, k)$, not all of which are zero, such that

$$\sum_{\substack{i=1\\(i\neq j\neq k)}}^{n^2+3} c(i, j, k)\, W_{ik} \equiv 0 \text{ on } X. \tag{11}$$

We observe that $W_{ik} = U_i - U_k = V_i^I - V_k^I = V_i^I\,[I_n = V_iV_k^I] = V_i^I[V_k - V_i]V_k^I = [Y_i - Y_j]^I[Y_k - Y_i][Y_k - Y_j]^I$. If this final value of W_{ik} is substituted into (11) and the resulting equation multiplied on the right by $-[Y_k - Y_j]$, equation (7) is obtained. The theorem follows when j and k are allowed to take on all possible values such that $i \neq j \neq k \neq i$.

Corollary I. *If $Y_1, Y_2, \cdots, Y_{n^2+2}$, are solutions of (6) such that $d(Y_i - Y_j) \neq 0$ on X and if Y is the general solution of (6), then*

$$[Y - Y_j]^I\,[Y - Y_k] = \sum_{\substack{i=1\\(i\neq j\neq k\neq i)}}^{n^2+2} c_i[Y_i - Y_j]^I\,[Y_i - Y_k],$$

where $c_1, \cdots, c_{n^2+2}$ are arbitrary constants and j and k are fixed indices.

Corollary II. *The case $n = 1$ in Theorem IV and Corollary I yields a well-known theorem concerning the anharmonic ratio of four solutions of the Riccati type differential equation.*†

Theorem V. *If equation (6) has a solution Y^* such that each element of Y^* is an even function of x, thus $Y^*(x) = Y^*(-x)$, and if Y_1, Y_2, Y_3, Y_4, are any four solutions of (6) such that $d(Y_i - Y^*) \neq 0$, $(i = 1, \cdots, 4)$, then the matrices $W_{ij}(x) = [Y_i - Y^*]^I - [Y_j - Y^*]^I = [Y_i - Y^*]^I \times [Y_j - Y_i][Y_j - Y^*]^I$ have the property that the sum of the main diagonal elements of $W_{ij}(x)W_{hk}(-x)$ is constant on X, $(i, j, h, k = 1, \cdots, 4)$.*

Proof. An examination of the proof of Theorem IV shows that $W_{ij}(x)$ is a solution of the equation

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† See Ince, *Ordinary Differential Equations*, London (1927), page 24.

$$dW/(dx) - Y^*W - WY^* = 0 \tag{12}$$

while $V_{ij}(x) \equiv W_{ij}(-x)$ satisfies the equation

$$dV/(dx) + VY^* + Y^*V = 0. \tag{13}$$

A comparison of equations (12) and (13) with equations (2) and (3) shows that these equations are adjoints. An application of Theorem III yields Theorem V.

THEOREM VI. *If equation* (6) *has a solution* $Y^*(x)$ *such that each element of* Y^* *is an odd function of* x, *thus* $Y^*(x) \equiv -Y^*(-x)$, *and if* $Y_1(x)$ *and* $Y_2(x)$ *are any two solutions of* (6) *such that* $d(Y_1 - Y^*) \neq 0$, $d(Y_2 - Y^*) \neq 0$ *on* X, *then the matrix* $W(x) = [Y_1 - Y^*]^I - [Y_2 - Y^*]^I = [Y_1 - Y^*]^I [Y_2 - Y_1] [Y_2 - Y^*]^I$ *is such that* $W(x) \equiv W(-x)$ *on* X.

Proof. The matrices $W(x)$ and $V(x) = W(-x)$ satisfy the equation

$$dU/(dx) - Y^*U - UY^* = 0. \tag{14}$$

Furthermore, $W(0) = V(0)$ and equation (14) has a *unique* solution which takes on the initial values $U(0) = W(0)$. It follows from this that $W(x) \equiv V(x) \equiv W(-x)$ on X.

The special cases of Theorems V and VI where $n = 1$ are of interest since equation (6) then becomes the generalized Riccati equation (after this equation has been subjected to suitable changes of the independent and dependent variables). Interesting examples can be constructed by building the Riccati differential equations satisfied by specific even or odd functions and applying Theorems V and VI to these.

In conclusion, it may be stated that the present paper emphasizes the point of view that the algebraic form of a differential system is important and that canonical forms for the system are not always desirable or obtainable.†

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† This point of view was definitely stated by Ritt on page iii of the introduction to his Colloquium lectures (*American Mathematical Society Colloquium Publications*, vol. XIV, New York, 1932).

SOLUTION OF $f(x, y) \frac{\partial z}{\partial x} + g(x, y) \frac{\partial z}{\partial y} = 0$ IN A NEIGHBORHOOD OF A SINGULAR POINT.

By H. H. Alden.

I. The problem of solving $\partial z/\partial x + F(x, y)\partial z/\partial y = 0$, where the real function $F(x, y)$ is continuous and has a continuous partial derivative with respect to y in an open region R, is equivalent to the problem of finding an integral* of $dy/dx = F(x, y)$ having partial derivatives with respect to x and y. This problem is treated in the small in standard books on differential equations, and Kamke † has shown that in any simply connected open subregion R' of R having no boundary points in common with the boundary of R, there exists an integral.

The equation

$$f(x, y)\partial z/\partial x + g(x, y)\partial z/\partial y = 0, \tag{1}$$

where f and g are real functions which are continuous and have continuous partial derivatives with respect to x and y in an open region R can be solved in a neighborhood of a point (α, β) of R if $f(\alpha, \beta)$ and $g(\alpha, \beta)$ are not both zero, since the equation can then be put into one of the forms

$$\frac{\partial z}{\partial x} + \frac{g(x, y)}{f(x, y)} \frac{\partial z}{\partial y} = 0, \quad \text{or} \; \frac{\partial z}{\partial y} + \frac{f(x, y)}{g(x, y)} \frac{\partial z}{\partial x} = 0$$

in a neighborhood of (α, β). In Picard's Traité d'Analyse, vol. III, pp. 1-12, there is discussed the existence of solutions of (1) under the hypothesis that f and g are analytic functions of the complex variables x and y in a neighborhood of a point (α, β) at which $f(\alpha, \beta) = g(\alpha, \beta) = 0$. Under this hypothesis, letting $f(x, y) = a_{10}(x - \alpha) + a_{01}(y - \beta) + \cdots$ and $g(x, y) = b_{10}(x - \alpha) + b_{01}(y - \beta) + \cdots$, and letting λ_1 and λ_2 be the roots of $(a_{10} - \lambda)(b_{01} - \lambda) - a_{01}b_{10} = 0$, it is proved that there exist two analytic functions $u_1(x, y)$ and $u_2(x, y)$, both of which vanish for $x = y = 0$, such that in a deleted neighborhood of (α, β), $u_1^{1/\lambda_1}u_2^{-1/\lambda_2}$ is a solution of (1) if λ_1 and λ_2 are distinct and if the line joining the complex numbers λ_1 and λ_2 in the complex plane does not pass through the origin. It will be noted that

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[*] An integral of this equation is a function $z(x, y)$ which is constant along each solution-curve of $dy/dx = F(x, y)$; the constant may vary from curve to curve.

[†] *Mathematische Annalen*, vol. 99, p. 602.

these conditions are not satisfied when $f = x$, $g = -y$, or when $f = y$, $g = -x$, although the equation $x\partial z/\partial x - y\partial z/\partial y = 0$ has the solution $z = xy$, and the equation $y\partial z/\partial x - x\partial z/\partial y = 0$ has the solution $z = x^2 + y^2$.

In this paper we propose to discuss under the following conditions the existence of solutions of (1) over (α, β):

(a) $f(x, y)$ and $g(x, y)$ are real functions which are continuous and have continuous first partial derivatives with respect to x and y in an open region R;

(b) f and g vanish together at only one point (α, β) of R;

(c) by a solution of (1) shall be understood a continuous function $z(x, y)$ which is constant in no circle in R, and for which first partial derivatives exist and reduce (1) to an identity in x and y.

We note that an integral of the system

$$dx/dt = f(x, y), \quad dy/dt = g(x, y) \tag{2}$$

which is a function of x and y alone is a solution of (1) if it has partial derivatives, and conversely a solution of (1) is an integral of (2). We are therefore to consider the existence of integrals of (2) in a neighborhood N of (α, β), where by an integral we shall now understand a function $J(x, y)$ satisfying the following conditions:

(i) J is continuous and has partial derivatives with respect to x and y;

(ii) J has the same value at all points of each orbit* of (2); the value may vary from orbit to orbit;

(iii) J is not constant in any circle in N.

From condition (a) above, we find that through each point of R there passes one and only one orbit of (2), and the unique orbit through (α, β) is $x = \alpha$, $y = \beta$; i. e., the point itself. The orientation of the orbits about (α, β) will largely determine whether or not an integral of (2) exists in a neighborhood of that point. Suppose first that each of the orbits in a neighborhood N of (α, β) has (α, β) as a limit point, as is the case when (α, β) is a knot point or a spiral point of (2). Then no integral exists in N. For suppose $J(x, y)$ satisfies (i) and (ii). Since (α, β) is a limit point of each orbit in N, $J(\alpha, \beta)$ is the limit of values of $J(x, y)$ for (x, y) on any orbit in N, according to (i). Hence on each orbit in N, $J(x, y) = J(\alpha, \beta)$, according to (ii), and hence in N, $J(x, y) \equiv J(\alpha, \beta)$, whence condition (iii) is not satisfied.

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* An orbit of (2) is the projection on the x, y plane of a solution curve of (2) in t, x, y space.

Suppose next that among the orbits there is a sequence of closed curves converging to (α, β), each being interior to the area bounded by the preceding one, and suppose that in the area between any two consecutive closed curves of the sequence the orbits are spirals asymptotic to the closed curves. Then no integral exists in any neighborhood of (α, β). For suppose $J(x, y)$ satisfies conditions (i) and (ii) in N, a neighborhood of (α, β). Let O_k be a closed orbit entirely in N, and let j be the value of $J(x, y)$ on O_k. Then also $J(x, y) = j$ for (x, y) on a spiral asymptotic to O_k, and hence $J(x, y) \equiv j$ in the entire area bounded by O_k and O_{k+1}, whence condition (iii) is not satisfied.

II. Two simple orientations remain to be discussed; namely, that in which all the orbits are closed curves with (α, β) interior to the area bounded by each, and that in which the orbits form a saddle. We shall prove here that if all the orbits are closed, and if A denotes a subregion of R bounded by one of the closed orbits, then there exists an integral in all of A. Without loss of generality we may assume that the point (α, β) is the origin. Let $r(x, y)$ be the greatest lower bound of $(\bar{x}^2 + \bar{y}^2)^{1/2}$ for all points $(\bar{x}, \bar{y})$ on the orbit through (x, y); i. e., let r be the least distance from the origin to the orbit through (x, y). Now r itself is not a solution of (1), for in the case where $f = y$ and $g = -x$, the orbits are circles and r is $(x^2 + y^2)^{1/2}$, so that the surface $z = r(x, y)$ is a cone with its vertex at the origin, and hence the partial derivatives of $r(x, y)$ do not exist at the origin. We shall prove, however, that r^2 is an integral $J(x, y)$ satisfying the three prescribed conditions. That this is a function of x and y is apparent from the fact that through every point (x, y) of A there passes an orbit entirely in A, and this orbit has a least distance from the origin. It is also evident that for each orbit of A, $J(x, y)$ has the same value at all points (x, y) of the orbit. Hence condition (ii) is satisfied.

Also condition (iii) is satisfied. For take any circle ρ in A, with center (x_1, y_1), and let (x_2, y_2) be any point in ρ within the orbit O_1 through (x_1, y_1). Then the orbit O_2 through (x_2, y_2) is within O_1. Since O_1 is a closed point set, there is at least one point p_1 on O_1 such that the line $\overline{op_1}$ joining the origin o to p_1 is of length r_1, where r_1 is the least distance from the origin to O_1. Now O_2 cuts $\overline{op_1}$ in a point p_2, and hence $\overline{op_2} < r_1$. Also, r_2, the least distance from the origin to O_2, satisfies the inequality $r_2 \leq \overline{op_2}$. Hence $r_2 < r_1$, and condition (iii) is satisfied, since every circle ρ in A contains two points for which J has different values. We have now only to show that $J(x, y)$ is continuous and has partial derivatives with respect to x and y in A.

III. $J(x, y)$ is continuous. At the origin, $r = 0$, and for a point (x, y)

other than the origin we have $0 < r \leqq |x| + |y|$, whence $\lim_{x,y \to 0,0} r^2 = 0$. Hence at the origin we have continuity. Consider next $J(x, y)$ at a point (x_1, y_1) in A, other than the origin. Let O_1 be the orbit through this point, and let r_1 be the least distance from the origin to O_1. Let $\{(x_n, y_n)\}$, $n = 2, 3, \cdots$ be any sequence whatever of points of A convergent to (x_1, y_1); let O_n be the orbit through (x_n, y_n), and let r_n be the least distance from the origin to O_n. It is known that for every $\epsilon > 0$ there exists a δ_ϵ such that for every (x, y) satisfying $|x - x_1| < \delta_\epsilon$, $|y - y_1| < \delta_\epsilon$ it is true that within ϵ of any point on O_1 there is a point on the orbit through (x, y), and within ϵ of any point on the orbit through (x, y) there is a point on O_1. Let $\epsilon > 0$ be given, and let n be so large that $|x_n - x_1| < \delta_\epsilon$ and $|y_n - y_1| < \delta_\epsilon$. Since there is a point on O_1 whose distance from the origin is r_1, we have that there is a point on O_n whose distance d_n from the origin satisfies $|d_n - r_1| < \epsilon$. Hence, from $r_n \leqq d_n$, we have

$$r_n - r_1 < \epsilon. \tag{3}$$

Also there exists no point p_n on O_n whose distance l_n from the origin satisfies $l_n < r_1 - \epsilon$. For if d is the distance of any point whatever on O_1 from the origin, then $r_1 \leqq d$, whence $r_1 - \epsilon \leqq d - \epsilon$. Hence from $l_n < r_1 - \epsilon$ would follow $l_n < d - \epsilon$, so that there would be no point on O_1 within ϵ of p_n, which is false. Hence $l_n \geqq r_1 - \epsilon$, where l_n is the distance of any point whatever on O_n from the origin. From this follows $r_n \geqq r_1 - \epsilon$, and this, with (3), gives $|r_1 - r_n| \leqq \epsilon$. Since the sequence $\{(x_n, y_n)\}$ was any sequence whatever in A converging to (x_1, y_1), it follows that $J(x, y)$ is continuous at (x_1, y_1), and since (x_1, y_1) was any point of A other than the origin, $J(x, y)$ is continuous throughout A.

IV. At the origin, J_x and J_y exist and are zero. For $J_x(0, 0) = \lim_{x \to 0} r \cdot (r/x)$. Now $|x|$ is the distance from the origin to some point on an orbit, and r is the least distance from the origin to the same orbit. Hence surely $r \leqq |x|$, whence $\lim_{x \to 0} r = 0$ and $\overline{\lim_{x \to 0}} |r/x| \leqq 1$. Therefore $\lim_{x \to 0} r \cdot (r/x) = 0$. Similarly $J_y(0, 0) = 0$.

Considerable difficulty is encountered in proving that the partial derivatives of $J(x, y)$ exist at points of A other than the origin. In the next eight sections of the paper we shall be concerned with establishing facts which when collected in section XIII will constitute a proof that $J_x(x, y)$ exists at every point (x, y) of A other than the origin.

V. Consider the differential equations

$$dx/d\tau = -g(x, y), \quad dy/d\tau = f(x, y). \tag{4}$$

These have an orbit through each point of A, and have the origin for their only singular point. At all other points their orbits are perpendicular to the orbits of (2) through those points. We shall continue to speak of the orbits of (2) as orbits, and shall call the orbits of (4) trajectories. Let the orbits be given by

$$\begin{aligned} x &= \phi(t, x_0, y_0), & y &= \psi(t, x_0, y_0) \\ x_0 &= \phi(0, x_0, y_0), & y_0 &= \psi(0, x_0, y_0) \end{aligned} \tag{5}$$

and let the trajectories be given by

$$\begin{aligned} x &= \lambda(\tau, x_0, y_0), & y &= \theta(\tau, x_0, y_0) \\ x_0 &= \lambda(0, x_0, y_0), & y_0 &= \theta(0, x_0, y_0). \end{aligned} \tag{6}$$

Then the functions ϕ, ψ, λ, and θ are continuous and have continuous partial derivatives with respect to each of their arguments for (x_0, y_0) and (x, y) and the arc of the orbit or trajectory joining them in A. In particular ϕ and ψ have this property for (x_0, y_0) in A, and $-\infty < t < +\infty$.

Suppose an arc C of a trajectory lies entirely in A. Then C cuts no orbit in more than one point. For suppose an orbit O is cut by C in at least two points P_1 and P_2. Let x_0 and y_0 in (6) be the coördinates of P_1, so that (6) with $\tau = 0$ give the point P_1, and let P_2 be given by (6) with $\tau = \tau_2$. Without loss of generality we may assume $\tau_2 > 0$, for if $\tau_2 < 0$, an interchange of the subscripts of P_1 and P_2 renders the statement true. Now let τ_3 be the least positive value of τ for which equations (6) give a point on O, and let this point be called P_3. Then at one of the points P_1 and P_3, C enters the area bounded by O with increasing τ, and at the other of these points C emerges from this area with increasing τ. We shall show this to be impossible. Let S_1 be the set of points on O such that the trajectory through each point of S_1 enters the area bounded by O with increasing τ, and let S_2 be the set of points on O such that the trajectory through each point of S_2 emerges from this area with increasing τ. Since neither S_1 nor S_2 is empty, and every point of O is in either S_1 or S_2, there must be at least one point Q on O such that in every neighborhood of Q there are points of S_1 and points of S_2. From the continuity of f and g it follows that both f and g must be zero at Q. This is not true however, since f and g both vanish only at the origin, and Q is not the origin. Hence we have a contradiction, and hence C cuts no orbit in more than one point.

Suppose next that C is a trajectory which has a point (x_0, y_0) in A. Then C cuts each orbit in A except the origin. For let O_0 be the orbit through (x_0, y_0), and let r_0 be the least distance of O_0 from the origin, and let $\underline{r}$ be the greatest lower bound of the least distances between the origin and the orbits which C cuts. Since C cuts O_0, it cuts some of the orbits within O_0, and hence $\underline{r} < r_0$, and every orbit for which the least distance r from the origin satisfies $\underline{r} < r \leq r_0$ is cut exactly once by C. If we can show $\underline{r} = 0$, we will have proved that every orbit within O_0 is cut just once. Suppose $\underline{r} > 0$. Then we need only show that C cuts the orbit $O_{\underline{r}}$ for which the least distance from the origin is $\underline{r}$, for if it cuts $O_{\underline{r}}$, it cuts orbits within $O_{\underline{r}}$, which contradicts the definition of $\underline{r}$. Now C has at least one limit point on $O_{\underline{r}}$, for it cuts every orbit between O_0 and $O_{\underline{r}}$ once and only once, and does not cut $O_{\underline{r}}$ by assumption. If P_1 is the only limit point which C has on $O_{\underline{r}}$, then P_1 is a singular point of (4); this is not true since the origin is the only singular point of (4), and the origin is not on $O_{\underline{r}}$. Hence C must have at least two limit points P_1 and P_2 on $O_{\underline{r}}$. On each of the two arcs of $O_{\underline{r}}$ having P_1 and P_2 for end points, select a point between P_1 and P_2. Let these two points be called $\bar{P}$ and $\bar{\bar{P}}$, and let the trajectory through $\bar{P}$ and that through $\bar{\bar{P}}$ be constructed. Each cuts all the orbits in a neighborhood of $O_{\underline{r}}$. Hence in order to pass from a neighborhood of P_1 to a neighborhood of P_2, C must cut one of these trajectories through $\bar{P}$ or $\bar{\bar{P}}$. But if C has a point on a trajectory, it coincides with that trajectory, and hence it cuts $O_{\underline{r}}$.

Similarly it is proved that C cuts all the orbits of A outside of O_0. We shall now be free in what follows to speak of *the* point in which a trajectory cuts an orbit, for we have shown that each trajectory in A cuts each orbit in A once and only once.

VI. Consider the functions

$$(7) \qquad x = \phi(t, \lambda(\tau, a, b), \theta(\tau, a, b)), \qquad y = \psi(t, \lambda(\tau, a, b), \theta(\tau, a, b))$$

where (a, b) is a point P of A other than the origin. According to (5) and (6), for $\tau = 0$ and t variable the right members of (7) give the orbit through P, and for $t = 0$ and τ variable they give the trajectory through P. For τ fixed and t variable they give the orbit through a point Q whose coördinates are $[\lambda(\tau, a, b), \theta(\tau, a, b)]$, whence Q is a point on the trajectory through P. We shall need to examine the right members of (7) with t fixed, say at t_1, and τ variable. For $\tau = 0$ we get a point S on the orbit through P. Hence for $|\tau|$ small, since the functions are continuous, we get points near S. We shall show that we get a continuous arc with a continuously turning tangent. Let Q be a point variable with τ, obtained as above by setting $t = 0$, $\tau = \tau$

in the right members of (7), and let T be the point given by setting $t = t_1$, $\tau = \tau$ in (7). For simplicity we shall write $\phi_{x_0}(Q, T)$ for $\phi_{x_0}(t_1, x_0, y_0)$ with x_0 and y_0 replaced by $\lambda(\tau, a, b)$ and $\theta(\tau, a, b)$ respectively after the differentiation; similar interpretations of the other symbols appearing are readily made.*

Now we have, by differentiating (7) with respect to τ,

$$(8)\qquad \begin{aligned} \partial x/\partial\tau &= -\phi_{x_0}(Q,T)g(Q) + \phi_{y_0}(Q,T)f(Q),\\ \partial y/\partial\tau &= -\psi_{x_0}(Q,T)g(Q) + \psi_{y_0}(Q,T)f(Q). \end{aligned}$$

The right members of (8) are continuous functions of τ, and they do not vanish together, for if both were zero, the equations

$$(9)\qquad \begin{aligned} \phi_{x_0}(Q,T)\xi + \phi_{y_0}(Q,T)\eta &= 0\\ \psi_{x_0}(Q,T)\xi + \psi_{y_0}(Q,T)\eta &= 0 \end{aligned}$$

would have a solution $(\xi, \eta) = (-g(Q), f(Q))$. Now

$$\begin{vmatrix} \phi_{x_0}(Q,T) & \phi_{y_0}(Q,T)\\ \psi_{x_0}(Q,T) & \psi_{y_0}(Q,T) \end{vmatrix} \neq 0$$

since it is the determinant of a fundamental set of solutions of the equations of variation for the orbit through Q.† Hence the only solution of (9) is $(\xi, \eta) = (0, 0)$. But f and g do not both vanish at Q, since they vanish together only at the origin, and Q is not the origin. Hence $(\partial x/\partial\tau)^2 + (\partial y/\partial\tau)^2 > 0$, and the equations (7) with t fixed and τ variable give a continuous curve with a continuously turning tangent.

Let μ be the angle made by the curve given by (7) with t fixed at t_1 and τ variable, with a parallel to the x axis. Then we have

$$(10)\qquad \begin{aligned} \cos\mu &= \frac{-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P)}{[(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2]^{1/2}}\\ \sin\mu &= \frac{-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P)}{[(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2]^{1/2}}. \end{aligned}$$

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* This designation is not entirely satisfactory, for $\phi_{x_0}(Q, T)$ might also be interpreted as $\phi_{x_0}(t_2, x_0, y_0)$, where t_2 is another value of t having all the properties we have demanded of t_1. To be definite therefore let us assume that t_1 is the least positive value of t for which $\phi(t, a, b)$ and $\psi(t, a, b)$ are the coördinates of S. Obviously there is a least such value. With the adoption of this convention $\phi_{x_0}(Q, T)$ becomes a well defined quantity. The only exception to the convention will be made in section XI, where a quantity t_k analogous to the above t_1 is explicitly defined in another way.

† See Moulton, *Differential Equations*, pp. 234-238.

We observe that for each point Q on the trajectory through P there is exactly one point T on the curve through S given by (7) with t fixed at t_1 and τ variable, and that Q and T lie on the same orbit. Hence if O is any orbit in A, and Q is the point in which the trajectory through P cuts O, there corresponds to Q exactly one point T on O in which the curve through S under consideration cuts O; i. e., the curve through S cuts each orbit of A in one and only one point. We shall therefore be able to legitimately speak of *the* point in which an orbit of A is cut by a curve of the type considered here.

VII. Consider the trajectory PQ and the curve ST of the previous section. Let the length of PQ be Δn and let the length of ST be Δc. We now prove that as Q approaches P along the trajectory through P, and T approaches S along the curve through S and T, with Q and T always on the same orbit,

$$(11)\quad \lim_{\Delta n \to 0} \frac{\Delta c}{\Delta n}$$
$$= \left[\frac{(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2}{f^2(P)+g^2(P)}\right]^{1/2}$$

Let the coördinates of S be (α, β) and those of T be (ξ, η). Then

$$(12)\quad \begin{aligned} \xi - \alpha &= \phi(t_1, \lambda(\tau, a, b), \theta(\tau, a, b)) - \phi(t_1, \lambda(0, a, b), \theta(0, a, b)) \\ \eta - \beta &= \psi(t_1, \lambda(\tau, a, b), \theta(\tau, a, b)) - \psi(t_1, \lambda(0, a, b), \theta(0, a, b)). \end{aligned}$$

Hence by the theorem of the mean,

$$(13)\quad \begin{aligned} \xi - \alpha &= [\phi_{x_0}(P', S')(-g(P')) + \phi_{y_0}(P', S')f(P')] \cdot \tau \\ \eta - \beta &= [\psi_{x_0}(P'', S'')(-g(P'')) + \psi_{y_0}(P'', S'')f(P'')] \cdot \tau \end{aligned}$$

where S' and S'' are points on the given curve ST between S and T, and P' and P'' are points on the trajectory PQ between P and Q. Hence

$$(14)\quad \begin{aligned} &[(\xi - \alpha)^2 + (\eta - \beta)^2]^{1/2} \\ &\quad = \tau \cdot [(-\phi_{x_0}(P', S')g(P') + \phi_{y_0}(P', S')f(P'))^2 \\ &\qquad + (-\psi_{x_0}(P'', S'')g(P'') + \psi_{y_0}(P'', S'')f(P''))^2]^{1/2}. \end{aligned}$$

Also

$$(15)\quad \begin{aligned} \Delta n &= \int_0^\tau [f^2(\lambda(\tau, a, b), \theta(\tau, a, b)) + g^2(\lambda(\tau, a, b), \theta(\tau, a, b))]^{1/2} d\tau \\ &= \tau \cdot [f^2(P''') + g^2(P''')]^{1/2}, \end{aligned}$$

where P''' is a point on the trajectory PQ between P and Q. Using the fact that for a curve with a continuously turning tangent, the limit of the length of an arc divided by the chord of the arc is unity, we have

$$(16)\qquad \lim_{\Delta n\to 0}\frac{\Delta c}{\Delta n}=\lim_{\Delta n\to 0}\frac{[(\xi-\alpha)^2+(\eta-\beta)^2]^{1/2}}{\Delta n}.$$

Now as $\Delta n\to 0$, $\tau\to 0$, and hence $P', P'', P'''\to P$, and $S', S''\to S$. Hence (14), (15), and (16) give (11). The right member of (11) is obviously a continuous function of the two points P and S on the same orbit. By the argument of section VI it is never zero, since P is not at the origin. Also, as $\Delta c\to 0$, $\Delta n\to 0$, since $\tau\to 0$. Hence we may write

$$(17)\quad \lim_{\Delta c\to 0}\frac{\Delta n}{\Delta c}$$
$$=\left[\frac{f^2(P)+g^2(P)}{(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2}\right]$$

VIII. Let U be the point in which the trajectory through S cuts the orbit through Q, and let Δl be the length of the trajectory SU. We now show

$$(18)\quad \lim_{\Delta c\to 0}\frac{\Delta l}{\Delta c}$$
$$=\frac{-g(S)(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))+f(S)(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))}{\{[f^2(S)+g^2(S)][(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2}$$

Let ν be the angle made by SU and a parallel to the x axis at S. Then

$$(19)\qquad \cos\nu=\frac{-g(S)}{(f^2(S)+g^2(S))^{1/2}}$$
$$\sin\nu=\frac{f(S)}{(f^2(S)+g^2(S))^{1/2}}.$$

As in the previous section, the arcs may be replaced by their chords, and since the angle SUT is a right angle, we have

$$\lim_{\Delta c\to 0}\frac{\Delta l}{\Delta c}=\cos(\mu-\nu)=\cos\mu\cos\nu+\sin\mu\sin\nu,$$

which, by (10) and (19), gives (18).

As $\Delta l\to 0$, $\tau\to 0$, whence $\Delta c\to 0$. Hence if we can show that $\lim\limits_{\Delta c\to 0}(\Delta l/\Delta c)\neq 0$, we will have that $\lim\limits_{\Delta l\to 0}(\Delta c/\Delta l)$ exists and is finite, and equals the reciprocal of the right member of (18). Now we have

$$(20)\qquad \Delta n/\Delta l=(\Delta n/\Delta c)\cdot(\Delta c/\Delta l).$$

Suppose that $\lim\limits_{\Delta c\to 0}(\Delta l/\Delta c)=0$, so that $\lim\limits_{\Delta l\to 0}(\Delta c/\Delta l)$ is infinite. From (17) we see that $\lim\limits_{\Delta c\to 0}(\Delta n/\Delta c)\neq 0$, since the right member of (17) is the quotient of a positive numerator and a finite denominator. Hence from (20) we see that our assumption that $\lim\limits_{\Delta c\to c} (\Delta l/\Delta c) = 0$ leads to the requirement $\lim\limits_{\Delta l\to 0} (\Delta n/\Delta l) = \infty$, and $\lim\limits_{\Delta n\to 0} (\Delta l/\Delta n) = 0$, since as $\Delta n \to 0$, $\tau \to 0$, and hence $\Delta l \to 0$. However, $\lim\limits_{\Delta n\to 0} (\Delta l/\Delta n) \neq 0$, as we now prove. Let (α, β) be the coördinates of S, and (a, b) those of P. Since P is on the orbit through S, and since this orbit is given by $x = \phi(t, \alpha, \beta)$, $y = \psi(t, \alpha, \beta)$, where $\alpha = \phi(0, \alpha, \beta)$ and $\beta = \psi(0, \alpha, \beta)$, we have that for some t, say t_2, $a = \phi(t_2, \alpha, \beta)$, and $b = \psi(t_2, \alpha, \beta)$. Now consider the equations

$$\begin{aligned} x &= \phi(t_2, \lambda(\tau, \alpha, \beta), \theta(\tau, \alpha, \beta)) \\ y &= \psi(t_2, \lambda(\tau, \alpha, \beta), \theta(\tau, \alpha, \beta)). \end{aligned} \tag{21}$$

By section VI, these give a curve through P having a continuously turning tangent. Let Δk be the length of the arc of this curve from P to the orbit through U. Then $\Delta l/\Delta n = (\Delta l/\Delta k) \cdot (\Delta k/\Delta n)$. Also, as $\Delta n \to 0$, $\tau \to 0$, whence $\Delta k \to 0$. From the requirement $\lim\limits_{\Delta n\to 0} (\Delta l/\Delta n) = 0$ it follows that either $\lim\limits_{\Delta k\to 0} (\Delta l/\Delta k) = 0$ or $\lim\limits_{\Delta n\to 0} (\Delta k/\Delta n) = 0$. Now $\lim\limits_{\Delta k\to 0} (\Delta l/\Delta k) \neq 0$, by a proof analogous to the proof in section VII that $\lim\limits_{\Delta c\to 0} (\Delta n/\Delta c)$ exists and is not zero. Also, by the method of deducing formula (18), we have

$$\lim_{\Delta n\to 0} \frac{\Delta k}{\Delta n}$$
$$= \frac{\{[f^2(P)+g^2(P)][(-\phi_{x_0}(S,P)g(S)+\phi_{y_0}(S,P)f(S))^2+(-\psi_{x_0}(S,P)g(S)+\psi_{y_0}(S,P)f(S))^2]}{-g(P)(-\phi_{x_0}(S,P)g(S)+\phi_{y_0}(S,P)f(S))+f(P)(-\psi_{x_0}(S,P)g(S)+\psi_{y_0}(S,P)f(S))}$$

The denominator of this expression is finite, and the numerator is not zero, for the first bracket of the numerator is positive since P is not at the origin, and the second bracket is not zero by the argument of section VI. Hence the requirement $\lim\limits_{\Delta n\to 0} (\Delta l/\Delta n) = 0$ is not satisfied, so the assumption $\lim\limits_{\Delta c\to 0} (\Delta l/\Delta c) = 0$ is false, whence $\lim\limits_{\Delta l\to 0} (\Delta c/\Delta l)$ exists and is finite and is given by

$$\lim_{\Delta l\to 0} \frac{\Delta c}{\Delta l} \tag{22}$$
$$= \frac{\{[f^2(S)+g^2(S)][-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))^2+(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))^2]\}}{-g(S)(-\phi_{x_0}(P,S)g(P)+\phi_{y_0}(P,S)f(P))+f(S)(-\psi_{x_0}(P,S)g(P)+\psi_{y_0}(P,S)f(P))}$$

In carrying through the above proof we have established the interesting fact that $\lim\limits_{\Delta n\to 0} (\Delta l/\Delta n)$ exists and is different from zero. This is analogous to the property $\partial y(x, a, b)/\partial b \neq 0$ of a solution $y(x, a, b)$, where $b = y(a, a, b)$, of the equation $dy/dy = F(x, y)$, where F is continuous and has a continuous partial derivative with respect to y.

IX. Let Δx be the distance along a parallel to the x axis from S to the orbit through Q. Then

$$\lim_{\Delta x \to 0} \frac{\Delta l}{\Delta x} = \frac{-g(S)}{(f^2(S) + g^2(S))^{1/2}} \tag{23}$$

since the limit is the cosine of the angle between the trajectory through S and a line parallel to the x axis at S.

X. Let O be a fixed orbit other than the singular point, and let r be its least distance from the origin o. There is at least one point q on O such that the distance $\overline{oq}$ from the origin to q is equal to r. Let Σ be the set of points on O such that the distance of each from the origin is r. Since q is in Σ, the set Σ is not empty. Now consider any sequence of orbits $\{O_n\}$ converging to O; let r_n be the least distance from the origin to O_n, and let Σ_n be the set of points on O_n such that the distance of each point from the origin is r_n. No set Σ_n is empty. Hence within every ϵ neighborhood of O there is an infinite number of points on sets Σ_n, and therefore there is at least one point on O which is a limit point of the points of the sets Σ_n. For suppose the points of the sets Σ_n had no limit point on O. Then about each point ζ of O there would exist an open set N_ζ containing ζ but containing no points of the sets Σ_n. Since O is a closed point set, by the Borel covering theorem there exist a finite number of points $\zeta_1, \zeta_2, \cdots, \zeta_m$ on O such that the set $N_{\zeta_1} + N_{\zeta_2} + \cdots + N_{\zeta_m}$ covers O. Let M be the complementary set of $N_{\zeta_1} + N_{\zeta_2} + \cdots + N_{\zeta_m}$. Then M is closed and has no points in common with O. Let h be the greatest lower bound of distances between pairs of points taken so that one point ζ of each pair is on O, and the other point z is in M. Then since O and M are closed, there exists a pair of points $\bar{\zeta}$ on O and $\bar{z}$ in M such that the distance between $\bar{\zeta}$ and $\bar{z}$ is h. Hence $h > 0$, for if $h = 0$, then $\bar{\zeta}$ and $\bar{z}$ coincide, which is impossible since M and O have no points in common. Letting $\epsilon = h/2$, we see that the assumption that O contains no limit points of points of the sets Σ_n gives the existence of an ϵ neighborhood of O which contains no points of the sets Σ_n, which is impossible since the orbits $\{O_n\}$ converge to O. Hence there is at least one point on O which is a limit point of points of the sets Σ_n. Let σ be the set of points p on O such that to each point p of σ there corresponds a sequence $\{O_n^{(p)}\}$ of orbits converging to O, such that p is a limit point of points of the corresponding sets $\Sigma_n^{(p)}$.

We assert that σ is contained in Σ. For suppose that π is a point in σ but not in Σ. Then the distance from the origin to π is $\geqq r + \delta$, where δ is sufficiently small but positive. Since the distance of points from the origin varies continuously along the orbit, there exists an arc s on O, containing π, and such that every point of s is distant at least $r + \delta/2$ from the origin. Since $\{O_n^{(\pi)}\}$ converges to O, for every $\delta > 0$ there exists an integer N_δ such that for $n > N_\delta$ the distances from the origin of all points on arcs s_n of $O_n^{(\pi)}$ converging to s are at least equal to $r + \delta/4$. On the other hand, since $r_n \rightarrow r$ by section III, we have that π is not a limit point of points of the sets $\Sigma_n^{(\pi)}$. This is a contradiction, and hence σ is contained in Σ; i. e., every point which is a limit point of points of least distance on orbits is a point of least distance on the orbit on which it lies.

XI. Let P be a point of σ on an orbit O, and let $\{O_k^{(P)}\}$ be a sequence of orbits converging to O, such that P is a limit point of points of the sets $\Sigma_k^{(P)}$. Let Q_k be the point in which the trajectory through P cuts the orbit $O_k^{(P)}$, and let Δ_{n_k} be the length of the arc PQ_k of the trajectory. Let Δr_k be the difference of the least distances from the origin of $O_k^{(P)}$ and O; i. e., let $\Delta r_k = r_k^{(P)} - r$. We shall prove $\lim\limits_{k\rightarrow\infty} (\Delta r_k/\Delta n_k) = 1$.

To prove this we shall first need another relation. Draw circles about the origin as center, with radii r and $r + \Delta r_k$. The former is tangent to O at P, and the latter is tangent to $O_k^{(P)}$ at a point M_k, which is such that the sequence $\{M_k\}$ converges to P. Construct the trajectory through M_k, and let N_k be the point in which it cuts O. Let Δm_k be the length of the arc M_kN_k of the trajectory. Then

$$\lim_{k\rightarrow\infty} (\Delta m_k/\Delta n_k) = 1. \tag{24}$$

For let the coördinates of M_k be (a_k, b_k), so that the equations of the orbit through M_k and Q_k are $x = \phi(t, a_k, b_k)$, $y = \psi(t, a_k, b_k)$, where $a_k = \phi(0, a_k, b_k)$ and $b_k = \psi(0, a_k, b_k)$. Let the coördinates of Q_k be (α_k, β_k), and let $|t_k|$ be* the least value of $|t|$ for which $\alpha_k = \phi(t, a_k, b_k)$ and $\beta_k = \psi(t, a_k, b_k)$. Consider the functions

$$\begin{aligned} x &= \phi(t_k, \lambda(\tau, a_k, b_k), \theta(\tau, a_k, b_k)) \\ y &= \psi(t_k, \lambda(\tau, a_k, b_k), \theta(\tau, a_k, b_k)) \end{aligned} \tag{25}$$

with τ variable. They are the equations of a curve with a continuously turning tangent, by section VI, and the curve passes through Q_k. Let R_k be the point in which this curve cuts the orbit through P, and let Δd_k be the length of the arc Q_kR_k of the curve. Then by the method of section VII we have

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* See first footnote on page 599.

$$(26)\quad \frac{\Delta m_k}{\Delta d_k} = \left[\frac{f^2(M''_k)+g^2(M''_k)}{(-\phi_{x_0}(M'_k,Q'_k)g(M'_k)+\phi_{y_0}(M'_k,Q'_k)f(M'_k))^2 + (-\psi_{x_0}(M'_k,Q'_k)g(M'_k)+\psi_{y_0}(M'_k,Q'_k)f(M'_k))^2}\right]^{1/2}$$

where M'_k and M''_k are points on the arc of the trajectory M_kN_k, and Q'_k is a point on the arc of the curve Q_kR_k. By hypothesis $M_k \to P$ as $k \to \infty$. Hence $M'_k, M''_k, Q_k, Q'_k \to P$ as $k \to \infty$, and by the choice of t_k, $\lim_{k\to\infty} t_k = 0$. From the continuity of the right member of (26) we have

$$(27)\quad \lim_{k\to\infty}\frac{\Delta m_k}{\Delta d_k} = \left[\frac{f^2(P)+g^2(P)}{(-\phi_{x_0}(P,P)g(P)+\phi_{y_0}(P,P)f(P))^2+(-\psi_{x_0}(P,P)g(P)+\psi_{y_0}(P,P)f(P))^2}\right]$$

Since $\phi_{x_0}(P,P)=1$, $\phi_{y_0}(P,P)=0$, $\psi_{x_0}(P,P)=0$, and $\psi_{y_0}(P,P)=1$, it follows that

$$(28)\qquad \lim_{k\to\infty}(\Delta m_k/\Delta d_k)=1.$$

Also,

$$(29)\qquad \lim_{k\to\infty}(\Delta d_k/\Delta n_k)=1,$$

as can be seen by considering the sequence of curvilinear right triangles Q_kPR_k of which Δd_k and Δn_k are sides. From $\Delta m_k/\Delta n_k = (\Delta d_k/\Delta n_k)\cdot(\Delta m_k/\Delta d_k)$ and from (28) and (29), equation (24) follows.

In order to prove that $\lim_{k\to\infty}(\Delta r_k/\Delta n_k)=1$, we shall first suppose that every Δr_k is positive, then that every Δr_k is negative. If the equality is proved for each of these two cases, it is clear that the result is true in the general case. Suppose then first that $\Delta r_k > 0$ for every k. From a figure it is obvious that $\Delta n_k \geqq \Delta r_k$, whence

$$(30)\qquad \varlimsup_{k\to\infty}(\Delta r_k/\Delta n_k)\leqq 1.$$

We shall also prove

$$(31)\qquad \varliminf_{k\to\infty}(\Delta r_k/\Delta n_k)\geqq 1.$$

Draw the chord M_kN_k, and extend it to the circle of radius r. Let $\overline{\Delta m_k}$ be the length of the chord M_kN_k, and let Δs_k be the length of the chord from M_k to the circle of radius r. Now $\overline{\Delta m_k}\leqq \Delta s_k$. Hence $\Delta r_k/\overline{\Delta m_k}\geqq \Delta r_k/\Delta s_k$, and also

$$\varliminf_{k\to\infty}\frac{\Delta r_k}{\Delta m_k}=\varliminf_{k\to\infty}\frac{\Delta r_k}{\overline{\Delta m_k}}\cdot\lim_{k\to\infty}\frac{\overline{\Delta m_k}}{\Delta m_k}=\varliminf_{k\to\infty}\frac{\Delta r_k}{\overline{\Delta m_k}}\geqq\varliminf_{k\to\infty}\frac{\Delta r_k}{\Delta s_k}=\varliminf_{k\to\infty}\cos\gamma_k=1,$$

where γ_k is the angle between Δs_k and the radius to the circle through M_k at M_k. In the above we have used the equalities $\lim\limits_{k\to\infty} (\overline{\Delta m_k}/\Delta m_k) = 1$ and $\lim\limits_{k\to\infty} \cos\gamma_k = 1$, both of which are true because of the uniform continuity of the slope functions of Δm_k and $O_k^{(P)}$ in a neighborhood of P. From $\Delta r_k/\Delta n_k = (\Delta r_k/\Delta m_k)\cdot(\Delta m_k/\Delta n_k)$, and by the use of (24) and the last result, we have proved (31). But (30) and (31) together give, for every $\Delta r_k > 0$,

$$\lim_{k\to\infty} (\Delta r_k/\Delta n_k) = 1. \tag{32}$$

For every $\Delta r_k < 0$, obviously $-\Delta m_k \geqq -\Delta r_k$, so that $\overline{\lim\limits_{k\to\infty}} (\Delta r_k/\Delta m_k) \leqq 1$, and hence by (24)

$$\overline{\lim_{k\to\infty}} (\Delta r_k/\Delta n_k) \leqq 1. \tag{33}$$

Also, extend the trajectory PQ_k to the circle of radius $r + \Delta r_k$, and let the length of the arc from P to the circle of radius $r + \Delta r_k$ be $-\Delta u_k$. Then $-\Delta u_k \geqq -\Delta n_k$, and $\Delta r_k/\Delta n_k \geqq \Delta r_k/\Delta u_k$, whence $\underline{\lim\limits_{k\to\infty}} (\Delta r_k/\Delta n_k) \geqq \underline{\lim\limits_{k\to\infty}} (\Delta r_k/\Delta u_k)$. Now $\underline{\lim\limits_{k\to\infty}} (\Delta r_k/\Delta u_k) = 1$, as can be seen by considering the sequence of curvilinear right triangles of which $-\Delta r_k$ and $-\Delta u_k$ are sides. Hence we have

$$\underline{\lim_{k\to\infty}} (\Delta r_k/\Delta n_k) \geqq 1, \tag{34}$$

and from (33) and (34) we see that (32) holds also for every $\Delta r_k < 0$, and hence (32) is true for the general case where the Δr_k's are not all of the same sign.

XII. Let P be the point of σ on O as chosen in the last section, and let Q be the point in which the trajectory through P cuts a neighboring orbit. Let Δr be the difference between the least distances of the orbits through Q and P from the origin, and let Δn be the length of the arc PQ of the trajectory. We want now to show

$$\lim_{\Delta n\to 0} (\Delta r/\Delta n) = 1. \tag{35}$$

In the previous section we have shown that a sequence of orbits converging to O can be so chosen that the limit of $\Delta r/\Delta n$ evaluated on the orbits of this sequence is 1. Here however we are to prove that over every sequence of orbits whatever converging to O, equation (35) is true. Let γ be any point of σ other than P, and let $\{O_k^{(\gamma)}\}$ be a sequence of orbits converging to O, such that γ is a limit point of points of the corresponding sets $\Sigma_k^{(\gamma)}$. Let Δs_k

be the length of the arc of the trajectory through γ between γ and $O_k^{(\gamma)}$. Let Δn_k be the length of the arc of the trajectory through P between P and the same orbit $O_k^{(\gamma)}$, and let Δr_k be the difference between the least distances of $O_k^{(\gamma)}$ and O from the origin. Then

$$\Delta r_k/\Delta n_k = (\Delta r_k/\Delta s_k) \cdot (\Delta s_k/\Delta n_k). \tag{36}$$

By section XI, $\lim_{k\to\infty} (\Delta r_k/\Delta s_k) = 1$. By section VIII, $\lim_{k\to\infty} (\Delta s_k/\Delta n_k)$ exists, for it was there shown that, using the notation of that section, $\lim_{\Delta n\to 0} (\Delta l/\Delta n)$ exists, where $\Delta n \to 0$ in any manner whatsoever. Hence we have that

$$\lim_{k\to\infty} (\Delta r_k/\Delta n_k) = \lim_{k\to\infty} (\Delta s_k/\Delta n_k) = \lim_{\Delta n\to 0} (\Delta s/\Delta n), \tag{37}$$

where Δs is the length of the arc of the trajectory through γ between γ and a variable orbit, and Δn is the length of the arc of the trajectory through P between P and the same variable orbit. From $|\Delta r_k| \leq |\Delta n_k|$, it follows that $\lim_{k\to\infty} (\Delta r_k/\Delta n_k) \leq 1$, whence, by (37),

$$\lim_{\Delta n\to 0} (\Delta s/\Delta n) \leq 1. \tag{38}$$

In the same way, by interchanging γ and P in the argument, we have

$$\lim_{\Delta n\to 0} (\Delta s/\Delta n) \geq 1. \tag{39}$$

From (38) and (39) we have

$$\lim_{\Delta n\to 0} (\Delta s/\Delta n) = 1, \tag{40}$$

whence by (37) we get

$$\lim_{k\to\infty} (\Delta r_k/\Delta n_k) = 1. \tag{41}$$

Suppose now that (35) were not true. Then there must be at least one sequence of orbits $\{\bar{O}_k\}$ converging to O, such that the limit of $\Delta r/\Delta n$ taken over this sequence exists and is different from 1. Let Γ be a point of σ on O such that a subsequence $\{\bar{O}_k^{(\Gamma)}\}$ of the sequence $\{\bar{O}_k\}$ has Γ for a limit point of points of the corresponding sets $\Sigma_k^{(\Gamma)}$. The value of $\lim_{\Delta n\to 0} (\Delta r/\Delta n)$ over this subsequence is the same as the value over the whole sequence, and hence is different from 1. However the argument of this section, with Γ replacing γ, shows that over the subsequence the limit is 1. Hence the assumption that (35) is not true leads to a contradiction, whence (35) is proved.

XIII. We are now prepared to prove that $J_x(x, y)$ exists for (x, y) any point of A other than the origin. Let the point (x, y) be called S;

conforming with the notation of sections VI, VII, IX, and XII, with P any point of σ on O, the orbit through S, we write

$$(42) \qquad \Delta r/\Delta x = (\Delta r/\Delta n)\cdot(\Delta n/\Delta c)\cdot(\Delta c/\Delta l)(\Delta l/\Delta x).$$

By (23), as $\Delta x \to 0$, $\Delta l \to 0$; by (22), as $\Delta l \to 0$, $\Delta c \to 0$; by (17), as $\Delta c \to 0$, $\Delta n \to 0$; and by (35), as $\Delta n \to 0$, $\Delta r \to 0$. Hence, by (42)

$$(43) \quad \lim_{\Delta x\to 0}(\Delta r/\Delta x) = \lim_{\Delta n\to 0}(\Delta r/\Delta n)\cdot\lim_{\Delta c\to 0}(\Delta n/\Delta c)\cdot\lim_{\Delta l\to 0}(\Delta c/\Delta l)\cdot\lim_{\Delta x\to 0}(\Delta l/\Delta x),$$

and by (35), (17), (22), and (23), this gives

$$(44) \quad \lim_{\Delta x\to 0}\frac{\Delta r}{\Delta x} = \frac{-g(S)\,(f^2(P)+g^2(P))^{\frac{1}{2}}}{\begin{array}{l}-g(S)\,(-\phi_{x_0}(P,S)\,g(P)+\phi_{y_0}(P,S)\,f(P))\\ \qquad\qquad +f(S)\,(-\psi_{x_0}(P,S)\,g(P)+\psi_{y_0}(P,S)f(P))\end{array}}.$$

Hence we have evaluated $\lim_{\Delta x\to 0}(\Delta r/\Delta x)$. It is interesting to note that while the result apparently depends upon the choice of the point P in σ on O, this of course is not the case. For the value obtained by using another point γ of σ on O is given, using the notation of section XII, by taking limits in the equation

$$\Delta r/\Delta x = (\Delta r/\Delta s)\cdot(\Delta s/\Delta n)\cdot(\Delta n/\Delta c)\cdot(\Delta c/\Delta l)\cdot(\Delta l/\Delta x)$$

and since $\lim_{\Delta n\to 0}(\Delta s/\Delta n) = 1$, by (40), we have $\lim_{\Delta s\to 0}(\Delta r/\Delta s) = \lim_{\Delta n\to 0}(\Delta r/\Delta n)$ $= 1$ by (35), and hence substituting γ for P in (44) leaves the value of the expression unchanged.

XIV. Formula (44) gives $\lim_{\Delta x\to 0}(\Delta r/\Delta x)$ explicitly as a function of two points P and S. Since P is determined (not necessarily uniquely) by S, the right member of (44) is essentially a function of S only; i. e., of x and y only. We will now show that the right member of (44) is continuous at every point S of A other than the origin. Let $\{S_n\}$ be any sequence whatever of points in A converging to S. Using the notation of section X, let σ_n be the set of points on the orbit through S_n of the type of σ, and let P_n be any point whatever in σ_n. By a proof analogous to that in section X, there is at least one point P in σ on O, the orbit through S, which is a limit point of the points P_n. Hence P is the limit of a sequence of points $\{P_{n_k}\}$ converging to P, where the sequence $\{P_{n_k}\}$ is a subsequence of the sequence $\{P_n\}$. Let

$\{S_{n_k}\}$ be the corresponding subsequence of the sequence $\{S_n\}$. At S_{n_k} we have, by (44),

$$(45)\quad \lim_{\Delta x \to 0} \frac{\Delta r}{\Delta x}\Big|_{S_{n_k}} = \frac{-g(S_{n_k})(f^2(P_{n_k}) + g^2(P_{n_k}))^{1/2}}{\begin{array}{l}-g(S_{n_k})(-\phi_{x_0}(P_{n_k}, S_{n_k})g(P_{n_k}) + \phi_{y_0}(P_{n_k}, S_{n_k})\ f(P_{n_k}))\\ \quad + f(S_{n_k})(-\psi_{x_0}(P_{n_k}, S_{n_k})g(P_{n_k}) + \psi_{y_0}(P_{n_k}, S_{n_k})f(P_{n_k}))\end{array}}.$$

Since $S_{n_k} \to S$ and $P_{n_k} \to P$, and the right member of (44) is continuous when considered as a function of two independent points P and S on the same orbit, we get

$$\lim_{S_{n_k} \to S}\left(\lim_{\Delta x \to 0} \frac{\Delta r}{\Delta x}\Big|_{S_{n_k}}\right) = \lim_{\Delta x \to 0} \frac{\Delta r}{\Delta x}\Big|_{S}.$$

Suppose now that $\lim\limits_{\Delta x \to 0} (\Delta r / \Delta x)$ is not continuous at some point S in A other than the origin. Then there exists a sequence of points $\{\bar{S}_n\}$ converging to S, such that the limit as $n \to \infty$ of the values of the right member of (44) evaluated at $\bar{S}_n$ exists and is different from the value of the same expression at S. The limit of the values taken over any subsequence $\{\bar{S}_{n_k}\}$ of $\{\bar{S}_n\}$ has the same value as the limit taken over $\{\bar{S}_n\}$. However by the argument just given there exists a subsequence $\{\bar{S}_{n_k}\}$ such that the limit of the values of the right member of (44) taken over this subsequence is the value at S. Hence no such sequence $\{\bar{S}_n\}$ as was assumed to exist can exist, and hence the right member of (44) is continuous at every point S of A other than the origin.

XV. We have shown that if (x, y) is a point S in A other than the origin,

$$(46)\quad J_x(x, y) = \frac{-2rg(S)(f^2(P) + g^2(P))^{1/2}}{\begin{array}{l}-g(S)(-\phi_{x_0}(P, S)g(P) + \phi_{y_0}(P, S)\ f(P))\\ \quad + f(S)(-\psi_{x_0}(P, S)g(P) + \psi_{y_0}(P, S)f(P))\end{array}}.$$

We shall now simplify the form of the denominator of the right member of this equation. Let the denominator be denoted by q, and let P be a fixed point (a, b). Recalling that S is given by $x = \phi(t, a, b)$, $y = \psi(t, a, b)$, where $a = \phi(0, a, b)$ and $b = \psi(0, a, b)$, for some value of t, we see that as t varies, S varies on the orbit through P. Noting further that (f, g), (ϕ_{x_0}, ψ_{x_0}), and (ϕ_{y_0}, ψ_{y_0}) are solutions of the equations of variation for the orbit through P, we see that q has a derivative with respect to t, and it is given by

$$
\begin{aligned}
(47)\quad dq/dt = {} & g_x(S)f(S)g(P)\phi_{x_0}(P,S) + g_y(S)g(S)g(P)\phi_{x_0}(P,S) \\
& - g_x(S)f(S)f(P)\phi_{y_0}(P,S) - g_y(S)g(S)f(P)\phi_{y_0}(P,S) \\
& - f_x(S)f(S)g(P)\psi_{x_0}(P,S) - f_y(S)g(S)g(P)\psi_{x_0}(P,S) \\
& + f_x(S)f(S)f(P)\psi_{y_0}(P,S) + f_y(S)g(S)f(P)\psi_{y_0}(P,S) \\
& + f_x(S)g(S)g(P)\phi_{x_0}(P,S) + f_y(S)g(S)g(P)\psi_{x_0}(P,S) \\
& - f_x(S)g(S)f(P)\phi_{y_0}(P,S) - f_y(S)g(S)f(P)\psi_{y_0}(P,S) \\
& - g_x(S)f(S)g(P)\phi_{x_0}(P,S) - g_y(S)f(S)g(P)\psi_{x_0}(P,S) \\
& + g_x(S)f(S)f(P)\phi_{y_0}(P,S) + g_y(S)f(S)f(P)\psi_{y_0}(P,S) \\
= {} & (f_x(S) + g_y(S)) \cdot q.
\end{aligned}
$$

Hence $q = K \exp [\int_0^t (f_x + g_y)dt]$, where K is a constant, and the integration is along the orbit from P to S. To evaluate K, let $t = 0$. Then S is at P, and $q(P) = f^2(P) + g^2(P)$. Hence we have

$$
(48)\qquad q = (f^2(P) + g^2(P)) \cdot \exp [\int_0^t (f_x + g_y)dt]
$$

and we may write for any point S in A with coördinates $(x, y) \neq (0, 0)$,

$$
(49)\quad J_x(x,y) = \frac{-2rg(S)}{(f^2(P) + g^2(P))^{1/2} \cdot \exp [\int_0^t (f_x + g_y)dt]}.
$$

Similarly it can be shown that

$$
(50)\quad J_y(x,y) = \frac{2rf(S)}{(f^2(P) + g^2(P))^{1/2} \cdot \exp [\int_0^t (f_x + g_y)dt]}
$$

for $(x, y) \neq (0, 0)$. We have, therefore, a function $J(x, y) \equiv r^2$ which satisfies conditions (i), (ii), and (iii), and hence the problem of finding a solution of (1) has been solved for the case where all the orbits of (2) are closed curves about the singular point. The solution $J(x, y)$ has been shown to have the further property that, except possibly at the singular point, $J_x(x, y)$ and $J_y(x, y)$ are also continuous. Whether or not these partial derivatives are continuous at the origin we are unable to say. No example has yet been found by the writer in which they are not continuous, and a condition sufficient to insure their continuity is the following: Let $\omega(x, y)$ be the period of the functions ϕ and ψ defining the orbit through the point (x, y). If

$$
\lim_{x,y \to 0,0} f_x(x, y) \cdot \omega(x, y)
$$

$$\lim_{x,y\to 0,0} f_y(x,y)\cdot\omega(x,y)$$

$$\lim_{x,y\to 0,0} g_x(x,y)\cdot\omega(x,y)$$

$$\lim_{x,y\to 0,0} g_y(x,y)\cdot\omega(x,y)$$

are all finite, then $\lim_{x,y\to 0,0} J_x(x,y) = \lim_{x,y\to 0,0} J_y(x,y) = 0$. This condition is not necessary, however, as the example $f(x,y) = y\exp\{-[1/(x^2+y^2)]\}$, $g(x,y) = -x\exp\{-[1/(x^2+y^2)]\}$ readily shows.

XVI. It is interesting to observe that in the region A with the origin deleted, which region we shall call A', solutions of (1) in the large can be readily constructed without the difficulties either of the construction of the above function $J(x,y)$ or of those encountered in Kamke's paper. We shall give several functions which are solutions of (1) in A'. First, the function $\omega(x,y)$ already mentioned is a solution, which, however, may not satisfy condition (iii). By the definition of $\omega(x,y)$ we have

$$\begin{aligned}\phi(\omega(x,y),x,y)-x&=0\\ \psi(\omega(x,y),x,y)-y&=0\end{aligned}$$

from which

$$\begin{aligned}f(x,y)\cdot\omega_x(x,y)+\phi_{x_0}(\omega(x,y),x,y)-1&=0\\ g(x,y)\cdot\omega_x(x,y)+\psi_{x_0}(\omega(x,y),x,y)\quad&=0\end{aligned}$$

whence finally

$$\omega_x(x,y)=\frac{f(x,y)-f(x,y)\phi_{x_0}(\omega(x,y),x,y)-g(x,y)\psi_{x_0}(\omega(x,y),x,y)}{f^2(x,y)+g^2(x,y)}.$$

Similarly we get

$$\omega_y(x,y)=\frac{g(x,y)-g(x,y)\psi_{y_0}(\omega(x,y),x,y)-f(x,y)\phi_{y_0}(\omega(x,y),x,y)}{f^2(x,y)+g^2(x,y)}.$$

Since $\omega(x,y)$ is constant along each orbit, and has partial derivatives (which are continuous in A'), it is a solution of (1) with condition (iii) removed.

Second, the length $L(x,y)$ of a trajectory T from a fixed point (μ,ν) on T to the point in which it cuts the orbit through (x,y) is a solution of equation (1), and satisfies condition (iii). We have already shown that T cuts each orbit of A' once and only once, so that the function $L(x,y)$ is well defined and satisfies condition (iii). That it satisfies equation (1) follows by using equations (17), (20), (22), and (23).

Third, the length $l(x,y)$ of the orbit through (x,y) is a solution in A' with condition (iii) removed. To prove this we note first that

$$l(x, y) = \int_0^{\omega(x,y)} (f^2(\phi, \psi) + g^2(\phi, \psi))^{1/2} dt$$

where $\phi = \phi(t, x, y)$ and $\psi = \psi(t, x, y)$. It is now evident that the function $l(x, y)$ has partial derivatives with respect to x and y (which are continuous in A'), and since $l(x, y)$ is constant along each orbit it is a solution of (1).

Fourth, the area $A(x, y)$ contained within the orbit through the point (x, y) is a solution of (1) in A', and we note particularly that condition (iii) is satisfied by this solution. To prove that $A(x, y)$ has partial derivatives with respect to x and y (which are continuous in A') we write

$$A(x, y) = \tfrac{1}{2} \int_0^{\omega(x,y)} (\phi g(\phi, \psi) - \psi f(\phi, \psi)) dt$$

where $\phi = \phi(t, x, y)$ and $\psi = \psi(t, x, y)$. The result follows immediately.

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ON THE DISTRIBUTION OF CONJUGATE POINTS ON A CLOSED GEODESIC.

By Daniel C. Lewis, Jr.[*]

Introduction. We consider the Jacobi differential equation for a closed geodesic of length 2π. Using suitable coördinates (t, u) we may write it in the well known form

$$d^2u/dt^2 + k(t)u = 0, \tag{0.1}$$

where $k(t)$ is periodic with the period 2π. Moreover $k(t)$ is precisely the total curvature at the point $(t, 0)$ and is therefore analytic, if, as we assume, the surface is analytic. The conjugate points of a given point are given by the zeros of a solution of (0.1) which vanishes at the given point without vanishing identically. There certainly exist cases when any solution has an infinite number of zeros; for example, if $k(t)$ is everywhere positive. We restrict ourselves to these cases.

Let $\chi(t)$ denote the first point after t which is conjugate to t. The transformation $t' = \chi(t)$ is then easily seen, by the Sturm separation theorem, to be monotonic increasing. Furthermore it is continuous, and even analytic with an analytic inverse.[†] If we take t_0 arbitrarily and let $t_n = \chi(t_{n-1})$, we know that there exists a Poincaré rotation number β, independent of t_0 such that $2\pi\beta = \lim_{n=\infty} (t_n/n)$. Furthermore we know from Denjoy's fundamental result [‡] that a necessary and sufficient condition that these conjugate points t_n should be everywhere dense on the closed geodesic is that β should be irrational.

The distribution function, $\Delta(\tau)$, for these conjugate points is defined as follows: Let $N(n, \tau)$ denote the number of the points $t_1, t_2, \cdots, t_n$, which after being reduced modulo 2π lie on the open interval $0 < t < \tau \leqq 2\pi$. Then $\Delta(\tau) = \lim_{n=\infty} [N(n, \tau)/n]$, if this limit exists.

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[†] Let $u(t, \tau)$ be the solution such that $u(\tau, \tau) = 0$ and $\partial u/\partial t = 1$ for $t = \tau$. Then, if $u(t'_1, \tau_1)$ also vanishes $(t'_1 \neq \tau_1)$, we know by the uniqueness theorem for differential equations that $\partial u/\partial t$ can not vanish at the point $t = t'_1$, $\tau = \tau_1$; so that the equation $u(t', \tau) = 0$ can be solved analytically for t' as a function, $\chi(\tau)$, of τ in the neighborhood of $\tau = \tau_1$, $t' = t'_1$.

[‡] A. Denjoy, "Sur les caractéristiques à la surface du tore," *Comptes Rendus*, vol. 194 (1932), pp. 830-833. The existence of a uniform analytic inverse to $\chi(t)$ shows that $d\chi/dt$ can never vanish and hence $\log (d\chi/dt)$ is of bounded variation as required by Denjoy.

It is the principal object of this paper to show that, if β is irrational, then $\Delta(\tau)$ not only exists but is analytic. We even write down a simple explicit expression for $\Delta(\tau)$ in terms of a fundamental set of solutions of (0.1).

Our main result is hardly more than a reflection of the fact that it is possible, by a linear transformation on the variables u and du/dt with coefficients that are analytic periodic functions of t, to transform the equation (0.1) into the form $d^2u/dt^2 + ku = 0$ where k is a constant.* Nevertheless, the methods are of considerable interest in themselves and yield somewhat more general theorems concerning the distribution of the characteristics of a torus. These developments will be found in the first three sections. In the concluding section we make the final application to the problem here set forth. It is also possible, by some obvious slight changes, to obtain analogous results for the general second order linear differential equation whose coefficients are periodic but not necessarily analytic. In the latter case, $\Delta(\tau)$ is not, of course, analytic but possesses a high degree of regularity depending upon the regularity of the coefficients.

Let $u(t)$ denote a solution of (0.1) which does not vanish identically. It follows that $u(t)$ and $v(t) = du/dt$ can not vanish simultaneously. Now set $x(t) = \tan^{-1}[v(t)/u(t)]$ and determine $x(t)$ as a single valued function of t by continuity from an arbitrary determination at $t = 0$. It follows that $x(t)$ satisfies the differential equation,

$$dx/dt = -k(t)\cos^2 x - \sin^2 x. \tag{0.2}$$

It is this reduction which leads to the consideration of the characteristics of a torus.

It is well known that an equation of the form (0.2) gives rise to a rotation number α, such that, if $x(t)$ is any solution of (0.2), then $\lim_{t=\infty} [x(t)/t] = \alpha$. Before passing on to the more general theory of the next three sections, it will be convenient to deduce the relation connecting α and the other rotation number β:

We consider a solution $u(t) \not\equiv 0$ vanishing for $t = t_0, t_1, t_2, \cdots$. It can be easily shown that $t_n \to \infty$ as $n \to \infty$. We may take $x(t_0) = \frac{1}{2}\pi$. By continuity we then infer that $x(t_n) = \frac{1}{2}\pi - n\pi$, $n = 1, 2, \cdots$. But

$$\begin{aligned} 2\pi\beta \cdot \alpha &= \lim_{n=\infty} [t_n/n][x(t_n)/t_n] = \lim_{n=\infty} [x(t_n)/n] \\ &= \lim_{n=\infty} [\tfrac{1}{2}(\pi/n) - \pi] = -\pi, \end{aligned}$$

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* Cf. for example G. D. Birkhoff, "Dynamical systems," *American Mathematical Society Colloquium Publications*, vol. 9, chapter 3, first part of § 9.

and hence $\alpha\beta = -\frac{1}{2}$. Thus we obtain another necessary and sufficient condition that the conjugate points be everywhere dense on the geodesic, viz., that α be irrational. Since β is obviously positive, we also obtain the result (useful for later purposes) that α must be negative.

1. *Definition of the two dimensional and linear distribution functions.* We consider the torus obtained by reducing the points of the xy-plane modulo 2π upon the square, $0 \leqq x < 2\pi$, $0 \leqq y < 2\pi$.

Let E denote any Borel measurable point set on this torus.

Consider also on the torus the continuous curve, $x = x(t)$, $y = y(t)$, $[0 \leqq t < \infty]$ and let $\Phi(T, E) = (1/T)$ times the measure of the point set of points t on the interval $0 < t < T$ for which $[x(t), y(t)] \subset E$. If the limit of $\Phi(T, E)$ as $T \rightarrow \infty$ exists whenever E is a rectangle whose sides are parallel to the x and y axes, with the exception perhaps of certain rectangles whose sides lie on an at most denumerable set of singular lines, the curve is said to possess the *distribution function* $\Phi(E) = \lim_{T=\infty} \Phi(T, E)$. The domain of definition of $\Phi(E)$ thus contains any set E which can be constructed from an at most denumerable set of rectangles (closed or open), none of whose sides lie on the singular lines, by the operations of addition, multiplication and the taking of complements.*

If $\Phi(E)$ can be represented in the form, $\int\int_E \psi(x, y)\, dx dy$, of a Lebesgue integral, then there are no singular lines and $\psi(x, y)$ is called the *distribution density*. We shall meet cases where such a distribution density exists and is even analytic.

Consider, on the torus, the circle $y = y_1$, and, on this circle, the open arc $0 < x < x_1 \leqq 2\pi$. Let us now suppose that the curve $[x(t), y(t)]$ cuts the circle an infinite number of times but that the points on the t axis for which $y(t) = y_1 \pmod{2\pi}$ have no limit point (other than $t = \infty$). Starting from $t = 0$, we follow the curve in the sense for which t increases until we have crossed the circle $y = y_1$ exactly N times. Denote by $\nu(N, x_1, y_1)$ the number of these crossings which take place on the interval $0 < x < x_1$ and let

$$\eta(x_1, y_1) = \lim_{N=\infty} \frac{\nu(N, x_1, y_1)}{N}, \text{ if this limit exists.}$$

This function $\eta(x_1, y_1)$ we shall call the *linear distribution function* for the circle $y = y_1$.

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* For a discussion of the singular lines of a monotone absolutely additive set function, cf. J. Radon, "Theorie und Anwendungen der absolut additiven Mengenfunktionen," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften*, Wien, vol. 128 (1919), pp. 1083-1121.

By interchanging the rôles of x and y we similarly define the linear distribution function $\xi(x_1, y_1)$.

The linear distribution functions, in contrast to the two dimensional distribution function $\Phi(E)$, are independent of the parametric representation of the curve in question. This is true in the sense that $\eta(x_1, y_1)$ is invariant under any change of parameter of the type, $t^* = t^*(t)$, where $t^*(0) = 0$ and $t^*(t)$ is continuous and monotonically increasing for $0 \leqq t < \infty$.

2. *The relations between the linear distribution functions and the distribution density in the case of the characteristics on a torus.* We now limit attention to curves which are solutions of a system of differential equations of the type,

$$dx/dt = X(x, y), \qquad dy/dt = Y(x, y), \tag{2.1}$$

where X and Y are continuous and periodic in both x and y with the period 2π. We assume certain further conditions of regularity for X and Y, so that there exists a unique solution

$$x = f(t, x_0, y_0), \qquad y = g(t, x_0, y_0) \tag{2.2}$$

taking on the values x_0, y_0 for $t = 0$ and such that f and g are of class C'' in x_0, y_0 as well as t. These conditions are satisfied, for example, if X and Y are of class C''.

Let us assume that any solution curve on the torus cuts the circle $y = y_1$ infinitely often. Then we proceed to prove the following theorem:

If a solution curve possesses a continuous non-vanishing distribution density, $\psi(x, y)$, and if Y does not vanish for $y = y_1$, then the linear distribution function $\eta(x_1, y_1)$ exists and is given by †

$$\eta(x_1, y_1) = \frac{\int_0^{x_1} \psi(x, y_1) Y(x, y_1)\, dx}{\int_0^{2\pi} \psi(x, y_1) Y(x, y_1)\, dx}. \tag{2.3}$$

Proof. Consider the locus defined by the equations, $x = f(\tau, x_0, y_1)$, $y = g(\tau, x_0, y_1)$, as τ and y_1 are held constant while x_0 varies on the interval $0 \leqq x \leqq 2\pi$. If τ is positive but sufficiently small, this locus is obviously a closed curve C_τ on the torus lying uniformly near the circle, $y = y_1$, but never touching it. The narrow band shaped region bounded by C_τ and the circle we denote by $R_\tau(y_1)$.

---

† As a matter of fact $\eta(x_1, y_1)$ exists under very much lighter hypotheses, but it is then not possible to establish such a simple explicit formula.

Now consider the curve segments $x = f(\sigma, 0, y_1)$, $y = g(\sigma, 0, y_1)$, and $x = f(\sigma, x_1, y_1)$, $y = g(\sigma, x_1, y_1)$, where the parameter σ is supposed to vary on the interval $0 \leqq \sigma \leqq \tau$. We define the open region $U_\tau(x_1, y_1)$ as the part of $R_\tau(y_1)$ which is cut out by these two curves and which has the point $(\frac{1}{2}x_1, y_1)$, say, on its boundary.

Evidently,

$$\frac{\nu(N, x_1, y_1)}{N} = \frac{\tau\nu(N, x_1, y_1)}{\tau N}$$

$$= \frac{\text{meas. of pt. set on interval } 0 \leqq t \leqq T(N) \text{ for which } [x(t), y(t)] \subset U_\tau(x_1, y_1)}{\text{meas. of pt. set on interval } 0 \leqq t \leqq T(N) \text{ for which } [x(t), y(t)] \subset \mathrm{R}_\tau(y_1)}$$

$$= \frac{\Phi[T(N), U_\tau(x_1, y_1)]}{\Phi[T(N), R_\tau(y_1)]},$$

where $T(N)$ denotes a suitably chosen value of t between the N-th crossing and the $N + 1$-th as t increases from zero. It is assumed that τ has been chosen so small that $Y(x, y)$ does not vanish in $U_\tau(x_1, y_1)$. $T(N)$ evidently $\to \infty$ as $N \to \infty$, since $T(N) \geqq \tau N$ which $\to \infty$ for τ fixed. Hence, passing to the limit, we obtain from the definitions of η, Φ, and ψ the following result:

$$(2.4) \qquad \eta(x_1, y_1) = \frac{\Phi[U_\tau(x_1, y_1)]}{\Phi[R_\tau(y_1)]} = \frac{(1/\tau) \displaystyle\iint_{U_T(x_1, y_1)} \psi(x, y)\, dx dy}{(1/\tau) \displaystyle\iint_{R_T(y_1)} \psi(x, y)\, dx dy.}$$

Since $\eta(x_1, y_1)$ is independent of τ, we shall pass to the limit as $\tau \to 0$. In order to evaluate this limit let us define the region $V_\tau(x_1, y_1)$ as the part of $R_\tau(y_1)$ which is cut out by the circles $x = 0$ and $x = x_1$ and which has the point $(\frac{1}{2}x_1, y_1)$ on its boundary. The difference in measure of $U_\tau(x_1, y_1)$ and $V_\tau(x_1, y_1)$ can be easily appraised and is found to be less in absolute value than an infinitesimal of the second order in τ. Hence we may write

$$(2.5) \qquad \left| (1/\tau) \iint_{U_T(x_1, y_1)} \psi(x, y)\, dx dy - (1/\tau) \iint_{V_T(x_1, y_1)} \psi(x, y)\, dx dy \right| < \epsilon,$$

as long as $0 < \tau < \delta(\epsilon)$, where ϵ is a prescribed positive number. Here we use the fact that $\psi(x, y)$ is bounded since it is continuous on a closed manifold.

Since $\partial f/\partial x_0(\tau, x_0, y_1)$ is uniformly continuous in τ and x_0 in any closed region of (τ, x_0) space and since it is identically 1 for $\tau = 0$, we have $\partial f/\partial x_0 > 0$ for τ positive but sufficiently small. Hence there is no difficulty in eliminating the parameter x_0 from the equations defining C_τ, viz.,

$$x = f(\tau, x_0, y_1), \quad y = g(\tau, x_0, y_1).$$

By Taylor's theorem we have

$$x - x_0 = \tau \frac{\partial f}{\partial \tau}(\tau', x_0, y_1), \qquad 0 \leqq \tau' \leqq \tau,$$

$$y - y_1 = \tau \frac{\partial g}{\partial \tau}(0, x_0, y_1) + \tfrac{1}{2}\tau^2 \frac{\partial^2 g}{\partial \tau^2}(\tau'', x_0, y_1), \qquad 0 \leqq \tau'' \leqq \tau,$$

$$\frac{\partial g}{\partial \tau}(0, x, y_1) = \frac{\partial g}{\partial \tau}(0, x_0, y_1) + (x - x_0)\frac{\partial^2 g}{\partial x \partial \tau}(0, \bar{x}, y_1), \qquad 0 \leqq \bar{x} \leqq 2\pi.$$

Combining these we easily see that

$$y = y_1 + \tau \frac{\partial g}{\partial \tau}(0, x, y_1) + \tau^2 A(x, \tau) = y_1 + \tau Y(x, y_1) + \tau^2 A(x, \tau),$$

where A is bounded for τ sufficiently small and $0 \leqq x \leqq 2\pi$. Hence, we find

$$(1/\tau) \iint\limits_{V_\tau(x_1, y_1)} \psi(x, y)\, dx dy = \pm (1/\tau) \int_0^{x_1} \left\{ \int_{y_1}^{y_1 + \tau Y(x, y_1) + \tau^2 A(x, \tau)} \psi(x, y)\, dy \right\} dx$$

$$= \pm \int_0^{x_1} [Y(x, y_1) + \tau A(x, \tau)] \psi(x, \bar{y})\, dx, \quad | y_1 - \bar{y} | \leqq | \tau Y(x, y) + \tau^2 A(x, \tau)$$

where $\pm$ is taken $+$ or $-$ according as $Y(x, y_1)$ is positive or negative. On account of the uniform continuity of $\psi(x, y)$ it is seen that the integrand of the last written integral converges uniformly as τ approaches zero. Hence we can pass to the limit under the sign of integration. It follows from (2.5) that $\lim\limits_{\tau=0} (1/\tau) \iint\limits_{U_\tau(x_1, y_1)} \psi(x, y)\, dx dy = \pm \int_0^{x_1} Y(x, y_1) \psi(x, y_1)\, dx$. In particular, by taking $x_1 = 2\pi$, we get also

$$\lim_{\tau=0} (1/\tau) \iint\limits_{R_\tau(y_1)} \psi(x, y)\, dx dy = \pm \int_0^{2\pi} Y(x, y_1) \psi(x, y_1)\, dx.$$

Substituting in (2.4) we see that (2.3) has been proved.

Similarly we may prove the formula,

$$\xi(x_1, y_1) = \frac{\int_0^{y_1} X(x_1, y) \psi(x_1, y)\, dy}{\int_0^{2\pi} X(x_1, y) \psi(x_1, y)\, dy}, \tag{2.6}$$

under the same hypotheses with the rôles of x and y interchanged.

3. *The distribution function for the characteristics on a torus.* If Y is nowhere zero, as we hereby assume, we can take y for the independent variable t in the differential equations (2.1), so that from now on we write

$$dx/dt = X(x, t); \quad Y(x, y) \equiv 1, \quad y = t. \tag{3.1}$$

According to the introductory remarks of the preceding section, we assume that the solution $x = x(t, x_0)$, which takes on the initial value x_0 for $t = 0$, admits a continous 2nd partial derivative with respect to x_0. It is known that any such solution is of the form $\alpha t + 0(1)$ as $t \rightarrow \infty$, where α is independent of both t and x_0.*

If we set $x_1 = x(2\pi, x_0)$, we have also $x_0 = x(-2\pi, x_1)$; and we surely know that both dx_1/dx_0 and dx_0/dx_1 are continuous and of bounded variation. Hence dx_1/dx_0 can be neither zero nor infinite and therefore $\log(dx_1/dx_0)$ is of bounded variation. It follows from a result of Denjoy † that, if α is irrational, any solution curve must be everywhere dense on the torus. We hereafter always assume α to be irrational. We may now apply Bohl's result,‡ which allows us to write any particular solution in the form, $x = \alpha t + F(\alpha t, t)$, where $F(x, y)$ is continuous and periodic in both x and y with the period 2π and is such that $x + F(x, y)$ is monotonic in x and nowhere constant. We can go even further and state that then the general solution may be written in the form,

$$x = \alpha t + c + F(\alpha t + c, t), \tag{3.2}$$

where c is a constant of integration and is thus a function of x_0 given implicitly by the equation $x_0 = c + F(c, 0)$.

If $\lim_{T=\infty} (1/T) \int_0^T e^{imx(t)+iny(t)} dt$ exists for all integral values of m and n (positive, negative, or zero), then the curve $x = x(t)$, $y = y(t)$ possesses a distribution function $\Phi(E)$ on the torus.§ Furthermore $\Phi(E)$ must satisfy the relations,

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* Cf. H. Poincaré, *Oeuvres complètes*, t. 1, pp. 137-158; E. E. Levi, "Sur les équations différentielles périodiques," *Comptes Rendus*, vol. 153 (1911), pp. 799-802; P. Bohl, "Ueber die hinsichtlich der unabhängigen und abhängigen Variabeln periodische Differentialgleichung erster Ordnung," *Acta Mathematica*, vol. 40 (1916), pp. 321-336; A. Weil, "On systems of curves on a ring-shaped surface," *The Journal of the Indian Mathematical Society*, vol. 19 (1931), pp. 109-114.

† A. Denjoy, "Sur les courbes définies par les équations différentielles à la surface du tore," *Journal de Mathematique*, 9e. série, vol. 11 (1932), pp. 334-375.

‡ *Loc. cit.*

§ Cf. E. K. Haviland and A. Wintner, "A note on the Kronecker-Weyl theorem," *American Journal of Mathematics*, vol. 56 (1934), pp. 17-24. It should be noted that Haviland and Wintner use

$$\lim_{T=\infty} (1/2T) \int_{-T}^{+T} \cdots \text{ instead of our } \lim_{T=\infty} (1/T) \int_0^T \cdots,$$

with a corresponding modification in the definition of $\Phi(E)$. These differences are, however, unessential.

$$(3.3)\qquad \lim_{T=\infty}\,(1/T)\int_0^T e^{imx(t)+iny(t)}dt=\int\!\!\int e^{imx+iny}d_{xy}\Phi(E),$$

where the integral on the right is to be taken in the sense of Radon-Stieltjes over the whole torus. These relations determine $\Phi(E)$ uniquely, at least for all point sets E which are sums of non-singular rectangles pertaining to Φ.

We purpose to exhibit a set function $\Phi(E)$ which satisfies (3.3) in case $x(t)=\alpha t+c+F(\alpha t+c,t)$, $y(t)=t$. In this case the left members certainly exist, since the $e^{imx(t)+iny(t)}$ are almost periodic.

For this purpose we need two lemmas:

Lemma 1. *The equation* $x=\bar{x}+F(\bar{x},y)$ *can be solved for* $\bar{x}$ *in the form*

$$(3.4)\qquad \bar{x}=G(x,y)$$

where G *is continuous in both* x *and* y *and monotonic increasing with respect to* x *and such that* $G(x+2\pi,y)\equiv G(x,y)+2\pi$.

The proof of this lemma follows in a simple manner from the properties already announced of $F(x,y)$, and is left to the reader.

Lemma 2. *Let* $L(x,y)$ *be continuous and periodic in* x *and* y *with the period* 2π *and let* α *be irrational; then*

$$(3.5)\qquad \lim_{T=\infty}\,(1/T)\int_0^T L(\alpha t,t)\,dt=(1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi}L(x,y)\,dxdy.$$

Proof. First suppose that $L(x,y)$ is a trigonometric polynomial of the form,

$$L(x,y)=\sum_{p,q}a_{p,q}e^{ipx+iqy};\quad L(\alpha t,t)=\sum_{p,q}a_{p,q}e^{ip\alpha t+iqt}.$$

Then an explicit evaluation yields $a_{0,0}$ for both sides of (3.5). This proof depends upon the fact that $p\alpha+q$ (where p and q are integers) can never vanish if α is irrational, except for $p=0$, $q=0$.

If $L(x,y)$ is not a trigonometric polynomial, it can nevertheless be approximated uniformly by means of such polynomials and it is easy to complete the proof.

We now take $L(x,y)=e^{im[x+c+F(x+c,y)]+iny}$ and therefore obtain:

$$\lim_{T=\infty}\,(1/T)\int_0^T e^{im[\alpha t+c+F(\alpha t+c,t)]+int}dt$$
$$=(1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi}e^{im[x+c+F(x+c,y)]+iny}dxdy.$$

Introducing $\bar{x}=x+c$ and remembering that the integrand is periodic with period 2π, this becomes

$$(1/4\pi^2)\int_0^{2\pi}\left[\int_0^{2\pi} e^{im[\bar{x}+F(\bar{x},y)]+iny}d\bar{x}\right]dy,$$

which by Lemma I is obviously the same as

$$(1/4\pi^2)\int_0^{2\pi}\left[\int_0^{2\pi} e^{imx+iny}d_xG(x,y)\right]dy,$$

where the integral in the brackets is a Stieltjes integral depending on a parameter y.

Thus, referring back to (3.3), we see that *every solution curve* $x=\alpha t+c+F(\alpha t+c,t)$, $y=t$ *possesses a distribution function* $\Phi(E)$, *which is defined as follows in case* E *is a rectangle with sides* $x=x'$, $x=x''$, $y=y'$, $y=y''$ $(0\leqq x'\leqq x''\leqq 2\pi;\ 0\leqq y'\leqq y''\leqq 2\pi)$:

$$\Phi(E)=(1/4\pi^2)\int_{y'}^{y''}[G(x'',y)-G(x',y)]dy.$$

This definition may readily be extended by the additive property for all Borel measurable point sets.

In case $G(x,y)$ *admits a continuous partial derivative with respect to* x, *we have the continuous distribution density*, $\psi(x,y)\equiv(\partial G/\partial x)/4\pi^2$.

It should also be noted that neither Φ nor ψ depend upon c.

4. *Application to equation* (0.2). We write equation (0.2) in the form of a system:

$$\begin{aligned} dx/dt &= X(x,y)\equiv -k(y)\cos^2 x-\sin^2 x\\ dy/dt &= Y(x,y)\equiv 1. \end{aligned}\tag{4.1}$$

Referring back to section 1 and the definition of the distribution function $\Delta(\tau)$ for the conjugate points it is evident that

$$\Delta(y)=\tfrac{1}{2}[\xi(\tfrac{1}{2}\pi,y)+\xi(-\tfrac{1}{2}\pi,y)].$$

But we have already seen in the preceding two sections how $\xi(x,y)$ may be computed when once a particular solution of (0.2) has been put in the normal form of Bohl, $x=\alpha t+F(\alpha t,t)$; at least if the doubly periodic function $F(x,y)$ is such that $x+F(x,y)$ is monotonic in x and satisfies certain simple conditions of regularity which are certainly fulfilled if it is analytic. We shall now proceed to compute $F(x,y)$ and show that it satisfies the required conditions, always assuming, of course, that the rotation number α (or β) is irrational.

We start with the remark that we may exclude the case in which the

Jacobi differential equation has real characteristic exponents. Otherwise, according to a result of Levi-Civita, α would be an integer.*

According to Floquet, we can write down two linearly independent solutions of (0. 1) in the form,

$$\begin{cases} u_1 = A(t) \cos \lambda t - B(t) \sin \lambda t \\ v_1 = C(t) \cos \lambda t - D(t) \sin \lambda t \end{cases} \qquad \begin{cases} u_2 = B(t) \cos \lambda t + A(t) \sin \lambda t \\ v_2 = D(t) \cos \lambda t + C(t) \sin \lambda t, \end{cases}$$

where A, B, C, D are real analytic periodic functions with the period 2π, such that $AD - BC \equiv \iota$, where ι is ± 1. $2\pi\lambda$ is simply the absolute value of either of the two characteristic exponents, $2\pi\lambda(-1)^{1/2}$ and $-2\pi\lambda(-1)^{1/2}$. λ is thus determined so far only up to an additive integer. We may further require, however, that $\tan^{-1}[C(t)/A(t)]$ shall pass continuously through a zero angle (instead of through some non-zero integral multiple of 2π) as t increases from 0 to 2π. $\tan^{-1}[D(t)/B(t)]$ will then have the same property as follows from the fact that $AD - BC$ is nowhere zero. Under these circumstances λ is uniquely determined and $\iota\lambda = -\alpha$.† Since α is negative and λ is positive, this shows that $\iota = +1$.

We know that $x = \tan^{-1}[v_1(t)/u_1(t)]$ is a particular solution of (0. 2). It may be written in the form, $x = \alpha t + \phi(t)$, where

$$\phi(t) = -\alpha t + \tan^{-1}\left(\frac{C(t) \cos \alpha t + D(t) \sin \alpha t}{A(t) \cos \alpha t + B(t) \sin \alpha t}\right),$$

the branch of the inverse tangent being determined from an arbitrary initial determination in such a way that $\phi(t)$ is continuous. It is well known that $\phi(t)$ is bounded.‡

Let us set formally

$$(4.2) \quad F(x, y) \equiv -x + \tan^{-1}\left(\frac{C(y) \cos x + D(y) \sin x}{A(y) \cos x + B(y) \sin x}\right).$$

This formula may be used to determine a single valued function in the following way:

In the first place the numerator and denominator of the expression appearing in the parenthesis can never vanish simultaneously, since the determinant $AD - BC$ is never zero.

We decide upon a fixed determination of the inverse tangent for $x = 0$,

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[*] "Sur les équations linéaires à coefficients periodiques et sur la moyen mouvement du noeud lunaire," *Annales de l'école normale supérieure*, 3e. série, vol. 28 (1911), pp. 325-376, especially p. 350.

[†] Levi-Civita, *loc. cit.*, especially pages 340-342.

[‡] Cf. Levi-Civita, *loc. cit.*, or the more general result of E. E. Levi, *loc. cit.*

$y=0$, and then determine the proper branch of the inverse tangent at any other point (x', y') so that $F(x, y)$ is continuous along a curve connecting (x', y') with the origin. The value thus assigned to $F(x', y')$ is independent of the choice of the curve, since any two such curves can be continuously deformed into each other without passing a point where the numerator and denominator vanish simultaneously, whereas the *a priori* possible values of $F(x', y')$ differ from each other by integral multiples of π.

$F(x, y)$ is now defined for all real values of x and y and is obviously analytic. We next prove that it admits the period 2π in both x and y. In the first place

$$F(x, y) = F(0, y) + \int_0^x (\partial F/\partial x)\, dx.$$

But $F(0, y) \equiv \tan^{-1}[C(y)/A(y)]$ is periodic in y on account of the choice of A and C already explained, while $\partial F/\partial x$ is also periodic. It follows that $F(x, y)$ is periodic in y. In the second place, we must have $F(x+2\pi, y) = F(x, y) + m\pi$, where m is either zero or an integer. It is independent of x and y since $F(x+2\pi, y) - F(x, y)$ is continuous. Furthermore, if $m \neq 0$, it is easily seen that $\phi(t) \equiv F(\alpha t, t)$ would not be bounded, periodicity in y having already been established. Hence $m=0$, and $F(x, y)$ is also periodic in x.

The fact that $x + F(x, y)$ is monotonic in x can be proved by showing that its derivative with respect to x is everywhere positive. Remembering that $AD - BC \equiv 1$, we easily get

$$[\partial/\partial x][x + F(x, y)] = [(A \cos x + B \sin x)^2 + (C \cos x + D \sin x)^2]^{-1} > 0.$$

We now proceed to calculate the actual distribution functions.

The first step is to solve the equation $x = \bar{x} + F(\bar{x}, y)$ for $\bar{x}$ in terms of x and y. This is easily done and the result is:

$$\bar{x} = G(x, y) \equiv \tan^{-1}\left(\frac{A(y) \sin x - C(y) \cos x}{D(y) \cos x - B(y) \sin x}\right).$$

Hence the distribution density on the torus is given, according to § 3, by: $4\pi^2\psi(x, y) = \partial G/\partial x \equiv [(A \sin x - C \cos x)^2 + (D \cos x - B \sin x)^2]^{-1}$. Now referring back to formulas (4.1) and (2.6) we find that

$\xi(\tfrac{1}{2}\pi, y) \equiv \xi(-\tfrac{1}{2}\pi, y) \equiv \Delta(y) = M \int_0^y \frac{dy}{[A(y)]^2 + [B(y)]^2}$, where M is the reciprocal of $\int_0^{2\pi} \frac{dy}{[A(y)]^2 + [B(y)]^2}$. We may make one more slight simplification by observing that $[A(t)]^2 + [B(t)]^2 \equiv u_1^2 + u_2^2$.

We can check our result with the help of the Kronecker-Weyl Theorem * in the following way: It is evident that

$$u(t, t_0) \equiv A \cos \lambda(t - t_0) - B \sin \lambda(t - t_0) \equiv u_1 \cos \lambda t_0 + u_2 \sin \lambda t_0$$

satisfies (0.1), while $U(T, t_0) \equiv \cos \lambda(T - t_0)$ may be regarded as the corresponding solution of $d^2U/dT^2 + \lambda^2 U = 0$. It is now shown directly that, if the transformation, $T = T(t)$, transforms the zeros of $u(t, t_0)$ into the zeros of $U(T, t_0)$, then

$$\frac{dT}{dt} = \frac{1/\lambda}{[A(t)]^2 + [B(t)]^2},$$

even though λ is rational. In order to do this we have merely to eliminate t_0 from the equations $u(t, t_0) = 0$ and $U(T, t_0) = 0$, solve for T, differentiate and reduce with the help of the identity

$$AD - BC \equiv AB' - BA' + \lambda A^2 + \lambda B^2 \equiv 1.$$

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* For the connection with the Kronecker-Weyl Theorem, see the note of Lewis and Wintner in this volume of the *American Journal of Mathematics*, pp. 407-410.

ON THE THEORY OF ABSOLUTELY ADDITIVE DISTRIBUTION FUNCTIONS.

By E. K. HAVILAND.

Introduction. Recently * the author considered the problem of the asymptotic distribution of a movement in an n-space, e. g., the distribution of a complex-valued almost-periodic function $(n=2)$.† The methods used are those of classical probability theory. In particular, it turns out that in the case of the distribution of the Riemann ζ-function ‡ and similar problems in the analytic theory of numbers, the distribution (or repartition) function § is built up, in the sense of classical probability theory, from statistically independent two-dimensional distributions, each of which represents the distribution of an individual term of the almost-periodic development under consideration.

The purpose of the present paper is to develop, without reference to movements, an exact treatment of general multi-dimensional statistical methods which since the time of Laplace have rested upon a purely formal foundation.¶ Although in one dimension the results are already well established,‖ the extension of the theorems in question to more than one dimension is, in general, by no means immediate, as will be explained later. In fact, extensive use will be made of the Radon theory of monotone absolutely additive set functions,** which are considered as defining distributions in n-dimensional space, so that the present paper may be regarded as a contribution to the theory of these functions and certain integrals associated with them. It may be pointed out that it is not, in general, permissible to suppose that

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* Cf. E. K. Haviland, *loc. cit.*, I and II. (References appear at the end of this article.) In II, $\lim\limits_{T=\infty}\int_{-T}^{T}\cdots\, dt$ should obviously be replaced by $\lim\limits_{T=\infty}\int_{-T}^{T}\cdots\, dt\,/\,2T$.

† The corresponding problem for $n=1$ has been treated by A. Wintner, *loc. cit.*, I, especially pp. 156-157, and III. The extension to the case $n>1$ is, however, not immediate.

‡ Cf. A. Wintner, *loc. cit.*, IV, especially pp. 328-329, where references to the literature are given.

§ As defined, e. g., by the author, *loc. cit.*, I, p. 554.

¶ Cf., e. g., J. Hadamard, *op. cit.*, §§ 430-433.

‖ With perhaps the exception of the analogue of Theorem V of the present paper, for which I am not aware of an exact general published proof.

** Cf. J. Radon, *loc. cit.*, I, especially pp. 1295-1342, and *loc. cit.*, II, especially pp. 1092-1094.

Radon distributions may be represented by means of Lebesgue integrals, i. e., that they are absolutely continuous, as, e. g., it is clear that the partial distribution functions mentioned above in connection with the Riemann ζ-function are not absolutely continuous, and in some other applications there occur even "point spectra," where the Radon distribution is not only not absolutely continuous, but not even continuous.

It has previously been shown [*] that in the case of distribution functions which are zero for all sets lying outside an arbitrarily large but fixed rectangle the Radon momentum problem is determined, in the sense that two distribution functions having the same momenta are identical save at most on their discontinuity lines. Furthermore, if a sequence of distribution functions ϕ_n is such that the momenta approach a limit as n becomes infinite, then $\{\phi_n\}$ converges to a distribution function, provided all ϕ_n are zero for sets lying outside a sufficiently large but fixed rectangle.[†] In the first part of the present paper, it will be shown that the foregoing restriction on the distribution functions can be removed, and it turns out that the question is, as in the one-dimensional case,[‡] rather closely connected with the problem of determinateness in the sense just mentioned. For the determination of the momentum problem, a sufficient and "almost" necessary condition will be obtained.

The second portion of the paper concerns the symbolical product [§] of two general distribution functions, corresponding to the addition of statistically independent distributions. The existence of this symbolical product is established and it is shown that the Fourier transform of the symbolical product of two distribution functions is the product of the Fourier transforms of the individual functions. In a recent paper,[¶] S. Bochner proves by methods differing substantially from those here used that the product of two characteristic functions is again a characteristic function, where by a characteristic function is meant the Fourier transform of a distribution function. He establishes the existence of the symbolical product and proves the relation mentioned above between the symbolical product and Fourier transforms only in the special case that one of the distribution functions is absolutely continuous, so that it can be represented as a Lebesgue integral. As we have pointed out above, such a representation is not always possible, so that the existence of the symbolical product in the general case must be based, as in Theorem IV of the present paper, on a detailed consideration of Radon integrals.

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[*] Cf. E. K. Haviland, *loc. cit.*, I, pp. 550-551. Also C. A. Fischer, *loc. cit.*, p. 40 and T. H. Hildebrandt and I. J. Schoenberg, *loc. cit.*, p. 324.

[†] Cf. E. K. Haviland, *loc. cit.*, I, p. 552. The proof for a sequence of distribution functions is the same as that for a continuum.

[‡] Cf. A. Wintner, *loc. cit.*, II; M. Fréchet and J. Shohat, *loc. cit.*

[§] Also termed "Faltung" or "convolution."

[¶] Cf. S. Bochner, *loc. cit.*

The final portion of this paper treats of the spectra of distribution functions and affords an illustration of some of the important differences between distribution functions in one dimension and those in more than one dimension.

We shall suppose for definiteness that $n = 2$, where n is the number of dimensions of the space in which our set functions are defined, as our discussions admit immediate extension to space of a higher number of dimensions. The case $n = 2$, on the other hand, is for several reasons essentially more complicated than the case $n = 1$. For instance, in the two-dimensional case the discontinuities of the set function do not arise solely from the so-called "point spectrum." Moreover, in one dimension, the set function appears disguised as a point function, so that its true rôle is not as clear as in Radon integrals, at least if the latter are defined over sets more complicated than rectangles.

1. *The momenta of distribution functions.*

Definition 1. The monotone absolutely additive set function* $\phi(E)$ is said to be a distribution function if $0 \leqq \phi(E) \leqq 1$ and

$$\phi(S) = \int \cdots \int_S d_{x_1} \cdots {}_{x_n} \phi(E) = 1,$$

where S denotes the whole n-space.

For theoretical purposes, of course, we are interested in distribution functions primarily insofar as they are monotone absolutely additive set functions of bounded total variation such that any set of them are of uniformly bounded total variation, and our theorems are essentially theorems on functions with these properties.

If, in addition, $\phi(E) = 0$ for all sets E outside a fixed finite rectangle K, we shall say that $\phi(E)$ is damped. In the present paper, we shall not suppose distribution functions damped unless this fact is explicitly stated.

By reasoning precisely similar to that used by Radon in the case of set functions defined for sets lying in a finite rectangle,† we may assign to $\phi(E)$ a point function defined by $F(\xi, \eta) = \phi(R_{\xi\eta})$, where $R_{\xi\eta}$: $(-\infty < x < \xi; -\infty < y < \eta)$. Then it follows,‡ since ϕ is a distribution function, that F, and hence by definition ϕ, is continuous save at most for points on a finite or denumerable set of lines $x = \xi_i$, $y = \eta_j$ corresponding to the discontinuities of $F(x, +\infty)$, $F(+\infty, y)$ respectively. These lines $x = \xi_i$, $y = \eta_j$ are called singular lines of the distribution function ϕ.

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* For definition, cf. J. Radon, *loc. cit.*, I, p. 1299.

† Cf. J. Radon, *loc. cit.*, I, pp. 1304-1305.

‡ Cf. J. Radon, *loc. cit.*, II, p. 1093.

Definition 2. A rectangle is said to be a singular rectangle of ϕ if at least one of its sides is a singular line of ϕ. By a rectangle, singular or non-singular, we shall always mean a rectangle with sides parallel to the coördinate axes.

It is clear that if R is a non-singular rectangle of ϕ and R_1 a second rectangle all of whose sides are sufficiently close to the sides of the former, then $| \phi(R_1) - \phi(R) | < \epsilon$.

Remark. The singular lines or singular rectangles of an arbitrary absolutely additive set function are the singular lines or singular rectangles of the monotone set functions representing its total positive and its total negative variation.*

Definition 3. A sequence of distribution functions $\{\phi_n(E)\}$ is said to converge to a distribution function $\phi(E)$ if $\lim_{n=\infty} \phi_n(R) = \phi(R)$ for all non-singular rectangles R of ϕ.

Definition 4. A sequence of rectangles $\{R_n\}$ is said to be everywhere dense if for every given rectangle $R: (x^1 < x < x^2;\ y^1 < y < y^2)$, not necessarily belonging to the sequence, and for every $\epsilon > 0$ there exists an n such that for the corresponding R_n we have

$$| x^1 - x^1{}_n |, \quad | x^2 - x^2{}_n |, \quad | y^1 - y^1{}_n |, \quad | y^2 - y^2{}_n |$$

all less than ϵ.

If ϕ^{I} and ϕ^{II} are two distribution functions such that $\phi^{\mathrm{I}}(R_n) = \phi^{\mathrm{II}}(R_n)$ on an everywhere dense sequence of rectangles $\{R_n\}$, then $\phi^{\mathrm{I}}(E) = \phi^{\mathrm{II}}(E)$ for every rectangle non-singular with respect to ϕ^{I} and ϕ^{II}. Furthermore, ϕ^{I} and ϕ^{II} have the same singular lines, hence the same singular rectangles.† In what follows, we shall regard two distribution functions as the same and call them essentially identical if their corresponding point functions are identical save at most on their singular lines.

The Fourier transform $\Lambda(s, t; \phi)$ of the distribution function $\phi(E)$ is defined by

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* Cf. E. K. Haviland, *loc. cit.*, I, p. 550.

† The proof is effectively the same as in the damped case. Cf. E. K. Haviland, *loc. cit.*, I, pp. 549-550.

$$\Lambda(s, t; \phi) = \int\!\!\int_S \exp\,[i(sx + ty)]\,d_{xy}\phi(E), \tag{1}$$

where s, t, x, y are all real and S denotes the entire xy-plane.

By the integral $\int\!\!\int_S f(x, y)\,d_{xy}\phi(E)$ of a continuous function $f(x, y)$ with respect to the distribution function ϕ is meant the limit as $R \to S$ of $\int\!\!\int_R f(x, y)\,d_{xy}\phi(E)$, R being an arbitrarily large rectangle, provided that this limit exists. The integral over R exists by a theorem of Radon * and the limit of this integral as $R \to S$ clearly exists for every bounded function f, inasmuch as ϕ is of finite total variation. In particular, $\Lambda(s, t; \phi)$ exists for all real values of s and t.

Definition 5. The Radon momentum problem associated with a distribution function $\phi(E)$ is said to be determined provided that all momenta

$$\mathrm{M}(p, q; \phi) = \int\!\!\int_S x^p y^q d_{xy}\phi(E)$$

exist and are such that any other distribution function ψ for which $\mathrm{M}(p, q; \psi) = \mathrm{M}(p, q; \phi)$; $p, q = 0, 1, 2, \cdots$, must be essentially identical with ϕ.

The author has previously shown that if, as n becomes infinite, $\lim \mathrm{M}(p, q; \phi_n)$ exists for all non-negative integers p, q, and for a sequence of uniformly damped distribution functions $\phi_n(E)$, for which the momentum problem is necessarily determined, then $\lim\limits_{n=\infty} \phi_n(R) = \phi(R)$, whence $\phi(E)$ is a damped distribution function for which R is a non-singular rectangle. In case the momentum problem for the limiting function ϕ is determined, a situation for which a sufficient and "almost" necessary condition will be given below, the foregoing result may be extended to the case of non-damped distribution functions. The first step in the extension is given by

LEMMA I.† *Let there be given a sequence of distribution functions* $\{\psi_n(E)\}$ *with the following properties:*

(A) *The momenta* $\mathrm{M}(p, q; \psi_n)$ *of all orders* $\mathrm{p} = 0, 1, 2, \cdots$;

---

* The integral is that defined by J. Radon, *loc. cit.*, I, pp. 1322-1324. This is an extension of the Riemann-Stieltjes integral in one dimension. For the latter, cf. E. W. Hobson, *op. cit.*, pp. 506-516 and H. Lebesgue, *op. cit.*, pp. 252-313.

† These propositions represent the extension to n dimensions, $n > 1$, of results obtained in the one-dimensional case by A. Wintner, *loc. cit.*, II, and by M. Fréchet and J. Shohat, *loc. cit.*

$q = 0, 1, 2, \cdots$ *exist for* $n = 1, 2, \cdots$ *(or at least from a certain rank* n *on, possibly depending on* p, q*).*

(B) *The quantities* $\mathrm{M}(p, q; \psi_n)$ *for any fixed* p, q *lie, when they exist, between two fixed limits independent of* n *(but possibly dependent on* p *and* q*).*

Then there exists a subsequence $\{\phi_i(E) = \psi_{n_i}(E)\}$ *such that*

(i) *As* $i \rightarrow \infty$, $\lim \mathrm{M}(p, q; \phi_i)$ *exists* ($= \mu_{pq}$, *say*) *for all non-negative values of* p, q, *and*

(ii) $\{\phi_i(R)\}$ *converges to* $\phi(R)$, *where* $\phi(E)$ *is a distribution function, on all non-singular rectangles* R *of* ϕ.

Proof. In virtue of the uniform boundedness of the $\mathrm{M}(p, q; \psi_n)$ for all fixed p and q, from $\{\mathrm{M}(0, 0; \psi_n)\}$ a subsequence $\{\mathrm{M}(0, 0; \psi_{l_j})\}$ having a limit μ_{00} may be selected. From $\{\mathrm{M}(1, 0; \psi_{l_j})\}$ a subsequence $\{\mathrm{M}(1, 0; \psi_{m_j})\}$ converging to a limit μ_{10} may then be selected. Proceeding in this way, we obtain by the diagonal principle of Cantor a sequence $\{\mathrm{M}(p, q; \psi_{l_1}), \mathrm{M}(p, q; \psi_{m_2}), \cdots\}$ converging to a finite limit μ_{pq} for all fixed non-negative values of p, q, where the labels $l_1, m_2, \cdots$ are independent of p, q. The corresponding distribution functions $\gamma_1(E) = \psi_{l_1}(E)$, $\gamma_2(E) = \psi_{m_2}(E), \cdots$ are seen to satisfy (i).

By applying the Compactness Theorem of Radon * to the sequence $\{\gamma_n(E)\}$, we extract from it a subsequence $\{\phi_i(E)\}$ converging to the monotone absolutely additive set function $\phi(E)$ on every non-singular rectangle of the latter.

It remains to be shown that ϕ is actually a distribution function, i. e., that $\phi(S) = 1$. To this end we consider the square $R : (-a < x < a; -a < y < a)$, where $a > 1$. $S - R$ may be divided into two (disconnected) parts $R_1 : (|x| \geqq a; -\infty < y < +\infty)$ and $R_2 : (-a < x < a; |y| \geqq a)$ such that in the former $|x| \geqq a$ while in the latter $|y| \geqq a$. Then by Schwarz' inequality,

$$\left[\int\int_{R_1} x^p y^q d_{xy}\psi_n(E)\right]^2 \leqq \int\int_{R_1} x^{2p} y^{2q} d_{xy}\psi_n(E) \cdot \int\int_{R_1} d_{xy}\psi_n(E)$$

$$\leqq \int\int_{R_1} x^{2p} y^{2q} d_{xy}\psi_n(E) \leqq \mathrm{M}(4p + 2, 2q; \psi_n)/a^{2p+2}$$

and similarly

$$\left[\int\int_{R_2} x^p y^q d_{xy}\psi_n(E)\right]^2 \leqq \mathrm{M}(2p, 4q + 2; \psi_n)/a^{2q+2},$$

---

* Cf. J. Radon, *loc. cit.*, I, p. 1337. A statement of these theorems more convenient for our purposes is given by the author, *loc. cit.*, I, pp. 551-552. The proof of the Compactness Theorem is there given for a finite rectangle but may be extended to the whole plane by the diagonal process of Cantor.

these appraisals being obviously analogous to a device of Tschebycheff.* If now we choose as our $\{\psi_n(E)\}$ the sequence $\{\phi_i(E)\}$ converging to $\phi(E)$, we may infer for i sufficiently large that

$$0 \leqq \mathrm{M}(4p+2, 2q; \phi_i) < \mu_{4p+2,2q} + 1,$$

and

$$0 \leqq \mathrm{M}(2p, 4q+2; \phi_i) < \mu_{2p,4q+2} + 1.$$

If, in particular, we set $p = q = 0$, we obtain

$$\left[\int\int_{R_1} d_{xy}\phi_i(E)\right]^2 < (\mu_{20}+1)/a^2 \text{ and } \left[\int\int_{R_2} d_{xy}\phi_i(E)\right]^2 < (\mu_{02}+1)/a^2.$$

But μ_{20} and μ_{02} are from their definitions non-negative, so

$$0 \leqq \int\int_{R_1} d_{xy}\phi_i(E) < (\mu_{20}+1)^{1/2}/a \text{ and } 0 \leqq \int\int_{R_2} d_{xy}\phi_i(E) < (\mu_{02}+1)^{1/2}/a.$$

Hence

$$0 \leqq \int\int_{S-R} d_{xy}\phi_i(E) < [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a.$$

As $\int\int_{S-R} = \int\int_S - \int\int_R$ and $\phi_i(E)$ is a distribution function, the preceding inequality is equivalent to

$$0 \leqq 1 - \phi_i(R) < [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a.$$

Letting i become infinite, we see that

$$0 \leqq 1 - \phi(R) \leqq [(\mu_{20}+1)^{1/2} + (\mu_{02}+1)^{1/2}]/a,$$

so that

$$\lim_{a=\infty} \phi(R) = \phi(S) = 1, \qquad \text{q. e. d.}$$

Remark. We shall later have occasion to make use of the fact that for any given non-negative values of p and q, the convergence of

$$\int\int_R x^p y^q d_{xy}\psi_n(E) \text{ to } \int\int_S x^p y^q d_{xy}\psi_n(E)$$

is uniform with respect to n, at least from a certain rank n on.

Proof. As in the proof of the preceding lemma,

$$\begin{aligned}\left|\int\int_{S-R} x^p y^q d_{xy}\psi_n(E)\right| &\leqq \left|\int\int_{R_1} x^p y^q d_{xy}\psi_n(E)\right| + \left|\int\int_{R_2} x^p y^q d_{xy}\psi_n(E)\right| \\ &\leqq [\mathrm{M}(4p+2, 2q; \psi_n)]^{1/2}/a^{p+1} + [\mathrm{M}(2p, 4q+2; \psi_n)]^{1/2}/a^{q+1} \\ &\leqq [\mathrm{M}_{4p+2,2q}]^{1/2}/a^{p+1} + [\mathrm{M}_{2p,4q+2}]^{1/2}/a^{q+1},\end{aligned}$$

---

* Cf. P. L. Tschebycheff, *loc. cit.*, p. 691 and A. Kolmogoroff, *op. cit.*, p. 37.

since by condition (B) of the lemma

$$0 \leqq \mathrm{M}(4p+2, 2q; \psi_n) \leqq \mathrm{M}_{4p+2,2q} \text{ and } 0 \leqq \mathrm{M}(2p, 4q+2; \psi_n) \leqq \mathrm{M}_{2p,4q+2},$$

q. e. d.

We are now in a position to prove

THEOREM I (MOMENTUM THEOREM). *Let there be given a sequence of distribution functions* $\{\phi_n(E)\}$ *satisfying conditions* (A) *and* (B) *of Lemma I. Then if* $\lim_{n=\infty} \mathrm{M}(p, q; \phi_n)$ *exists,* $= \mu_{pq}$ *say, for all non-negative values of* p *and* q, *there exists at least one distribution function, say* $\phi(E)$, *such that* $\mu_{pq} = \mathrm{M}(p, q; \phi)$, *and a subsequence* $\{\phi_{n_i}(E)\}$ *can be extracted from the given sequence of distribution functions so that* $\lim_{i=\infty} \phi_{n_i}(R) = \phi(R)$ *for all non-singular rectangles* R *of* $\phi(E)$.

If, in addition, the sequence $\{\mu_{pq}\}$ *is such that* ϕ *is uniquely determined by it, then the sequence* $\{\phi_n(E)\}$ *itself converges as* n *becomes infinite to* $\phi(R)$ *on any non-singular rectangle* R *of* ϕ.

Proof. By Lemma I, the existence of the subsequence $\{\phi_{n_i}\}$ converging to a ϕ is assured.

Now if a sequence of distribution functions $\{\phi_n(E)\}$ is such that

(a) $\lim_{n=\infty} \phi_n(Q) = \phi(Q)$ for every non-singular rectangle Q of the distribution function ϕ, and

(b) $\int\int_R f(x, y)\, d_{xy}\phi_n(E) \to \int\int_S f(x, y)\, d_{xy}\phi_n(E)$ uniformly with respect to n as $R \to S$, $f(x, y)$ being continuous in every finite rectangle R, then the Radon Term by Term Integration Theorem holds for the infinite domain S, so that

$$(2) \qquad \lim_{n=\infty} \int\int_S f(x, y)\, d_{xy}\phi_n(E) = \int\int_S f(x, y)\, d_{xy}\phi(E).$$

For if R'' be a closed rectangle and R' an open rectangle lying in the interior of R'', it follows from the Term by Term Integration Theorem for a finite domain * that

$$\left| \int\int_{R''-R'} f(x, y)\, d_{xy}\phi(E) \right| = \left| \lim_{n=\infty} \int\int_{R''-R'} f(x, y)\, d_{xy}\phi_n(E) \right|$$

---

* Cf. J. Radon, *loc. cit.*, I, p. 1337. A statement of these theorems more convenient for our purposes is given by the author, *loc. cit.*, I, pp. 551-552. The proof of the Compactness Theorem is there given for a finite rectangle but may be extended to the whole plane by the diagonal process of Cantor.

and the right-hand side of this equation is $< \epsilon$ for R' sufficiently large by (b). Hence the integral on the right of (2) exists. Furthermore,

$$\left| \int\int_S f(x, y)\, d_{xy}\phi(E) - \int\int_S f(x, y)\, d_{xy}\phi_n(E) \right| \leqq \left| \int\int_{S-R} f(x, y)\, d_{xy}\phi(E) \right|$$
$$+ \left| \int\int_{S-R} f(x, y)\, d_{xy}\phi_n(E) \right| + \left| \int\int_R f(x, y)\, d_{xy}\phi(E) - \int\int_R f(x, y)\, d_{xy}\phi_n(E) \right| < \epsilon$$

if R be taken sufficiently large and the theorem then applied for a finite domain. Moreover, by the Remark following Lemma I, the foregoing conditions (a) and (b) are satisfied if we replace $\{\phi_n(E)\}$ by $\{\phi_{n_i}(E)\}$ and $f(x, y)$ by $x^p y^q$. Hence

$$\mu_{rs} = \lim_{n=\infty} \mathrm{M}(p, q; \phi_n) = \lim_{i=\infty} \mathrm{M}(p, q; \phi_{n_i}) = \lim_{i=\infty} \int\int_S x^p y^q d_{xy}\phi_{n_i}(E)$$
$$= \int\int_S x^p y^q d_{xy}\phi(E).$$

This proves the first paragraph of the present theorem. The last paragraph is proved as follows:

Suppose there exists a non-singular rectangle R_0 of ϕ such that $\{\phi_n(R_0)\}$ does not converge to $\phi(R_0)$. Then a subsequence $\{\gamma_k(R_0) = \phi_{n_k}(R_0)\}$ can be extracted such that as k becomes infinite $\{\gamma_k(R_0)\}$ converges to a certain number $A \neq \phi(R_0)$. On the other hand, we have seen that the sequence $\{\gamma_k(E)\}$ gives rise to a subsequence $\{\delta_j(E)\}$ which converges as $j \rightarrow \infty$ to a distribution function $\delta(E)$ on all non-singular rectangles of the latter. Now $\delta(E)$ has the same momenta μ_{pq} as has $\phi(E)$, and therefore, since by hypothesis the momentum problem corresponding to $\{\mu_{pq}\}$ is determined,* $\lim \delta_j(R_0) = \delta(R_0) = \phi(R_0)$ when $j \rightarrow \infty$, since R_0 is a non-singular rectangle of ϕ. This contradicts the fact that $\{\delta_j(R_0)\}$ is a subsequence of $\{\gamma_k(R_0)\}$, which converges, but not to $\delta(R_0)$ and hence completes the proof of the theorem.

The condition that the momentum problem corresponding to $\{\mu_{pq}\}$ be determined is not only sufficient for the limiting relation $\lim \phi_n(E) = \phi(E)$ as $n \rightarrow \infty$ on any non-singular rectangle of the latter, but it is necessary as well. For if $\phi(E)$ and $\psi(E)$ be two distinct solutions of the momentum problem in question, then $\{\phi_n(E)\}$ should converge simultaneously to $\phi(E)$ and $\psi(E)$ at all non-singular rectangles of ϕ and ψ. But this is impossible,

---

* The momentum problem associated with $\{\mu_{pq}\}$ is said to be determined if

$$\mu_{pq} = \int\int_S x^p y^q d_{xy}\phi^{\mathrm{I}}(E) = \int\int_S x^p y^q d_{xy}\phi^{\mathrm{II}}(E)\,;\ p, q, = 0, 1, 2, \cdots,$$

implies ϕ^{I} and ϕ^{II} are essentially identical.

as there is at least one non-singular rectangle R_0 common to ϕ and ψ such that $\phi(R_0) \neq \psi(R_0)$.

In view of the rôle thus played by the determinateness of the momentum problem, we prove

THEOREM II.* *A sufficient condition that a distribution function be determined by its momenta* $G_{pq} = \mathrm{M}(p, q; \phi)$ *is that*

$$\left(\sum_{\nu=0}^{2m}\binom{2m}{\nu}\left|G_{2m-\nu,\nu}\right|\right)^{1/(2m)} = o(m). \tag{3}$$

This sufficient condition is, furthermore, almost necessary in that ϕ_1 *need not equal* ϕ_2 *if* $o(m)$ *is replaced by* $o(m^{1+\epsilon})$.

Proof. Let $\phi_1(E)$, $\phi_2(E)$ be distribution functions. Then as these functions are uniquely determined by their Fourier transforms,† the momentum problem is determined under conditions such that $\Lambda(s, t; \phi_1) = \Lambda(s, t; \phi_2)$, where s, t, x, y are all real.

Inasmuch as the formula

$$f(x+h) = f(x) + hf'(x) + \cdots + h^{n-1}(n-1)!^{-1}f^{(n-1)}(x) + (n-1)!^{-1}\int_0^h f^{(n)}(x+t)(h-t)^{n-1}dt$$

is obtained merely by successive partial integrations of the integral occurring in the remainder term, the formula is valid for complex functions of a real variable also, so that we obtain in the present case

$$e^{i\xi} = 1 + i\xi + \cdots + (i\xi)^{n-1}(n-1)!^{-1} + f_n(\xi) \tag{4}$$

where

$$|f_n(\xi)| = \left|i^n(n-1)!^{-1}\int_0^\xi e^{it}(\xi-t)^{n-1}dt\right| \leqq (n-1)!^{-1}\int_0^\xi |(\xi-t)|^{n-1}dt = |\xi|^n/n!$$

Setting $n = 2m$ and $\xi = sx + ty$ in (4), integrating first with respect to ϕ_1, then with respect to ϕ_2 and subtracting, we obtain, since all momenta are equal by hypothesis,

---

* The present proof of the sufficient condition is based on a treatment by F. P. Cantelli of the one-dimensional case under the assumption of absolutely continuous distribution functions. Cf. F. P. Cantelli, *loc. cit.*, pp. 155-157. In the one-dimensional case it is known that the theorem holds even if $o(m)$ and $o(m^{1+\epsilon})$ are replaced by $O(m)$. Cf. T. Carleman, *op. cit.*, p. 81.

† Cf. S. Bochner, *loc. cit.*, p. 402, Theorem 14. Another proof of this theorem will be given below.

$$
\begin{aligned}
&| \Lambda(s,t;\phi_1) - \Lambda(s,t;\phi_2) | \\
&= \left| \int\int_S \exp[i(sx+ty)]d_{xy}\phi_1(E) - \int\int_S \exp[i(sx+ty)]d_{xy}\phi_2(E) \right| \\
&\leqq \int\int_S | f_{2m}(sx+ty) | \, d_{xy}\phi_1(E) + \int\int_S | f_{2m}(sx+ty) | \, d_{xy}\phi_2(E) \\
&\leqq (2m)\,!^{-1} \int\int_S (sx+ty)^{2m} d_{xy}\phi_1(E) + (2m)\,!^{-1} \int\int_S (sx+ty)^{2m} d_{xy}\phi_2(E) \\
&= 2[(2m)\,!]^{-1} \sum_{\nu=0}^{2m} \binom{2m}{\nu} s^{2m-\nu} t^{\nu} G_{2m-\nu,\nu},
\end{aligned}
$$

so that on placing $b = \max(|s|, |t|)$, where s and t are fixed,

$$
\Lambda(s,t;\phi_1) = \Lambda(s,t;\phi_2) \text{ if, for a fixed value of } b,
$$

$$
b^{2m}(2m)\,!^{-1} \sum_{\nu=0}^{2m} \binom{2m}{\nu} | G_{2m-\nu,\nu} | = o(1),
$$

hence, a fortiori, if

$$
(2m)\,!^{-1/(2m)} \left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} | G_{2m-\nu,\nu} | \right]^{1/(2m)} = o(1),
$$

a condition implied by (3), inasmuch as by the Stirling Formula

$$
(2m)\,!^{1/(2m)} = O(m).
$$

As applications of the foregoing determinateness condition, we consider

1. The case in which the distribution function is damped.* In this case,

$$
| G_{2m-\nu,\nu} | = \left| \int\int_S x^{2m-\nu} y^{\nu} d_{xy}\phi(E) \right| \leqq R^{2m},
$$

provided $\phi(E) = 0$ for all sets E outside $(-R < x < R; -R < y < R)$. Hence

$$
\left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} | G_{2m-\nu,\nu} | \right]^{1/(2m)} \leqq 2R = o(m),
$$

so that the condition (3) is satisfied.

2. The case in which

$$
\begin{aligned}
\phi(E) = \int\int_E f(x,y)\,dxdy, \text{ where } 0 &\leqq f(x,y) \\
&= O\,(\exp[-(x^2+y^2)^{\alpha}]),\ 0 < \alpha,\ (x^2+y^2 \to +\infty).
\end{aligned}
$$

It is easy to see that (3) is satisfied by the momenta G_{pq} of such a ϕ if

$$
(5) \qquad \left[\sum_{\nu=0}^{2m} \binom{2m}{\nu} | H_{2m-\nu,\nu} | \right]^{1/(2m)} = o(m)
$$

---

* For a direct treatment of this case, cf. E. K. Haviland, *loc. cit*, I, pp. 552-553.

where

$$H_{pq} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^p y^q \exp\left[-(x^2+y^2)^\alpha\right] dxdy, \qquad (0 < \alpha).$$

Now

$$\sum_{\nu=0}^{2m} \binom{2m}{\nu} \left| H_{2m-\nu,\nu} \right| \leqq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x| + |y|)^{2m} \exp\left[-(x^2+y^2)^\alpha\right] dxdy$$

$$= 4 \int_0^{+\infty}\int_0^{+\infty} \left[(x+y)^2\right]^m \exp\left[-(x^2+y^2)^\alpha\right] dxdy$$

$$\leqq 2^{m+2} \int_0^{+\infty}\int_0^{+\infty} (x^2+y^2)^m \exp\left[-(x^2+y^2)^\alpha\right] dxdy$$

$$= 2^{m+2} \int_0^{\pi/2} d\vartheta \int_0^{+\infty} r^{2m+1} \exp\left[-r^{2\alpha}\right] dr$$

$$= 2^m \pi \Gamma([m+1]/\alpha)/\alpha.$$

The $2m$-th root of this expression is* $= O(m^{1/(2\alpha)})$, so that (5), hence (3), is satisfied if $\alpha > \frac{1}{2}$ but not if $\alpha \leqq \frac{1}{2}$. Consequently, for $\alpha > \frac{1}{2}$, we have determinateness. It is to be noticed that this includes the classical case $\alpha = 1$.

Furthermore, the sufficient condition (3) is almost necessary. To show this, we first observe that the one-dimensional distribution function may be regarded as a special case of the two-dimensional. For suppose $\sigma(x)$ a one-dimensional distribution function along $X : (-\infty < x < +\infty;\ y = 0)$, i. e., $\sigma(x)$ is a monotone non-decreasing function of x there and $\sigma(-\infty) = 0$, $\sigma(+\infty) = 1$. Then we can define a set function $\phi_1(R)$ for all (half-open) rectangles R of the xy-plane by setting $\phi_1(R) = \sigma(X \cdot R) = \sigma(x_2) - \sigma(x_1)$, x_1 and x_2 $(x_2 \geqq x_1)$ being the end points of the segment $X \cdot R$, if R contains a portion of the x-axis; $\phi_1(R) = 0$ otherwise. It is known † that ϕ_1 determines a monotone absolutely additive set function ϕ uniquely up to the singular lines of the latter. Moreover, $\phi_1(R) = \phi(R)$ on every non-singular rectangle R of ϕ, from which fact it follows, in particular, that $\phi(S) = 1$, so that ϕ is a distribution function.

Let $M : (-K \leqq x \leqq K;\ -K \leqq y \leqq K)$ be an arbitrarily large square in the plane and let $U : (-\infty < x < +\infty;\ -\epsilon \leqq y \leqq \epsilon)$, $0 < \epsilon < 1$, be a strip enclosing the x-axis. Furthermore, we denote $M \cdot U$ by V. Then by the inequality of Schwarz,

---

* Cf. E. Artin, *op. cit.*, p. 23, equation (26).

† This follows from the reasoning of J. Radon, *loc. cit.*, I, pp. 1305-1313 and pp. 1338-1339 if it be observed that to a monotone additive interval function there may be made to correspond a monotone point function whose continuity points are everywhere dense. Cf. also S. Bochner, *loc. cit.*, pp. 387-388; E. K. Haviland, *loc. cit.*, III, Note 3.

$$\left[\int\int_M x^p y^q d_{xy}\phi(E)\right]^2 \leqq \int\int_M x^{2p} y^{2q} d_{xy}\phi(E) \cdot \int\int_M d_{xy}\phi(E)$$
$$\leqq \int\int_{M-V} x^{2p} y^{2q} d_{xy}\phi(E) + \int\int_V x^{2p} y^{2q} d_{xy}\phi(E).$$

But the integral over $M - V$ is zero by the definition of ϕ, and

$$\int\int_V x^{2p} y^{2q} d_{xy}\phi(E) \leqq \epsilon^{2q} \int\int_S x^{2p} d_{xy}\phi(E) = \epsilon^{2q} \mathrm{M}(2p, 0; \phi).$$

It follows that $\int\int_M x^p y^q d_{xy}\phi(E) < \eta$ if $q > 0$, or as M is arbitrarily large and η arbitrarily small,

$$\text{(6a)} \qquad \mathrm{M}(p, q; \phi) = 0, \qquad p = 0, 1, 2, \cdots; \; q = 1, 2, \cdots$$

If $q = 0$, the definition of ϕ shows that

$$\int\int_M x^p d_{xy}\phi(E) = \int_{-K}^{K} x^p d\sigma(x).$$

Hence, as the right-hand side approaches a limit as $M \rightarrow S$,

$$\text{(6b)} \qquad \mathrm{M}(p, 0; \phi) = \int\int_S x^p d_{xy}\phi(E) = \int_{-\infty}^{+\infty} x^p d\sigma(x), \; p = 0, 1, 2, \cdots$$

In consequence, the one-dimensional distribution functions represented by

$$\sigma(x) = D_h \int_{-\infty}^{x} \exp\,[-x^\nu]\,[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)]\,dx;$$

where $\qquad -1 \leqq h \leqq 1, \qquad \nu = 2\beta/(2\beta + 1),$

β being a positive integer, and

$$D_h^{-1} = \int_{-\infty}^{+\infty} \exp\,[-x^\nu]\,[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)]\,dx,$$

may be regarded as two-dimensional distribution functions $\phi(E)$ whose momenta are given by

$$\mathrm{M}(p, q; \phi) = 0, \quad q > 0;$$
$$\mathrm{M}(p, 0; \phi) = \int_{-\infty}^{+\infty} x^p \exp\,[-x^\nu]\,[1 + h\cos(x^\nu \tan \tfrac{1}{2}\nu\pi)]\,dx.$$

But it is known [*] that

$$\int_{-\infty}^{+\infty} x^p \exp\,[-x^\nu]\cos(x^\nu \tan\tfrac{1}{2}\nu\pi)\,dx = 0$$

for every p, so that the momentum problem for such a function $\phi(E)$ is not determined, inasmuch as ϕ depends on h while the momenta do not. In this case, the left-hand side of (3) becomes

$$\begin{aligned}[\,|\,G_{2m0}\,|\,]^{1/(2m)}, \quad &\text{where} \quad |\,G_{2m0}\,| = G_{2m0} = \mathrm{M}(2m, 0\,;\phi)\\ &= \int_{-\infty}^{+\infty} x^{2m}\exp[-x^\nu]\,dx = 2\int_0^{+\infty} x^{2m}\exp[-x^\nu]\,dx = 2\Gamma([2m+1]/\nu)/\nu\\ &= o(m^{1+\epsilon}),\end{aligned}$$

where $\epsilon > 0$ may be made arbitrarily small by taking β sufficiently large. As we do not obtain determinateness in this case, it follows that (3) is almost necessary.

In obtaining the above sufficient condition for determinateness, we have made use of a theorem of Bochner [†] which states that if $\phi_1(E)$ and $\phi_2(E)$ be two distribution functions such that $\Lambda(s, t\,;\phi_1) = \Lambda(s, t\,;\phi_2)$ for all points (s, t) (which is the case if they are equal for a set of points everywhere dense), then $\phi_1(E) = \phi_2(E)$ up to their singular lines. The following proof [‡] of this proposition seems to be more direct than the original and is more closely related to the methods, viz., those of Radon integrals, of the present paper.

By use of the one-dimensional Fourier cosine inversion formula, it may be shown that

$$\begin{aligned}\int_{-\infty}^{+\infty}(\pi\eta s^2)^{-1}\sin\eta s\sin(h+\eta)s\exp[isx]\,ds &= E_\eta(x, h) = E_\eta(-x, h)\\ &= \begin{cases} 1, \text{ if } 0 \leqq x \leqq h,\\ 1-(x-h)/2\eta, \text{ if } h \leqq x \leqq h+2\eta\\ 0, \text{ if } h+2\eta \leqq x\end{cases}\end{aligned}$$

where $h > 0$, $\eta > 0$. Then setting

$$D(s, t\,; x, y) = (\pi^2\eta^2 s^2 t^2)^{-1}\sin\eta s\sin\eta t\sin(h+\eta)s\sin(k+\eta)t\exp[i(sx+ty)],$$

we obtain

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[*] Cf. W. Stekloff, *loc. cit.*, pp. 50-51.

[†] Cf. S. Bochner, *loc. cit.*, p. 402, Theorem 14.

[‡] The method of this proof has been suggested by the treatment of a similar problem in one dimension by G. Pólya, *loc. cit.*, pp. 107-108.

$$(7)\qquad E_\eta(x,h)E_\eta(y,k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D(s,t;x,y)\,ds\,dt = G_\eta(x,y;h,k)$$

$$= \begin{cases} 1, \text{ if } (x,y) \text{ is in } I_1, \\ 0, \text{ if } (x,y) \text{ is outside } I_2, \end{cases}$$

where I_1 is the rectangle $(-h \leqq x \leqq h; -k \leqq y \leqq k)$ and I_2 is the rectangle $(-(h+2\eta) \leqq x \leqq h+2\eta;\ -(k+2\eta) \leqq y \leqq k+2\eta)$.

Furthermore, $0 \leqq G_\eta(x,y;h,k) \leqq 1$ if (x,y) is in $I_2 - I_1$. As $G_\eta(x,y;h,k)$ is the product of two continuous functions of x and y, it is a continuous function of those variables. Finally, we note that although G_η was defined as an iterated integral, it may be regarded as a double integral since the latter exists when taken over any finite rectangle R and is absolutely convergent as $R \rightarrow T$, the st-plane.

As $\Lambda(s,t;\phi_i)$, $(i=1,2)$, are continuous* functions of s and t together throughout the st-plane and $|\Lambda(s,t;\phi_i)| \leqq 1$ for all (s,t),

$$\int\int_R D(s,t;-u,-v)\Lambda(s,t;\phi_1)\,dsdt$$
$$= \int\int_R D(s,t;-u,-v)\Lambda(s,t;\phi_2)\,dsdt$$

and hence, as both integrals converge absolutely,

$$(8)\qquad \int\int_T D(s,t;-u-v)\Lambda(s,t;\phi_1)\,dsdt$$
$$= \int\int_T D(s,t;-u-v)\Lambda(s,t;\phi_2)\,dsdt.$$

If we set

$$(9)\qquad H(s,t;x,y) = D(s,t;x-u,\ y-v),$$

these integrals may be separated with the use of two arbitrary non-singular rectangles J, K in the xy-plane and the st-plane respectively into

$$\int\int_K\Big[\int\int_J H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]dsdt$$
$$+\int\int_{T-K}\Big[\int\int_J H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]dsdt$$
$$+\int\int_T\Big[\int\int_{S-J} H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]dsdt, \qquad (i=1,2).$$

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* For

$$\int\int_R \exp\,[i(sx+ty)]\,d_{xy}\phi_i(E)$$

is a continuous function of s and t together, R being any finite rectangle, and ϕ_i is a distribution function.

As the last two integrals may be made arbitrarily small in absolute value by taking J and K sufficiently large,

$$(10)\quad \left| \int\int_T \left[\int\int_S H(s,t;x,y)\, d_{xy}\phi_i(E) \right] ds dt - \int\int_K \left[\int\int_J H(s,t;x,y)\, d_{xy}\phi_i(E) \right] ds dt \right| < \tfrac{1}{2}\epsilon$$

for suitable J, K. Similarly, if the non-singular rectangles U and V are taken sufficiently large,

$$(11)\quad \left| \int\int_S \left[\int\int_T H(s,t;x,y)\, ds dt \right] d_{xy}\phi_i(E) - \int\int_U \left[\int\int_V H(s,t;x,y)\, ds dt \right] d_{xy}\phi_i(E) \right| < \tfrac{1}{2}\epsilon \qquad (i = 1, 2).$$

Since (10) and (11) clearly hold if all four rectangles are sufficiently large, we may choose

$$(12)\qquad K = V \text{ and } J = U.$$

Again, if $f(s,t;x,y)$ be a continuous function of $s,t;x,y$ together in

Σ: $(-M_1 \leqq s \leqq M_1;\ -M_1 \leqq t \leqq M_1;\ -M_2 \leqq x \leqq M_2;\ -M_2 \leqq y \leqq M_2)$,

if Σ_1 and Σ_2 denote the rectangles

$(-M_1 \leqq s \leqq M_1; -M_1 \leqq t \leqq M_1)$ and $(-M_2 \leqq x \leqq M_2; -M_2 \leqq y \leqq M_2)$,

respectively and if $\phi(E)$ is a monotone absolutely additive set function defined in Σ_2, then

$$(13)\quad \int\int_{\Sigma_1} \left[\int\int_{\Sigma_2} f(s,t;x,y)\, d_{xy}\phi(E) \right] ds dt = \int\int_{\Sigma_2} \left[\int\int_{\Sigma_1} f(s,t;x,y)\, ds dt \right] d_{xy}\phi(E)$$

since both differ arbitrarily little from

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_j; x_k, y_k)\phi(E_k)\Delta_j,$$

provided the subdivisions E_k, Δ_j are taken sufficiently fine.

Since the preceding conditions are satisfied by taking $\phi = \phi_i(E)$, $(i = 1, 2)$, and $f(s,t;x,y) = H(s,t;x,y)$, we obtain, on combining (10), (11) and (13) and taking account of (12):

$$\int\int_T \Big[\int\int_S H(s,t;x,y)\,d_{xy}\phi_i(E)\Big]\,dsdt$$
$$= \int\int_S \Big[\int\int_T H(s,t;x,y)\,dsdt\Big]\,d_{xy}\phi_i(E), \qquad (i = 1, 2).$$

On combining this result with (9), (8) and (7) we see that

$$\int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\phi_1(E)$$
$$= \int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\phi_2(E)$$

or

$$(14) \qquad \int\int_S G_\eta(x-u, y-v; h, k)\,d_{xy}\psi(E) = 0,$$
$$\text{where} \quad \psi(E) = \phi_1(E) - \phi_2(E).$$

As $\psi(E)$ is an absolutely additive set function of bounded total variation, its non-singular rectangles are everywhere dense.* By a suitable choice of h, k and (u, v) in (7) and (9), the rectangle I_1 may be made to coincide with any non-singular rectangle R of ψ, so that from (14)

$$\int\int_{I_2-I_1} G_\eta(x-u, y-v; h, k)\,d_{xy}\psi(E) + \int\int_{I_1} d_{xy}\psi(E) = 0.$$

As the first integral may be made arbitrarily small by taking η in (7) sufficiently small,

$$\int\int_R d_{xy}\psi(E) = \psi(R) = 0$$

for every non-singular rectangle of $\psi = \phi_1 - \phi_2$.

It follows that $\phi_1(R) = \phi_2(R)$ on all rectangles R non-singular with respect to ψ, ϕ_1, and ϕ_2 and hence (since the singular lines are at most denumerable) on an everywhere dense set of rectangles in the plane. Now it is known that if ϕ_1 and ϕ_2 be two monotone absolutely additive set functions of bounded total variation in the entire plane and such that $\phi_1(R_n) = \phi_2(R_n)$ on the everywhere dense sequence $\{R_n\}$, then $\phi_1(R) = \phi_2(R)$ for every rectangle non-singular with respect to ϕ_1 and ϕ_2. Furthermore, ϕ_1 and ϕ_2 have the same singular lines and the same singular rectangles,† q. e. d.

2. *The symbolical product of distribution functions.* Although the Radon integral as defined ‡ for a continuous integrand has been sufficient for

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* The proof is essentially the same as that given by J. Radon, *loc. cit.*, II, p. 1093.

† The proof of this statement is essentially that given by the author, *loc. cit.*, I, p. 550, Theorem II.

‡ The integral is that defined by J. Radon, *loc. cit.*, I, pp. 1322-1324.

our previous purposes, it will not suffice for the considerations to follow. At the same time, it is neither convenient nor necessary to resort to the very general definition (corresponding to Lebesgue integrals) given by Radon [*] of the integral of functions measurable with respect to an absolutely additive set function. Accordingly, we shall extend the conception of the ordinary Radon integral to the case where the integrand, though bounded, is no longer continuous.

Let ϕ be an absolutely additive set function defined for a set [†] T of sets in the xy-plane. Let $f(x, y)$ be a bounded single-valued point function defined on the bounded set E_0 belonging to T. Let Π denote a division of E_0 into subsets $E_1, \cdots, E_m$, all belonging to T, and $D(\Pi)$ the greatest of the diameters of the sets $E_1, \cdots, E_n$. Finally, let U_k denote the least upper bound and L_k the greatest lower bound of $f(x, y)$ in E_k. Then if, as $D(\Pi) \to 0$,

$$\lim \sum_{k=1}^{n} U_k \phi(E_k) \quad \text{and} \quad \lim \sum_{k=1}^{n} L_k \phi(E_k)$$

exist and are equal, the Radon integral of f over E_0 with respect to ϕ is said to exist.

Definition 6. By the two-dimensional saltus $\omega(\xi, \eta)$ of $f(x, y)$ at a point (ξ, η) is meant

$$\lim_{\delta=0} \, [U_\delta(\xi, \eta) - L_\delta(\xi, \eta)],$$

where $U_\delta(\xi, \eta)$, $L_\delta(\xi, \eta)$ are respectively the least upper bound and the greatest lower bound of $f(x, y)$ in $R_\delta(\xi, \eta)$: $| \xi - x | < \delta$, $| \eta - y | < \delta$.

If a function $f(x, y)$ is defined in a closed rectangle J, we shall denote by G_ϵ the closed set of points in J at which $\omega(\xi, \eta) \geqq \epsilon$. If (ξ, η) is a boundary point of J, we agree to confine the points (x, y) of Definition 6 to points of J. Using this notation we shall prove

THEOREM III. *Let $f(x, y)$ be bounded and single-valued in*

$$J : (-M \leqq x \leqq M ; -M \leqq y \leqq M)$$

and $\phi(E)$ an absolutely additive set function defined in the same region. Then a necessary and sufficient condition that the Radon integral

$$\int\int_{E_0} f(x, y) \, d_{xy} \phi(E)$$

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[*] Cf. J. Radon, *loc. cit.*, I, p. 1329.

[†] For a definition of this set, cf. J. Radon, *loc. cit.*, I, p. 1299.

exist, where E_0 *is any point set in* J *belonging to the domain of definition of* ϕ, *is that the total variation*[*] *of* $\phi(E)$ *over the open* ρ*-neighborhood of* G_ϵ *should for every* ϵ *approach zero as* $\rho \to 0$.

Proof. We first prove the sufficiency of the condition. For this, one need consider only the case where $\phi(E)$ is monotone, inasmuch as any absolutely additive set function $\phi(E)$ may be expressed as the difference of two monotone absolutely additive set functions,[†] $\psi_1(E)$ and $\psi_2(E)$ and

$$\int\int_{E_0} f(x,y)\, d_{xy}\phi(E) = \int\int_{E_0} f(x,y)\, d_{xy}\psi_1(E) - \int\int_{E_0} f(x,y)\, d_{xy}\psi_2(E).$$

If $u = u(\Pi) = \sum_{k=1}^{m} U_k \phi(E_k)$ and $l = l(\Pi) = \sum_{k=1}^{m} L_k \phi(E_k)$ and if Π, Π' are two divisions of E_0 and $\Pi\Pi'$ the division consecutive to them, then $l(\Pi) \leqq l(\Pi\Pi') \leqq u(\Pi\Pi') \leqq u(\Pi')$, so that the least upper bound of the $l(\Pi)$ is not greater that the greatest lower bound of the $u(\Pi)$. Therefore the integral exists if $|u(\Pi) - l(\Pi)| < \epsilon$, provided $D(\Pi) < \delta$, where $\epsilon > 0$ is arbitrarily small and $\delta = \delta(\epsilon)$.

Let G_ϵ be as defined above. With centre (x,y), any point of G_ϵ, let a circle of radius 2ρ be taken, and let the open non-overlapping finite set formed by the coalescence of these circular regions be denoted by $N_{\epsilon,2\rho}$. It is the 2ρ-neighborhood of G_ϵ. Under the hypothesis regarding the total variation of $\phi(E)$, ρ can be so chosen that $0 \leqq \phi(\Delta_{\epsilon,2\rho}) < \zeta'$, where $\Delta_{\epsilon,2\rho} = J \cdot N_{\epsilon,2\rho}$. Then by taking the subdivision Π so fine that $D(\Pi) < \rho$, E_0 may (for this subdivision) be divided into two parts, E_0' and E_0'', where E_0' includes all points of E_0 in $N_{\epsilon,\rho}$, the ρ-neighborhood of G_ϵ, but is included in $N_{\epsilon,2\rho}$; and $E_0'' = E_0 - E_0'$. By E_k', E_k'' we denote those E_k composing E_0', E_0'' respectively. Accordingly,

$$u(\Pi) = u'(\Pi) + u''(\Pi), \qquad l(\Pi) = l'(\Pi) + l''(\Pi),$$

where we separate the contributions to the sums made by E_0' and by E_0''. Then

$$u(\Pi) - l(\Pi) = u'(\Pi) - l'(\Pi) + u''(\Pi) - l''(\Pi).$$

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[*] For the definition of total variation, cf. J. Radon, *loc. cit.*, I, p. 1302. These ρ-neighborhoods, being open, necessarily belong to the domain of definition, T, of ϕ. Cf. J. Radon, *ibid.*, p. 1308. The conditions of Theorem III of the present paper, being analogous to the du Bois-Reymond condition for Riemann-integrability, may be shown to be equivalent to conditions corresponding to those of Lebesgue for the existence of the Riemann integral. We use the former set of necessary and sufficient conditions as better suited for our applications.

[†] Cf. J. Radon, *loc. cit.*, I, p. 1303.

It follows from the definition of saltus that

$$\omega(\xi,\eta) \leqq U_\delta(\xi,\eta) - L_\delta(\xi,\eta) < \omega(\xi,\eta) + \epsilon,$$

provided $\delta \leqq \delta' = \delta'(\epsilon;\xi,\eta)$. Consequently, by the Heine-Borel Theorem, J is covered by a finite number, N, of open squares R_i: $|\xi_i - x| < \delta_i$, $|\eta_i - y| < \delta_i$ such that the fluctuation of $f(x,y)$ in R_i is $< \omega(\xi_i,\eta_i) + \epsilon$, where (ξ_i,η_i) is a definite point interior to R_i. No restriction is imposed by requiring that all R_i be less in diameter than a fixed arbitrarily preassigned quantity ζ. The sides of the rectangles R_i, if continued to meet the sides of J, divide J into a finite number of rectangles. Let d be the length of the smallest side to be found among these resulting rectangles. Any set E_k in J whose diameter is $< d$ is interior to one of the squares R_i, so that in E_k the fluctuation of $f(x,y)$ lies between $\omega(\xi_i,\eta_i)$ and $\omega(\xi_i,\eta_i) + \epsilon$, where (ξ_i,η_i) is some point of a square R_i of diameter $< \zeta$ which contains E_k in its interior. If $\zeta < \rho$, the R_i containing any E_k'' is such that $\omega(\xi_i,\eta_i) < \epsilon$. Accordingly, the E_k'' can be taken so fine that in each $U_k - L_k < 2\epsilon$, in which case

$$0 \leqq u''(\Pi) - l''(\Pi) \leqq 2\epsilon \int\int_{E_0} d\phi(E) \leqq 2\epsilon\phi(J).$$

Again,

$$0 \leqq u'(\Pi) - l'(\Pi) \leqq 2B\phi(\Delta_{\epsilon,2\rho}) < 2B\zeta',$$

where $B = \text{fin sup}\,|f(x,y)|$ in E_0.[*] It follows that

$$0 \leqq u(\Pi) - l(\Pi) < \epsilon', \quad \text{where} \quad \epsilon' = 2(\epsilon\phi(J) + B\zeta'),$$

provided $D(\Pi) < d < \rho$, the order of choice of the various quantities being as follows:

Given ϵ', we take $\epsilon \leqq \epsilon'/[4\phi(J)]$. This value of ϵ determines the region G_ϵ. Then ζ' is chosen $\leqq \epsilon'/[4B]$ whereby ρ is determined, and thereupon ζ. Finally, ϵ and ζ being known, d can be determined. It has therefore been shown that as $D(\Pi) \to 0$, $\lim \sum_{k=1}^{m} U_k\phi(E_k)$ and $\lim \sum_{k=1}^{m} L_k\phi(E_k)$ exist and are equal.

This limiting value is independent of the special choice of the sequence of divisions $\Pi_1, \Pi_2, \cdots$, since if $\Pi'_1, \Pi'_2, \cdots$ is a second sequence of divisions such that $D(\Pi'_i) \to 0$, the two divisions lead respectively to partial sums of the form

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[*] "fin sup" is used to denote the least upper bound. Cf. E. K. Haviland, *loc. cit.*, I, note 15.

$$S_m(\Pi)=\sum_{k=1}^{m} f(x_k,y_k)\phi(E_k) \quad\text{and}\quad S_{m'}(\Pi')=\sum_{l=1}^{m'} f(x'_l,y'_l)\,\phi(E'_l).$$

But

$$|S_m(\Pi)-S_{m'}(\Pi')|=|\sum_{k=1}^{m}\sum_{l=1}^{m'}[f(x_k,y_k)-f(x'_l,y'_l)]\phi(E_k\cdot E'_l)|$$
$$\leqq \vartheta(D(\Pi)+D(\Pi'))\phi(J)+2B\phi(\Delta_{\epsilon,2\rho}),$$

where $\vartheta(d)=\text{fin sup}\,|f(x,y)-f(x',y')|$ for all point pairs (x,y), (x',y') in $J-\Delta_{\epsilon,\rho}$ whose distance is $< d$. Hence $|S_m(\Pi)-S_{m'}(\Pi')|<\frac{1}{2}\epsilon_1$ if $D(\Pi)<\delta_1$, $D(\Pi')<\delta'_1$, say. But

$$|S_m(\Pi)-\int\int_{E_0} f(x,y)\,d_{xy}\phi(E)|<\tfrac{1}{2}\epsilon_1 \quad\text{if}\quad D(\Pi)<\delta_2,\ \text{say.}$$

Consequently, taking $D(\Pi)<\min(\delta_1,\delta_2)$ one obtains

$$|S_{m'}(\Pi')-\int\int_{E_0} f(x,y)\,d_{xy}\phi(E)|<\epsilon_1, \quad\text{if}\quad D(\Pi')<\delta'_1.$$

Hence as $D(\Pi')\to 0$,

$$\lim\sum_{l=1}^{m'} f(x'_l,y'_l)\phi(E_l)=\int\int_{E_0} f(x,y)\,d_{xy}\phi(E), \qquad \text{q. e. d.}$$

This concludes the proof of the sufficiency of the condition.

To show the necessity of the condition, it is sufficient to prove that if the condition is not fulfilled, $\int\int_J f(x,y)\,d_{xy}\phi(E)$ fails to exist. The proof is as follows:

Let J be divided by a network Σ_n into squares

$$S_{ij}^{(n)}:\ (M(i-1)2^{-n}\leqq x<Mi2^{-n};\ M(j-1)2^{-n}\leqq y<Mj2^{-n});$$
$$(i,j=-2^n+1,\cdots,2^n-1).$$
$$S_{ij}^{(n)}:\ (M(2^n-1)2^{-n}\leqq x\leqq M;\ M(j-1)2^{-n}\leqq y<Mj2^{-n});$$
$$(i=2^n,\ j=-2^n+1,\cdots,2^n-1).$$
$$S_{ij}^{(n)}:\ (M(i-1)2^{-n}\leqq x<Mi2^{-n};\ M(2^n-1)2^{-n}\leqq y\leqq M);$$
$$(j=2^n,\ i=-2^n+1,\cdots,2^n-1).$$
$$S_{ij}^{(n)}:\ (M(2^n-1)2^{-n}\leqq x\leqq M;\ M(2^n-1)2^{-n}\leqq y\leqq M);$$
$$(i=j=2^n).$$

where $n=0,1,2,\cdots$.

Let G_ϵ, as above, represent the set of points of J at which the saltus of $f(x,y)\geqq\epsilon$. There are at most a finite number of intersections within J of the lines of the network Σ_n at which points of G_ϵ are located. About each such intersection we draw a closed square D_p of length of side $< M2^{-n}$ and such that of the four squares of Σ_n in which D_p lies, each contains points of

G_ϵ not on the boundary of D_p or else contains no points of G_ϵ at all. Consider those regions $R_{ij}^{(n)}$ obtained by subtracting the D_p from the $S_{ij}^{(n)}$ which contain points of G_ϵ. If such an $R_{ij}^{(n)}$ contains points of G_ϵ only on its boundaries, such a point, say P_{ij}, may be made an interior point of $Q_{ij}^{(n)}$, the sum of $R_{ij}^{(n)}$ and a closed circular disc about P_{ij}, provided P_{ij} does not lie on a boundary of J. This disc may be made so small as not to deprive the adjacent $R_{ij}^{(n)}$ of points of G_ϵ. In this way, G_ϵ may be covered by a non-overlapping region $V_n = \Sigma\Sigma Q_{ij}^{(n)}$ so that

(a) The diameter of every $Q_{ij}^{(n)}$ is $\leqq 2^{-n}M(5)^{1/2}$;

(b) Each $Q_{ij}^{(n)}$ contains a point of G_ϵ in its interior or else on a boundary of $Q_{ij}^{(n)}$ that is also a part of the boundary of J.

Consequently $U_{ij} - L_{ij} \geqq \epsilon$ in each $Q_{ij}^{(n)}$, while by hypothesis $|\phi(V_n)| > K > 0$, K independent of n.

Therefore to every $\delta > 0$ there corresponds a subdivision Π such that $D(\Pi) < \delta$, but $u(\Pi) - l(\Pi) > \epsilon K > 0$, so $\int\int_J f(x, y)\, d_{xy}\phi(E)$ does not exist, q. e. d.

We proceed to apply Theorem III to the special case of distribution functions, for which we state

Lemma II. *If $\phi_1(E)$ and $\phi_2(E)$ be distribution functions, then there exists an everywhere dense*[*] *sequence of rectangles R_n such that the Radon integral*[†]

$$\Phi(Q) = \int\int_S \phi_1(Q - P_{xy})\, d_{xy}\phi_2(E)$$

exists for $Q = R_n$, $(n = 1, 2, 3, \cdots)$.

Proof. We prove the proposition first for the case where the integration is over an arbitrarily large closed finite rectangle J by showing that Theorem III is applicable. In what follows, those lines parallel to the coördinate axes containing discontinuities of a point function $f(x, y)$ will be referred to as exceptional lines of the function f, while by the singular lines of a set function $\phi(E)$ are meant, as before, the discontinuity lines parallel to the axes of the corresponding point function[‡] $F(x, y)$. In the present case, we note that the exceptional lines of the point function $f(x, y) = \phi_1(Q - P_{xy})$, where Q is a fixed rectangle, are those lines $x = x_i$, $y = y_j$ such that at least one side of $Q - P_{xy}$ lies on one of the singular lines $x = \xi'_i$, $y = \eta'_j$ of $\phi_1(E)$. As Q has but four sides and as the singular lines of $\phi_1(E)$ are at most denumerable,* it follows that so are the exceptional lines of the point function $f(x, y) = \phi_1(Q - P_{xy})$. Moreover, as $\phi_1(E)$ is bounded and monotone, the points at which its saltus,† i. e., the saltus of the corresponding point function, $\geqq \epsilon$ are confined to a fixed number $n = n(\epsilon)$ of its singular lines. Accordingly, the number of lines containing at least one point at which the saltus of $\phi_1(Q - P_{xy}) \geqq \epsilon$ is finite, $= 2n(\epsilon)$, for any given rectangle Q.

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[*] Cf. Definition 4.

[†] P_{xy} is the point with coördinates (x, y). $Q - P_{xy}$ is to be taken in the sense of vector addition and therefore represents the set of points which may be represented in the form $(\xi - x, \eta - y)$, where (ξ, η) is a point of Q. Cf. H. Bohr and B. Jessen, *loc. cit.*, p. 332.

[‡] Cf. J. Radon, *loc. cit.*, II, p. 1093.

If the singular lines of $\phi_2(E)$ be denoted by $x = \xi_p$, $y = \eta_q$, the number of sides of all rectangles Q for which $x_i = \xi_p$, $y_j = \eta_q$, $(i, j, p, q = 1, 2, \cdots)$ is clearly at most denumerable. For they are the lines $x = \xi_p + \xi'_i$, $(i, p = 1, 2, \cdots)$ and $y = \eta_q + \eta'_j$, $(j, q = 1, 2, \cdots)$. Consequently, there is an everywhere dense set of rectangles R_n such that the exceptional lines of $f(x, y) = \phi_1(R_n - P_{xy})$ are distinct from the singular lines of ϕ_2. When, however, the exceptional lines of a point function $f(x, y)$ are at most denumerable and are such that the points at which the saltus of the function $\geqq \epsilon$ are confined to a finite number $2n(\epsilon)$ of the lines, and when none of these exceptional lines coincide with any of the singular lines of the absolutely additive set function $\phi_2(E)$, the $2n(\epsilon)$ lines can be imbedded in strips so narrow that if W denote the sum of the portions of the strips in J, the value of $\phi_2(W)$ can be made arbitrarily small, while at all points of $J - W$ the saltus of $f(x, y)$ is less than ϵ. Thus by Theorem III

$$\Phi(Q) = \int\int_J \phi_1(Q - P_{xy}) \, d_{xy}\phi_2(E) \text{ exists for } Q = R_n \quad (n = 1, 2, \cdots).$$

From the definition of distribution functions, it follows that if J_1 be an arbitrarily large rectangle and J_2 a rectangle containing J_1 in its interior, then

$$\int\int_{J_1} \phi_1(R_n - P_{xy}) \, d_{xy}\phi_2(E)$$
$$\leqq \int\int_{J_2} \phi_1(R_n - P_{xy}) \, d_{xy}\phi_2(E) \leqq \int\int_S d_{xy}\phi_2(E) = 1.$$

Consequently,

$$\Phi(R_n) = \int\int_S \phi_1(R_n - P_{xy}) \, d_{xy}\phi_2(E) \quad \text{exists,} \quad (n = 1, 2, \cdots), \quad \text{q. e. d.}$$

The R_n for which Φ is defined include rectangles having infinite sides by virtue of the definition of ϕ_1. In particular, they include a sequence of rectangles $R_s : (-\infty < x < x_s;\ -\infty < y < y_s)$ such that the points (x_s, y_s) are everywhere dense in the plane. Moreover, these points are everywhere dense on the parallels to the axes through any one of them and they include the vertices of a set of rectangles everywhere dense in the plane. Then to Φ, defined on the R_n, we may assign a point function F^* defined only at the points (x_s, y_s) by the relation

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* Cf. J. Radon, *loc. cit.*, II, p. 1093.

† Cf. Definition 6.

$$F^*(x_s, y_s) = \Phi(R_s)\,;\ R_s\colon (-\infty < x < x_s;\ -\infty < y < y_s).$$

On the points (x_s, y_s), we shall prove that $F^*(x_s, y_s)$ possesses the following properties:

$$(\alpha)\quad F^*(x+h, y+k) - F^*(x+h, y) - F^*(x, y+k) + F^*(x, y) \geqq 0,$$

where $h \geqq 0$, $k \geqq 0$ and $(x+h, y+k)$, $(x+h, y)$, $(x, y+k)$, (x, y) are points of the everywhere dense sequence $\{(x_s, y_s)\}$.

$$(\beta)\qquad \lim_{h=0,\,k=0} F^*(x-h, y-k) = F^*(x, y),$$

where $h \geqq 0$, $k \geqq 0$ and (x, y), $(x-h, y-k)$ belong to the everywhere dense sequence.

$$(\gamma)\quad F^*(-\infty, y) = F^*(x, -\infty) = 0 \quad\text{and}\quad F^*(+\infty, +\infty) = 1,$$

where $F^*(-\infty, y) = \lim\limits_{x=-\infty} F^*(x, y)$, the (x, y) being points of the everywhere dense sequence, and similar interpretations are to be placed on $F^*(x, -\infty)$ and $F^*(+\infty, +\infty)$.

Proof of (α). Let us denote $x+h$, x, $y+k$, y by x_2, x_1, y_2, y_1 respectively, and the rectangle $(-\infty < x < x_i;\ -\infty < y < y_j)$ by R_{ij}. Then

$$\begin{aligned}
&F^*(x+h, y+k) - F^*(x+h, y) - F^*(x, y+k) + F^*(x, y)\\
&\quad = F^*(x_2, y_2) - F^*(x_2, y_1) - F^*(x_1, y_2) + F^*(x_1, y_1)\\
&\quad = \Phi(R_{22}) - \Phi(R_{21}) - \Phi(R_{12}) + \Phi(R_{11})\\
&\quad = \int\!\!\int_S [\phi_1(R_{22} - P_{xy}) - \phi_1(R_{21} - P_{xy})\\
&\qquad\qquad - \phi_1(R_{12} - P_{xy}) + \phi_1(R_{11} - P_{xy})]\, d_{xy}\phi_2(E)\\
&\quad = \int\!\!\int_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E) \geqq 0,
\end{aligned}$$

in virtue of the definition of ϕ_1 and ϕ_2, and since ϕ_1 is additive and

$$R_{22} - R_{21} - R_{12} + R_{11} = R\colon (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2), \qquad \text{q. e. d.}$$

Proof of (β). It is to be shown that if

$$R^0\colon (-\infty < x < x^0;\ -\infty < y < y^0)$$

and

$$R_s\colon (-\infty < x < x_s;\ -\infty < y < y_s),$$

then $0 \leqq \Phi(R^0) - \Phi(R_s) < \epsilon$ provided $0 \leqq x^0 - x_s < \delta$ and $0 \leqq y^0 - y_s < \delta$, $\delta = \delta(\epsilon; x^0, y^0)$, where R^0, R_s are rectangles of the everywhere dense set for which Φ is defined. We show this first for

$$\Phi_J(Q) = \int\int_J \phi_1(Q - P_{xy})\, d_{xy}\phi_2(E),$$

where J is a finite rectangle.

Let W be a closed rectangle so large that it contains the points (u, v), where $u = x^0 - x$, $v = y^0 - y$ for all (x, y) in J. With ϕ_1 there is associated a point function $F_1(u, v)$ such that

$$| F_1(u, v) - F_1(u', v') | < \tfrac{1}{2}\epsilon, \quad \text{provided} \quad \begin{array}{l} | u - u' | < \delta(\epsilon; u, v), \\ | v - v' | < \delta(\epsilon; u, v), \end{array}$$

for all points (u, v) of the plane (hence *a fortiori* of W) except at most the points of a finite number $\mu = \mu(\epsilon)$ of lines $u = u_i$ and a finite number $\nu = \nu(\epsilon)$ of lines $v = v_j$. To these there correspond $\mu(\epsilon)$ lines $x = \xi_i$ and $\nu(\epsilon)$ lines $y = \eta_j$ which by the choice of R^0 are non-singular lines of $\phi_2(E)$, so that they may be imbedded in a finite number of open strips X_i, Y_j so narrow that

$$0 \leqq \sum_{i=1}^{\mu} \phi_2(X_i) + \sum_{j=1}^{\nu} \phi_2(Y_j) < \epsilon/4.$$

To these strips there correspond strips U_i containing the u_i and V_j containing the v_j. By reasoning similar to that used in Theorem III, it follows that in the closed region

$$W(S - (\sum_{i=1}^{\mu} U_i + \sum_{j=1}^{\nu} V_j)) = W_1, \qquad \delta(\epsilon; u, v) > \delta'(\epsilon) > 0,$$

where $\delta'(\epsilon)$ is independent of the point (u, v). Hence

$$| F_1(x^0 - x, y^0 - y) - F_1(x_s - x, y_s - y) | < \tfrac{1}{2}\epsilon,$$

so that $| \phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy}) | < \frac{1}{2}\epsilon$ provided $| x^0 - x_s | < \delta'(\epsilon)$ and $| y^0 - y_s | < \delta'(\epsilon)$, where $\delta'(\epsilon)$ is independent of $(x^0 - x, y^0 - y)$ in W_1, and therefore of (x, y) in $J(S - (\sum_{i=1}^{\mu} X_i + \sum_{j=1}^{\nu} Y_j)) = J_1$. Consequently,

$$\left| \int\int_{J_1} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})] d_{xy}\phi_2(E) \right| \leqq \tfrac{1}{2}\epsilon \int\int_{J_1} d_{xy}\phi_2(E) \leqq \tfrac{1}{2}\epsilon.$$

Moreover,

$$\left| \int\int_{J-J_1} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})] d_{xy}\phi_2(E) \right|$$
$$\leqq 2 \int\int_{J-J_1} d_{xy}\phi_2(E) \leqq 2\left[\sum_{i=1}^{\mu} \phi_2(X_i) + \sum_{j=1}^{\nu} \phi_2(Y_j)\right] < \tfrac{1}{2}\epsilon,$$

so

$$\left| \int\int_{J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})] d_{xy}\phi_2(E) \right| < \epsilon.$$

Now suppose J so large that

$$\int\int_{S-J} d_{xy}\phi_2(E) < \tfrac{1}{2}\epsilon$$

and let $R^0 : (-\infty < x < x^0; -\infty < y < y^0)$ be a fixed rectangle for which Φ is defined. Then if $R_s : (-\infty < x < x_s; -\infty < y < y_s)$ be another rectangle for which Φ is defined and such that

$$| x^0 - x_s | < \delta'(\epsilon), \qquad | y^0 - y_s | < \delta'(\epsilon),$$

$$\begin{aligned} | F^*(x^0, y^0) - F^*(x_s, y_s) | & \\ \leqq & \left| \int\int_{S-J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})] d_{xy}\phi_2(E) \right| \\ + & \left| \int\int_{J} [\phi_1(R^0 - P_{xy}) - \phi_1(R_s - P_{xy})] d_{xy}\phi_2(E) \right| < 2\epsilon. \end{aligned}$$

Hence $F^*(x_s, y_s)$ possesses property (β), q. e. d.

Proof of (γ). To prove the first part of (γ), we note that by choosing a finite rectangle J so large that $\int\int_{S-J} \phi_1(R_s - P_{xy}) d_{xy}\phi_2(E) < \frac{1}{2}\epsilon$ and then choosing $R_s : (-\infty < x < x_s; -\infty < y < y_s)$ so distant that $\phi_1(R_s - P_{xy}) < \frac{1}{2}\epsilon$ for all P_{xy} in J, a result which can be brought about by taking $x_s < -M(\epsilon)$ or $y_s < -M(\epsilon)$, by virtue of the definition of ϕ_1 as a distribution function, we obtain for such a rectangle R_s

$$F^*(x_s, y_s) = \int\int_{S} \phi_1(R_s - P_{xy}) d_{xy}\phi_2(E) < \epsilon, \qquad \text{q. e. d.}$$

The proof of the second part of (γ) is as follows:*

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* Another proof of this statement is possible with the use of Fourier transforms, in virtue of Theorem V below.

$$\iint_S \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) \geqq \iint_K \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E),$$

where K is a fixed finite rectangle so large that $\iint_K d_{xy}\phi_2(E) > 1 - \epsilon$. R_s may then be taken so large, say $x_s > M(\epsilon)$, $y_s > M(\epsilon)$, that $\phi_1(R_s - P_{xy}) > 1 - \epsilon$ for all P_{xy} in K. Then

$\iint_S \phi_1(R_s - P_{xy})\, d_{xy}\phi_2(E) > (1 - \epsilon)^2$, i. e. $F^*(+\infty, +\infty) = 1$, q. e. d.

Now it is known [†] that if a function F^*, defined for a set of points everywhere dense in the plane and which are also such that they are everywhere dense on the parallels to the axes through any one of them and that they include the corners of an everywhere dense set of rectangles, possesses properties (α), (β), (γ), then there exists at least one function F defined in the whole plane and such that F fulfills conditions (α), (β), (γ) there and $F = F^*$ on all points of the everywhere dense set. To this function F there corresponds [‡] a monotone absolutely additive set function $\psi(E)$ such that $\psi(S) = 1$. Furthermore, F is determined by F^* up to the singular lines of F. We have thus proved

THEOREM IV. *Corresponding to every pair of distribution functions* ϕ_1, ϕ_2 *there exists a distribution function* $\psi(E)$ *such that* [§]

$$\psi(R) = \iint_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E)$$

for an everywhere dense set of rectangles R. *Furthermore,* ψ *is uniquely determined, up to its singular lines, by* ϕ_1 *and* ϕ_2.

An important property of $\psi(E)$ is given by

THEOREM V.

$$\Lambda(s, t; \psi) = \Lambda(s, t; \phi_1) \cdot \Lambda(s, t; \phi_2),$$

[†] Cf. J. Radon, *loc. cit.*, I, pp. 1338-1339. For our present purposes, it is necessary to make slight modifications in the reasoning because of the definition of F^* in the entire plane and to add the condition that as $h \to 0$, $k \to 0$, $\lim F^*(x - h, y - k) = F^*(x, y)$, where $(x - h, y - k)$ and (x, y) are points of the everywhere dense set for which F^* is defined.

[‡] Cf. J. Radon, *loc. cit.*, I, pp. 1305-1313. Slight modifications are necessary in the case of functions defined throughout the plane.

[§] This relation is often stated and used in its general form but without indication of a method of proof. Cf. e. g., R. v. Mises, *op. cit.*, p. 219.

Theorems IV and V of the present paper have been proved for the case of damped distribution functions by the author, *loc. cit.*, III.

where Λ *is the Fourier transform as introduced in* (1).

Proof. Let J be a rectangle so large that

$$0 \leqq \iint_{S-J} d_{xy}\sigma(E) < \epsilon/4; \qquad \sigma = \psi, \phi_1, \phi_2. \tag{15}$$

As all three integrals in the statement of the theorem clearly exist, they are by their definition as improper Radon integrals limits of approximating sums taken over suitably large rectangles, in which the E_k may be chosen as rectangles R_k of the everywhere dense set for which Φ is defined. By choosing the R_k sufficiently small and $R = \sum_{k=1}^{N} R_k$ sufficiently large that it encloses J, we may obtain

$$| \Lambda(s, t; \psi) - \sum_{k=1}^{N} \exp[i(sx_k + ty_k)]\psi(R_k) | < \tfrac{1}{2}\epsilon \tag{16}$$

and if in addition R be chosen so large that $R - P_j$ * $= \sum_{k=1}^{N} (R_k - P_j)$ includes J for all points P_j in J, we have for a sufficiently fine mesh

$$\begin{aligned} | \sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) \\ - \iint_{R-P_j} \exp[i(sx + ty)]d_{xy}\phi_1(E) | < 3\epsilon/4. \end{aligned}$$

Also

$$| \Lambda(s, t; \phi_1) - \iint_{R-P_j} \exp[i(sx + ty)]d_{xy}\phi_1(E) | < \epsilon/4,$$

so that

$$| \Lambda(s, t; \phi_1) - \sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) | < \epsilon.$$

In other words,

$$\sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) = \Lambda(s, t; \phi_1) + \zeta_{1j}, \tag{17}$$

where $| \zeta_{1j} | < \epsilon$, provided R is sufficiently large and its subdivision sufficiently fine. It is to be noted that the fineness of the R_k is independent of P_j, as is also ϵ, so that $N = N(\epsilon)$ is fixed. The meshes may then be further subdivided, if necessary, so that

$$| \psi(R_k) - \sum_{j=1}^{m} \phi_1(R_k - P_j)\phi_2(E_j) | < \epsilon/2N, \quad [k = 1, 2, \cdots, N = N(\epsilon)],$$

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* This represents vector addition as described in the second footnote on p. 646, R corresponding to Q and P_j to P_{xy}.

where $m = m(\epsilon)$. Hence

$$| \exp[i(sx_k + ty_k)]\psi(R_k) - \sum_{j=1}^{m} \exp[i(sx_k + ty_k)]\phi_1(R_k - P_j)\phi_2(E_j) | < \epsilon/2N.$$

On summing with respect to k, we have, with the help of (16) and after interchanging the order of the resulting double summation:

$$(18) \quad | \Lambda(s, t; \psi) - \sum_{j=1}^{m} \sum_{k=1}^{N} \exp[i\{s(x_k - x_j) + t(y_k - y_j)\}]\phi_1(R_k - P_j) \cdot \exp[i(sx_j + ty_j)]\phi_2(E_j) | < \epsilon.$$

Let the E_j be so fine (if this is not already the case) and $K = \sum_{j=1}^{m} E_j$ so large that

$$(19) \quad \sum_{j=1}^{m} \exp[i(sx_j + ty_j)]\phi_2(E_j) = \Lambda(s, t; \phi_2) + \zeta_2, \quad \text{where} \quad | \zeta_2 | < \epsilon.$$

Substituting (17) and (19) in (18), we obtain

$$| \Lambda(s, t; \psi) - \Lambda(s, t; \phi_1) \cdot \Lambda(s, t; \phi_2) - \zeta_2\Lambda(s, t; \phi_1) - \sum_{j=1}^{m} \zeta_{1j} \exp[i(sx_j + ty_j)]\phi_2(E_j) | < \epsilon.$$

But by virtue of the appraisals for ζ_2 and ζ_{1j} and the fact that the total variation of ϕ_1 and of ϕ_2 is 1, we obtain

$$| \zeta_2\Lambda(s, t; \phi_1) | < \epsilon$$

and

$$| \sum_{j=1}^{m} \zeta_{1j} \exp[i(sx_j + ty_j)]\phi_2(E_j) | \leqq \sum_{j=1}^{m} | \zeta_{1j} | \phi_2(E_j) < \epsilon.$$

Consequently,

$$| \Lambda(s, t; \psi) - \Lambda(s, t; \phi_1) \cdot \Lambda(s, t; \phi_2) | < 3\epsilon,$$

i. e., $\Lambda(s, t; \psi) = \Lambda(s, t; \phi_1) \cdot \Lambda(s, t; \phi_2)$, q. e. d.

Since distribution functions are determined up to their singular lines by their Fourier transforms, it follows from the preceding theorem that $\psi(E)$ as above defined remains unchanged by interchanging ϕ_1 and ϕ_2 in the definition of Φ. A direct proof of this proposition is also possible.

3. *The spectra of distribution functions.* With a view to investigating further the addition of distributions as expressed by the symbolical product, we now lay down the following definitions:

Definition 7. The spectrum of a distribution function $\phi(E)$ is the set of all points P such that $\phi(R) > 0$ for every open rectangle R containing P.

Definition 8. The point spectrum of the distribution function $\phi(E)$ is the set of points P such that $\phi(R) > \alpha > 0$ for every open rectangle R containing P, where $\alpha = \alpha_P$ is a positive lower bound independent of R.

We are now in a position to investigate the relations between the spectra of two distribution functions and the spectrum of their symbolical product,* making use of the agreement that if, in the vector addition of two sets of numbers, one of the sets be empty, then their sum is to be regarded as empty. We shall prove

THEOREM VI. *If $\psi(E)$ be the distribution function defined by*

$$\psi(R) = \int\int_S \phi_1(R - P_{xy})\, d_{xy}\phi_2(E)$$

for all rectangles R for which the integral exists, and if $S(\phi)$ denotes the spectrum of ϕ and $P(\phi)$ the point spectrum of ϕ, then the following relations hold:

$$\text{(a)}\quad S(\psi) = S(\phi_1) + S(\phi_2).$$
$$\text{(b)}\quad P(\psi) = P(\phi_1) + P(\phi_2).$$

Proof. As the proof of (a) follows the one-dimensional proof given by Wintner step by step, it is unnecessary to repeat it here, for the translation of the proof to two dimensions offers no difficulties. It is to be noted, however, that in the two dimensional case one cannot obtain a necessary and sufficient condition for the existence of the symbolical product in terms of the point spectra of ϕ_1 and of ϕ_2. For the purposes of the proof, though, it is sufficient that the integral definition of the symbolical product be valid for a set of rectangles everywhere dense.

The first step in the proof of (b) is to show that if a point (ξ, η) cannot be written in the form $(x_1, y_1) + (x_2, y_2)$, where (x_1, y_1) belongs to the point spectrum of ϕ_1 and (x_2, y_2) to the point spectrum of ϕ_2, then (ξ, η) cannot belong to the point spectrum of ψ; i. e. to any preassigned $\epsilon > 0$ there corresponds a rectangle R_ϵ containing (ξ, η) and such that $\psi(R_\epsilon) < \epsilon$.

We first consider the point spectrum of ϕ_1. Since ϕ_1 is monotone and $\phi_1(S) = 1$, it follows that the number of points $(\bar{x}_1, \bar{y}_1)$ in the plane such that $\phi_1(R) > \epsilon/2$ for every rectangle R about $(\bar{x}_1, \bar{y}_1)$ is finite, say $n = n(\epsilon/2)$. Consequently, for a fixed (ξ, η) there are only $n(\epsilon/2)$ points (x', y') such that $\phi_1(R - P_{x'y'}) > \epsilon/2$ for all R containing (ξ, η) and these are the points for which $(\xi, \eta) - (x', y') = (\bar{x}_1, \bar{y}_1)$ or $(\xi, \eta) = (\bar{x}_1, \bar{y}_1) + (x', y')$. It follows by hypothesis that (x', y') is not a point (x_2, y_2) of the point spectrum of ϕ_2. Consequently, about each of the $n(\epsilon/2)$ points (x'_i, y'_i), $[i = 1, 2, \cdots, n = n(\epsilon/2)]$, it is possible to take a rectangle R_i such that $\sum_{i=1}^{n} \phi_2(R_i) < \epsilon/4$. Moreover, let L be a rectangle so large that $\int\int_{S-L} d_{xy}\phi_2(E) < \epsilon/4$ and let $L - \sum_{i=1}^{n} R_i = L - U = L_1$. Then for all points (x, y) in L_1, we have $\phi_1(R - P_{xy}) < \epsilon/2$ provided R be sufficiently small. As (x, y) moves about L_1 (which may be taken as closed by supposing L closed, the R_i open), $(\xi, \eta) - (x, y) = (u, v)$ moves about a closed region L_2 which does not contain any points $(\bar{x}_1, \bar{y}_1)$ for which $\phi_1(R) > \epsilon/2$ for every R about $(\bar{x}_1, \bar{y}_1)$. Hence about every point (u, v) of L_2 it is possible to take a rectangle R_{uv} such that $\phi_1(R_{uv}) < \epsilon/2$. As L_2 is closed, it follows that there is a fixed rectangle $M(\epsilon/2)$ such that $\phi_1(R_{uv}) < \epsilon/2$ for all R_{uv} containing (u, v) and contained in a rectangle the size of $M(\epsilon/2)$. Hence if R_ϵ be a rectangle of the size of $M(\epsilon/2)$ with any (x, y) of L_1 as its centre, $\phi_1(R_\epsilon - P_{xy}) < \epsilon/2$. Now

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* In the one dimensional case, these relations have been derived by A. Wintner, *loc. cit.*, V.

$$\begin{aligned} \int\int_{S} \phi_1(R_\epsilon - P_{xy})\, d_{xy}\phi_2(E) &= \int\int_{S-L} + \int\int_{L-U} + \int\int_{U} \\ &\leqq \int\int_{S-L} d_{xy}\phi_2(E) + \int\int_{L_1} \phi_1(R_\epsilon - P_{xy})\, d_{xy}\phi_2(E) + \int\int_{U} d_{xy}\phi_2(E) \\ &< \epsilon/4 + \tfrac{1}{2}\epsilon \int\int_{L_1} d_{xy}\phi_2(E) + \epsilon/4 < \epsilon. \end{aligned}$$

Therefore, there exists a rectangle R_ϵ about (ξ, η) such that $\psi(R_\epsilon) < \epsilon$, wherefore (ξ, η) does not belong to the point spectrum of ψ. On the other hand, if $(\xi, \eta) = (x_1, y_1) + (x_2, y_2)$, the notation being as above, then (ξ, η) belongs to the point spectrum of ψ. Here, again, the proof is essentially the same as that in the one-dimensional case. This completes the proof of Theorem VI.

As has been pointed out above, the rôle played by the point spectrum in the existence of the symbolical product for the one-dimensional case is, in a sense, accidental and arises from the fact that the singular lines of the distribution functions, which in the two-dimensional case determine the existence of the symbolical product, coincide, in one dimension, with the point spectrum. We may, consequently, conclude with a few remarks on the singular lines.

In the first place, *all points of the point spectrum are necessarily situated at the intersection of two singular lines.* For it follows immediately from the definition of the point spectrum and that of the singular lines that the parallels to the coördinate axes through a point of the point spectrum are both singular lines.

On the other hand, *not every intersection of two singular lines is a point of the point spectrum, nor even of the spectrum.* For consider the distribution function $\phi(E)$ defined as follows:

$$\phi(R) = \tfrac{1}{2} \text{ for any rectangle } R: (x_1 \leqq x < x_2;\ y_1 \leqq y < y_2)$$

containing the point $(0, 0)$ and not the point $(1, 1)$ or else containing the point $(1, 1)$ and not the point $(0, 0)$, whereas $\phi(R) = 0$ for all rectangles R not containing $(1, 1)$ or $(0, 0)$. Thus $\phi(E)$ is a distribution function. To it corresponds the point function $F(x, y)$ where $F(x, y) = 0$ if $-\infty < x \leqq 0$ or $-\infty < y \leqq 0$; $F(x, y) = \frac{1}{2}$ if $0 < x \leqq 1$, $0 < y < +\infty$ or $0 < x < +\infty$, $0 < y \leqq 1$; $F(x, y) = 1$ in $(1 < x < +\infty;\ 1 < y < +\infty)$. Thus $x = 0$ and $y = 1$ are singular lines, but it may easily be computed [*] that $\phi(R) = 0$ for all rectangles sufficiently small about $(0, 1)$, so that $(0, 1)$ does not belong to the point spectrum nor even to the spectrum.

Moreover, *singular lines may exist even if the point spectrum is vacuous.* For consider the point function $F_1(x, y)$ defined as follows:

$F_1(x, y) = 0$ if $-\infty < x \leqq 0$ or $-\infty < y \leqq 0$.
$F_1(x, y) = y$ in $(0 < x < +\infty;\ 0 < y \leqq \frac{1}{2})$
$F_1(x, y) = \frac{1}{2}$ in $(0 < x < +\infty;\ \frac{1}{2} < y \leqq 1)$ and in $(0 < x \leqq \frac{1}{2};\ 1 < y < +\infty)$
$F_1(x, y) = x$ in $(\frac{1}{2} < x \leqq 1;\ 1 < y < +\infty)$
$F_1(x, y) = 1$ in $(1 < x < +\infty;\ 1 < y < +\infty)$.

As $F_1(x, y)$ possesses the properties (α), (β), (γ) defined above, it determines a distribution function $\phi_1(E)$. Both $x = 0$, and $y = 1$ are seen to be singular lines of ϕ_1, but an investigation of the point spectrum shows that it is vacuous.

Finally, the proof that $F^*(x_s, y_s)$, as defined above, possesses the property (β) contains effectively the more general result that $F(x, y) (= \psi(E))$ is continuous at all points (x, y) where x is not of the form $\bar{x}_n = \xi_i' + \xi_j''$ and y is not of the form $\bar{y}_n = \eta_k' + \eta_l''$; the lines $x = \xi_i'$, $y = \eta_k'$ being the singular lines of ϕ_1, and $x = \xi_j''$, $y = \eta_l''$ those of ϕ_2. All points of a line $x = \bar{\bar{x}}_n \neq \bar{x}_n$ must accordingly be continuity points of F if their ordinates are not of the form $y = \bar{y}_n$. As these continuity points are thus everywhere dense on the fixed line $x = \bar{\bar{x}}_n$, every finite length of $x = \bar{\bar{x}}_n$ can be imbedded in a rectangle R such that $\psi(R) < \epsilon$. Hence $x = \bar{\bar{x}}_n$ is not a singular line of ψ and the same is true of $y = \bar{\bar{y}}_n \neq \bar{y}_n$. It follows that at most the lines

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[*] Cf. J. Radon, *loc. cit.*, I, p. 1304.

$x = \bar{x}_n$, $y = \bar{y}_n$ are singular lines of $\psi(E)$. Again, if R be an arbitrarily narrow strip about $x = \bar{x}_n$, $R - \xi_j''$ is an arbitrarily narrow strip about $x = \xi_i'$. By hypothesis, $\phi_1(R - (\xi_j'', y))$[*] $> K_1$ for all y and any R. Furthermore, $\phi_1(R - P_{xy}) > K_1 > 0$ for all x in a strip $T: \xi_j'' - \delta < x < \xi_j'' + \delta$ whose width is less than that of R. But however narrow T, $\phi_2(T) > K_2 > 0$, so $\Phi(R) > K_1 K_2 > 0$ for arbitrarily narrow R and $\bar{x}_n$ is consequently a singular line of ψ. Similarly, $\bar{y}_n$ may be shown a singular line of ψ. *Consequently, the singular lines of ψ are precisely the lines $x = \bar{x}_n$, $y = \bar{y}_n$.*

As an application of the rules obtained for the addition of spectra, we may consider the addition of convex curves as treated by H. Bohr and B. Jessen.[†] Then the spectrum of the distribution function associated with a given curve is precisely the curve itself.[‡] By our rule, the spectrum of the distribution function belonging to the symbolical product of the distribution functions of two curves is the vector sum of the spectra of the individual curves, with the agreement that if one of the addends is the zero set, so also is the sum, a result in accordance with the definition of the distribution functon associated with the sum of the curves.[§]

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[*] This represents vector addition.

[†] Cf. H. Bohr and B. Jessen, *loc. cit.* An analytic treatment of the outer curve bounding the region formed in the addition of convex curves has been given by the author, *loc. cit.*, IV.

[‡] Cf. H. Bohr and B. Jessen, *loc. cit.*, pp. 338-339.

[§] Cf. H. Bohr and B. Jessen, *ibid.*, p. 340.

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ON ANALYTIC CONVOLUTIONS OF BERNOULLI DISTRIBUTIONS.

By Aurel Wintner.

By a distribution function $\sigma(x)$ is meant a monotone function defined for every real x and such that $\sigma(-\infty) = 0$, $\sigma(+\infty) = 1$. The inversion of the Fourier-Stieltjes transform

$$(1) \qquad L(t;\sigma) = \int_{-\infty}^{+\infty} \exp(itx)\, d\sigma(x) \qquad (-\infty < t < +\infty)$$

of σ is *

$$(2) \qquad \sigma(x) = \sigma(0) + \lim_{T=+\infty} \int_{-T}^{T} L(t;\sigma)[1 - \exp(-itx)]t^{-1}\, dt/2\pi i$$

where σ is supposed to be normalized by the condition

$$2\sigma(x) = \sigma(x+0) + \sigma(x-0).$$

It is easy to see † from (2) that if

$$(3) \qquad L(t;\sigma) = O(|\, t\,|^{-1/\epsilon}) \qquad (t \to \pm\infty)$$

holds for every $\epsilon > 0$ then all derivatives of $\sigma(x)$ exist for every real x. If (3) may be replaced by the sharper appraisal

$$(4) \qquad L(t;\sigma) = O(\exp(-A\,|\, t\,|)) \quad \text{where} \quad A > 0 \; (t \to \pm\infty)$$

then it is clear from (2) that

$$(4a) \qquad |\,\sigma^{(n)}(x)\,| < M \int_0^{+\infty} t^{n-1} \exp(-At)\, dt = M\Gamma(n)/A^n \quad (n = 1, 2, \cdots)$$

where M is a constant. This implies that σ may be developed in the vicinity of every real x into a power series having a radius of convergence $\geqq A$. Hence if

$$(5) \qquad L(t;\sigma) = O(\exp(-|\, t\,|^{\gamma})) \quad \text{where} \quad \gamma > 1 \qquad (t \to \pm\infty),$$

then $\sigma(x)$ is an entire function. In some problems † the Fourier-Stieltjes

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* P. Lévy, *Calcul des Probabilités*, Paris, 1925, pp. 163-172.

† A. Wintner, "Upon a statistical method in the theory of diophantine approximations," *American Journal of Mathematics*, vol. 55 (1933), pp. 309-331.

transform of σ appears as an infinite product the factors of which tend strongly to zero so that (3) is trivial. In the present note a problem will be treated where the factors do not approach zero, although even (5) may be satisfied.

Let $\tau(x;a)$ denote the distribution function which is $=0$ if $x<-a$ and $=1$ if $x>a$, finally $=1/2$ if $-a<x<a$. Thus τ is a symmetric Bernoulli distribution function and $L(t;\tau)=\cos(at)$. It is easy to see that if $\{a_n\}$ is a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n{}^2 < +\infty \tag{6}$$

then the infinite product $\cos(a_1t)\cos(a_2t)\cdots$ of the Fourier-Stieltjes transforms of the Bernoulli distribution functions $\tau(x;a_n)$ is uniformly convergent in every finite t-interval.* It follows therefore from a theorem first formulated by Lévy † that there exists a distribution function σ such that

$$L(t;\sigma)=\prod_{n=1}^{\infty}\cos(a_nt) \qquad \left(\sum_{n=1}^{\infty} a_n{}^2 < +\infty\right). \tag{7}$$

This $\sigma(x)$ is, in accordance with the product rule of convolutions ("Faltungen"), the distribution function of the probability belonging to $\xi_1+\xi_2+\cdots$ if $\tau(x;a_n)$ is the distribution function of the probability belonging to ξ_n, where $\xi_1,\xi_2,\cdots$ are statistically independent random variables. Our purpose is to delimit a class of sequences $\{a_n\}$ such that (7) satisfies one of the conditions (3), (4), (5).

A positive function $S(t)$, defined for all sufficiently large values of $t>0$, is said to be *slow* (Landau) if it is non-decreasing ‡ and such that

$$S(ct)/S(t)\to 1 \qquad (t\to+\infty)$$

holds for every fixed value of $c>0$. Thus $\log t$, $\log\log t,\cdots$ are slow functions and t^{ϵ} is slow only if $\epsilon=0$. The name of the slow functions is justified by the fact that

$$S(t)=O(t^{\epsilon}) \qquad (t\to+\infty) \tag{8}$$

holds for every slow function $S(t)$ and for every $\epsilon>0$.

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* The convergence of the product is meant in the sense that the limit may vanish if a factor is zero.

† *Op. cit.* For further references cf. a joint paper of B. Jessen and of the present author in the *Transactions of the American Mathematical Society* (not yet appeared).

‡ This restriction is not essential for our purposes. Generalized slow functions have been considered by J. Karamata, "Sur un mode de croissance régulière. Théorèmes fondamentaux," *Bulletin de la Société Mathématique de France*, vol. 61 (1933), pp. 1-8.

Let $\{b_n\}$ be a sequence of positive numbers and let the inequality $b_n \leqq t$ be satisfied for exactly $N = N(t)$ values of n. Suppose that the sequence $\{b_n\}$ is such that

$$N(t) \sim t^{\beta}(S(t))^{\pm 1} \qquad (t \to +\infty) \tag{9}$$

holds for some $\beta > 0$, for some slow function $S(t)$ and for a suitable choice of the sign of ± 1. Let $f(u)$ be any R-integrable function in the interval $0 \leqq u \leqq 1$. According to a theorem of Pólya * we have then the asymptotic relation

$$\sum_{b_n \leqq t} f(b_n/t) \sim C t^{\beta} S(t)^{\pm 1} \qquad (t \to +\infty), \tag{10}$$

where

$$C = \beta^{-1} \int_0^1 f(u) u^{-1+1/\beta}\, du. \tag{11}$$

Now let the sequence $\{a_n\}$ occurring in (7) be such that condition (9) is satisfied by the sequence $\{b_n\}$ if one chooses $b_n = 1/a_n$. Since

$$\log |L(t;\sigma)| = \sum_{n=1}^{\infty} \log |\cos(t/b_n)| \leqq \sum_{b_n \leqq t} \log |\cos(t/b_n)|,$$

we have

$$\log |L(t;\sigma)| \leqq \sum_{b_n \leqq t} \log \max(|\cos(t/b_n)|, 1/2).$$

This may be written in the form

$$\log |L(t;\sigma)| \leqq \sum_{b_n \leqq t} f(b_n/t) \tag{12}$$

by placing

$$f(u) = \log \max(|\cos(1/u)|, 1/2)$$

so that $f(u)$ is continuous and bounded in the interval $0 < u \leqq 1$. The value of (11) belonging to this $f(u)$ clearly is < 0. It follows therefore from (10) and (12) that

$$L(t;\sigma) = O(\exp(-A t^{\beta} S(t)^{\pm 1})), \quad \text{where} \quad A > 0. \tag{13}$$

In particular,

$$L(t;\sigma) = O[\exp(-t^{\beta-\epsilon})] \tag{14}$$

holds in virtue of (8) for every positive ϵ ($< \beta$). These estimates hold as $t \to +\infty$. Since $L(t;\sigma)$ is an even function, both estimates hold also as $t \to -\infty$ if one replaces t by $|t|$. Thus (3) is satisfied for every β (> 0) and (5) is satisfied for every $\beta > 1$. The situation becomes clearer by considering the following example:

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* Cf. G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 1, p. 68.

Let $a_n = n^{-\alpha}$ so that $\alpha > 1/2$ in virtue of (6). Since there are exactly $[t^{1/\alpha}]$ values of n satisfying the inequality $1/a_n \leqq t$, condition (9) is satisfied by placing $\beta = 1/\alpha$ and $S \equiv 1$. It follows therefore from (13) that

$$(15) \quad L(t;\sigma) = O(\exp(-At^{1/\alpha})), \quad \text{where} \quad A > 0 \qquad (\alpha > 1/2).$$

On comparing (15) with (3), (4) and (5) we see that $\sigma(x)$ has derivatives of arbitrarily high order if $\alpha > 1/2$ and that $\sigma(x)$ is regular analytic along the real axis if $1 \geqq \alpha > 1/2$, finally that $\sigma(x)$ is an entire function if $1 > \alpha > 1/2$. In the case $\alpha > 1$ the function $\sigma(x)$ cannot be regular analytic along the real axis since * the series $a_1 + a_2 + \cdots$ is convergent.†

Another interesting example is $a_n = 1/p_n$ where p_n denotes the n-th prime number. Condition (9) is satisfied by $\beta = 1$ and $(S(t))^{\pm 1} = 1/\log t$. Hence all derivatives of $\sigma(x)$ exist. It remains undecided whether $\sigma(x)$ is regular analytic along the real axis.

The example (15) shows that there exists for every $\gamma < 2$ a sequence $\{a_n\}$ such that the distribution function σ defined by (7) satisfies the condition (5). If σ is a Gaussian distribution function then its Fourier-Stieltjes transform has the form of a Gaussian density of probability so that there exist distribution functions satisfying (5) with $\gamma = 2$. This does not prove, however, that there exist sequences $\{a_n\}$ satisfying the condition (6) and such that the Fourier-Stieltjes transform of the corresponding Bernoulli convolution satisfies (5) with $\gamma = 2$. In fact, the Fourier-Stieltjes transform of a Gaussian distribution function has no zeros, hence it is not of the form (7).

The order of an entire function σ may be estimated by means of the appraisal (5) of the Fourier-Stieltjes transform. This remark holds not only for Bernoulli convolutions but for any distribution function σ satisfying (5). In fact, the argument leading to (4a) shows that (5) implies the inequality

$$(5\text{a}) \quad |\sigma^{(n)}(x)| < M \int_0^{+\infty} t^{n-1} \exp(-t^\gamma)\, dt = M\Gamma(n/\gamma)/\gamma \quad (n = 1, 2, \cdots),$$

where M is a constant. Hence it is clear from Stirling's formula that

$$|c_n|^{1/n} = O(n^{-1+1/\gamma}) \quad \text{where} \quad c_n = \sigma^{(n)}(0)/\Gamma(n+1).$$

It follows therefore from the Hadamard theory that the order of the entire function σ is $\leqq \gamma/(\gamma - 1)$.

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[*] Cf. A. Wintner, *loc. cit.*

[†] The case $a_n = 1/n$ may be treated by using "min" instead of "max." Cf. the joint paper of B. Jessen and of the present author, mentioned above. The treatment given there is based upon an "arithmetical accident" and fails even in the case $a_n = 1/n^a$ if $a \neq 1$.

It is known that if the order of an entire function is finite and not an integer then the genus is less than the order. Hence if there exists at all a distribution function σ satisfying (5) with a $\gamma > 2$ then this σ is an entire function of genus zero or one, although $0 < \sigma(x) < \sigma(x+h) < 1$ for every $x + h > x$. It may be mentioned * that there does or does not exist a distribution function σ such that

$$L(t;\sigma) = B\exp(-|t|^{\gamma}) \qquad (B = B_{\gamma} > 0) \tag{5b}$$

according as γ is or is not within the range $0 \leqq \gamma \leqq 2$.

A general method of estimating (7) by means of the difference $N(2t) - N(t)$ and with or without the assumption (9) will be given in a subsequent note.

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* Cf. P. Lévy, *op. cit.*